# Graphs without Odd Holes, Parachutes or Proper Wheels: A Generalization of Meyniel Graphs and of Line Graphs of Bipartite Graphs

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**Abstract**

We prove that the strong perfect graph conjecture holds for graphs that do not contain parachutes or proper wheels. This is done by showing the following theorem:

*If a graph $G$ contains no odd hole, no parachute and no proper wheel, then $G$ is bipartite or the line graph of a bipartite graph or $G$ contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

To prove this theorem, we prove two decomposition theorems which are interesting in their own rights. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

*Key words:* perfect graph, odd hole, strong perfect graph conjecture, decomposition, star cutset, 2-join, Meyniel graph, line graph of bipartite graph

*Running head:* WP-FREE GRAPHS

## 1 Introduction

A graph is *perfect* if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. A *hole* is a chordless cycle of length at least four. A hole is *odd* (*even*) if it contains an odd (even) number of nodes. A long standing conjecture of Berge [1] states that a graph $G$ is perfect if and only if neither $G$ nor its complement contains an odd hole. (The *complement* $\bar{G}$ of $G$ has node set $V(G)$ and two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in $G$). Berge's conjecture is known as the Strong Perfect Graph Conjecture. It was proved recently by Chudnovsky, Robertson, Seymour and Thomas [3]. This conjecture was already known to hold for several special classes of perfect graphs.

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For example, Meyniel [13] showed that if every odd cycle of $G$ is a triangle or contains at least two chords, then $G$ is perfect. These graphs are known as *Meyniel graphs.*

Another well-known example is the following. A graph $G$ is the *line graph* of a graph $H$ if $V(G) = E(H)$ and $v_i, v_j \in V(G)$ are adjacent if $e_i, e_j \in E(H)$ have a common endnode. If $G$ is the line graph of a bipartite graph $H$, then $G$ is perfect. (Indeed, the maximum degree of a node in $H$ is equal to the chromatic index of $H$ and this implies that the chromatic number of $G$ equals the size of its largest clique).

In this paper we introduce WP-free graphs (W stands for proper Wheel and P stands for Parachute: They will be defined later) and characterize the WP-free graphs that are perfect. Meyniel graphs and line graphs of bipartite graphs are perfect WP-free graphs.

WP-free graphs do not contain the complement of a hole $H$, $|H| \geq 7$. We show that if a WP-free graph contains no odd hole, then it is perfect. This is acheived by proving a structural theorem for even-signable WP-free graphs, a class of graphs that is larger than the class of WP-free graphs containing no odd hole. The proof of this theorem follows from two independent decomposition theorems, each interesting in its own right. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams [2]. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas [3] for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

### 1.1 Wheels, Parachutes and WP-Free Graphs.

A *wheel* $(H, v)$ consists of a hole $H$ together with a node $v$, called the *center*, that has at least three neighbors in $H$. If $v$ has exactly $k$ neighbors in $H$, the wheel is called a *$k$-wheel.*

**Definition 1.1** *A* T-wheel *(or* twin wheel*) is a 3-wheel $(H, v)$ such that the three neighbors of $v$ in $H$ are consecutive.*

*A wheel $(H, v)$ is a* $\Delta$-free wheel *(or* triangle-free wheel*) if the neighbors of $v$ in $H$ induce a stable set. That is, the graph induced by $(H, v)$ is a triangle-free graph.*

*A wheel $(H, v)$ is a* universal wheel *if $v$ is adjacent to every node of $H$.*

*A wheel $(H, v)$ is an* L-wheel *(or* line wheel*) if $(H, v)$ is the line graph of a cycle $C$ with a unique chord and $V(C)$ induces a triangle-free graph, i.e. the unique chord of $C$ is not a triangular chord. So $v$ has neighbors $a_1$, $a_2$, $b_1$ and $b_2$ in $H$, $H = a_1, P_1, b_1, b_2, P_2, a_2, a_1$ and $P_1$, $P_2$ are paths of length greater than 1.*

*A wheel that is in none of the above four classes is called a* proper wheel.

**Definition 1.2** *An* L-parachute *$LP(a_1b_1, a_2b_2, a_3, z)$ is a graph induced by an L-wheel $(H, a_3)$ where $H = a_1, b_1, \ldots, z, \ldots, b_2, a_2, \ldots, a_1$, where $a_1$, $a_2$, $b_1$, $b_2$ are the neighbors of $a_3$ in $H$, together with a chordless path $P = a_3, \ldots, z$ of length greater than 1. No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of $P$.*

*A* T-parachute *$TP(a_1, a_2, b_1, b_2, z)$ is a graph induced by a T-wheel $(H, a_2)$ where $H = b_1, a_1, b_2, \ldots, z, \ldots, b_1$, where $b_1$, $a_1$, $b_2$ are the neighbors of $a_2$ in $H$, together with a chordless path $P = a_2, \ldots, z$ of length greater than 1. No node of $H \setminus \{z, b_1\}$ may be adjacent to an intermediate node of $P$.*

*A* parachute *is either an L-parachute or a T-parachute.*

For an L-parachute or a T-parachute, let $P_1$, $P_2$ be respectively the $b_1z$-path and the $b_2z$-path in $H \setminus a_1$ and $C_1$, $C_2$ be the cycles induced by $P \cup P_1$ and $P \cup P_2$. Note that in a T-parachute or an L-parachute, the paths $P_1$ and $P_2$ may have length one.

In the definition below and throughout the rest of the paper, $G$ *contains* $G'$ if $G'$ is an induced subgraph of $G$ and $G$ is $G'$*-free* if $G$ does not contain $G'$.

**Definition 1.3** *A graph is* WP-free *if it contains neither a proper wheel nor a parachute.*

**Lemma 1.4** *Let $G$ be an L-parachute $LP(a_1b_1, a_2b_2, a_3, z)$ with the property that no proper subgraph of $G$ is a parachute or a proper wheel. Then $G$ is of one of the following types, see Figure 1.*

*type a) No intermediate node of $P$ is adjacent to $b_1$ or $b_2$.*

*type b) An intermediate node of $P$ is adjacent to $b_1$, $(C_2, b_1)$ is a $\Delta$-free wheel and $b_1$ is adjacent to $z$.*

*type c) An intermediate node of $P$ is adjacent to $b_1$, $(C_2, b_1)$ is a T-wheel and $b_1$ is adjacent to $z$.*

*type d) An intermediate node of $P$ is adjacent to $b_1$, $(C_2, b_1)$ is an L-wheel and $b_1$ is adjacent to $z$.*

*Proof:* If no intermediate node of $P$ is adjacent to $b_1$ or $b_2$, $G$ is of type a). Suppose an intermediate node of $P$ is adjacent to $b_1$, and $b_1$ is not adjacent to $z$. If the neighbor of $a_3$ in $P$ is the only intermediate node of $P$ that is adjacent to $b_1$, there is a smaller proper wheel with center $a_3$. Otherwise there is a smaller L-parachute. So $b_1$ must be adjacent to $z$ and therefore $(C_2, b_1)$ is a wheel, which is not proper by assumption and is not universal since $b_1$ and $b_2$ are nonadjacent. So $(C_2, b_1)$ is either a $\Delta$-free wheel or a T-wheel or an L-wheel and we have types b) or c) or d) in these three cases. $\square$

**Lemma 1.5** *Let $G$ be a T-parachute $TP(a_1, a_2, b_1, b_2, z)$ that is not an L-parachute and such that no proper subgraph of $G$ is a parachute or a proper wheel. Then $G$ is one of the following graphs, see Figure 2.*

*type a) No intermediate node of $P$ is adjacent to $b_1$ or $b_2$.*

*type b) An intermediate node of $P$ is adjacent to $b_1$, $(C_2, b_1)$ is a $\Delta$-free wheel and $b_1$ is adjacent to $z$.*

*type c) An intermediate node of $P$ is adjacent to $b_1$, $(C_2, b_1)$ is a T-wheel and $b_1$ is adjacent to $z$.*

*Proof:* If no intermediate node of $P$ is adjacent to $b_1$ or $b_2$ we have type a). Assume an intermediate node of $P$ is adjacent to $b_1$. Since no proper induced subgraph of $G$ is a parachute or a proper wheel, then $b_1$ is adjacent to $z$ and therefore $(C_2, b_1)$ is a wheel, which is not proper by assumption and is not universal since $b_1$ and $b_2$ are nonadjacent. If $(C_2, b_1)$ is an L-wheel then $G$ is also an L-parachute of type c. So $(C_2, b_1)$ is either a $\Delta$-free wheel or a T-wheel, and we have types b) or c). $\square$

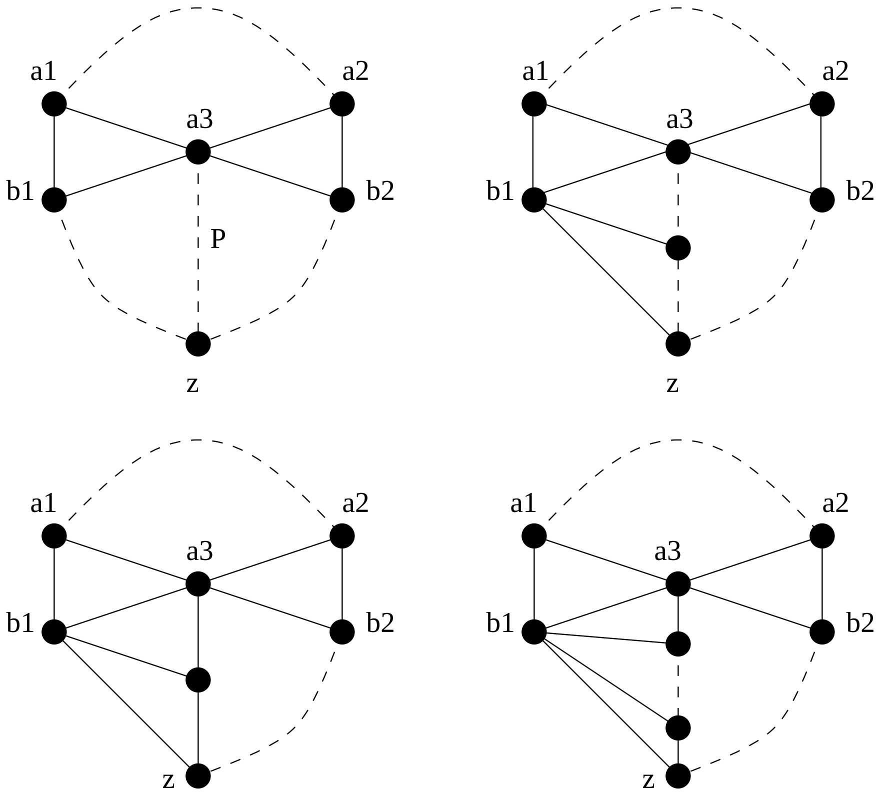


Figure 1: L-parachutes

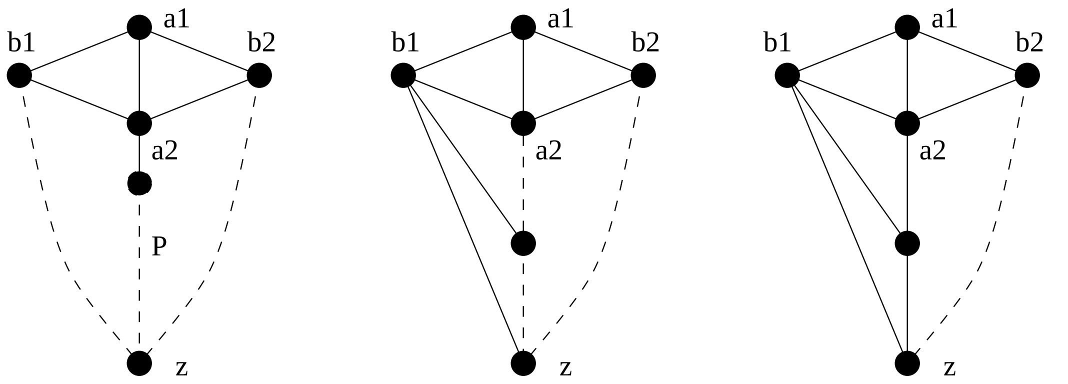


Figure 2: T-parachutes

A *cap* is a cycle $C$ of length at least 5 with a unique chord that is a triangular chord of $C$. A cap is *odd* if $C$ is odd.

**Remark 1.6** *A graph $G$ is Meyniel if and only if $G$ contains no odd hole and no odd cap.*

*Proof:* $G$ is not a Meyniel graph if and only if $G$ contains an odd cycle that is not a triangle and has at most one chord. Let $C$ be a smallest such cycle. $C$ is either an odd hole or an odd cap. $\square$

If $G$ contains a cap, $G$ contains an odd hole or an odd cap. So the class of cap-free graphs contains the class of Meyniel graphs. The structure of cap-free graphs is very similar to the structure of Meyniel graphs and was studied in [7]. Since every proper wheel and parachute contains a cap, the class of WP-free graphs contains the class of cap-free graphs.

A *diamond* is a cycle of length 4 with a unique chord. A *claw* is a graph on 4 nodes, one of them with degree 3 and the others with degree 1. The following characterization of the line graphs of bipartite graphs is due to Harary and Holtzmann [11]. It can be proven following the arguments of the proof of Remark 3.2.

**Remark 1.7** *$G$ is the line graph of a bipartite graph if and only if $G$ contains no odd hole, no claw and no diamond.*

It is straightforward to check that if $G$ is a proper wheel or a parachute, then $G$ contains a claw or a diamond. This implies the following remark:

**Remark 1.8** *The class of WP-free graphs containing no odd hole includes the class of Meyniel graphs and the class of line graphs of bipartite graphs.*

### 1.2 Even-Signable Graphs

We study even-signable WP-free graphs, a class of graphs that includes WP-free graphs containing no odd hole.

A graph $G$ is *signed* if its edges are given *odd* or *even* labels. A subset of $E(G)$ is odd (resp. even) if it contains an odd (resp. even) number of edges labeled odd. A graph $G$ is *even-signable* if there exists a signing of its edges such that every triangle is odd and every hole is even. These graphs were introduced in [6]. More results can be found in [7]. Note that, if $G$ contains no odd hole, then $G$ is even-signable since all its edges can be labeled odd. Also, if $G$ is triangle-free, then $G$ is even-signable since all its edges can be labeled even. It is shown in [7] that, if one can efficiently test whether $G$ is even-signable, then one can also efficiently test whether $G$ contains an odd hole.

The graphs in Figure 3 are relevant in this paper. Solid lines represent edges and dotted lines represent paths of length at least one. The first three graphs are referred to as *3-path configurations* ($3PC$'s). The first graph is called a $3PC(x,y)$ (or $3PC(\cdot,\cdot)$), where node $x$ and node $y$ are connected by three paths $P_1, P_2$ and $P_3$. The second is called a $3PC(xyz,u)$ (or $3PC(\Delta,\cdot)$), where $xyz$ is a triangle and $P_1$, $P_2$ and $P_3$ are three paths with endnodes $x$, $y$ and $z$ respectively and a common endnode $u$. The third is called a $3PC(xyz,uvw)$ (or $3PC(\Delta,\Delta)$), consists of two node disjoint triangles $xyz$ and $uvw$ and paths $P_1$, $P_2$ and $P_3$

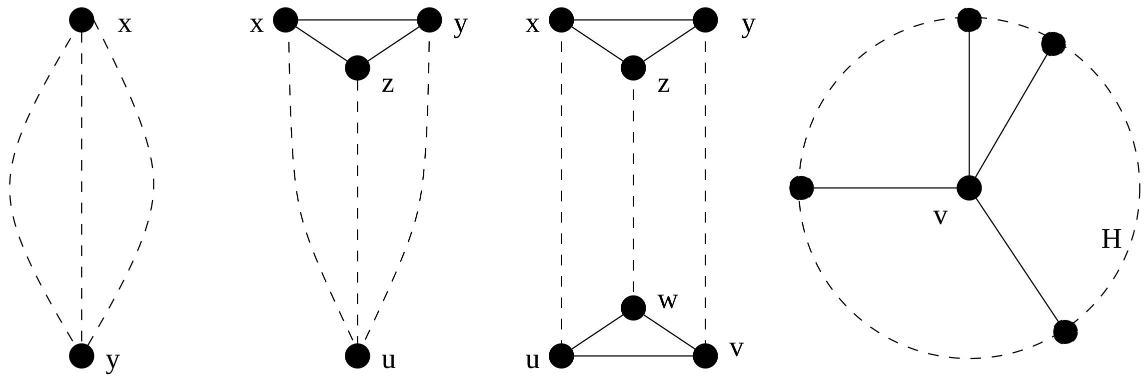


Figure 3: 3-path configurations and wheel

with endnodes $x$ and $u$, $y$ and $v$ and $z$ and $w$ respectively. In all three cases, the nodes of $P_i \cup P_j$ induce a hole for $i \neq j$. This implies that all paths of a $3PC(\cdot,\cdot)$ have length greater than one, and at most one path of a $3PC(\Delta,\cdot)$ has length one.

A wheel $(H, v)$ is an *odd wheel* if it contains an odd number of triangles: Since $H$ is a hole, every triangle of $(H, v)$ contains $v$ and two adjacent nodes of $H$. So a wheel $(H, v)$ is odd if the subgraph of $H$, induced by the neighbors of $v$, contains an odd number of edges.

A consequence of a theorem of Truemper [14] is the following co-NP characterization of even-signable graphs.

**Theorem 1.9** *A graph is even-signable if and only if it contains no $3PC(\Delta,\cdot)$ and no odd wheel.*

A derivation of this result and a discussion of Truemper's theorem can be found in [7] and [8]. We find it convenient to work with even-signable graphs because the graphs of Theorem 1.9 are easy to spot when proving results.

### 1.3 The Main Theorem

In a graph $G$, a node set $S$ is a *cutset* if the graph $G \setminus S$ is disconnected. A node set $S$ is a *star* if it consists of a node $x$ and neighbors of $x$. Chvátal [4] showed that a minimally imperfect graph cannot contain a star cutset.

A graph $G$ has an *extended 2-join* if $V(G)$ can be partitioned into subsets $V_A$, $V_B$ and $U$ ($U$ possibly empty), such that $A_1, A_2 \in V_A$, $B_1, B_2 \in V_B$ are nonempty disjoint sets with the following properties: (i) every node of $A_1$ is adjacent to every node of $B_1$, every node of $A_2$ is adjacent to every node of $B_2$ and these are the only adjacencies between $V_A$ and $V_B$, (ii) every node of $U$ is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$, (iii) the connected components of $G(V_A)$ meet both $A_1$ and $A_2$ and, if $|A_1| = |A_2| = 1$ then $V_A$ does not induce a chordless path and, (iv) the connected components of $G(V_B)$ meet both $B_1$ and $B_2$ and, if $|B_1| = |B_2| = 1$ then $V_B$ does not induce a chordless path.

An extended 2-join is called *extended strong 2-join* when, in addition, both $A_1 \cup B_1$, $A_2 \cup B_2$ induce cliques. When $U = \emptyset$, the extended 2-join reduces to the *2-join* introduced by Cornuéjols and Cunningham [9].

In this paper, we prove the following result.

**Theorem 1.10** *Let $G$ be an even-signable WP-free graph that is not a triangle-free graph nor the line graph of a triangle-free graph. Then $G$ contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

**Corollary 1.11** *Let $G$ be a WP-free graph that contains no odd hole. Then $G$ is a bipartite graph or the line graph of a bipartite graph or $G$ contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

This result, together with the next two theorems, implies that the Strong Perfect Graph Conjecture holds for WP-free graphs.

**Theorem 1.12** [4] *A minimally imperfect graph cannot contain a star cutset.*

The following theorem follows from a result of Conforti, Cornuéjols, Gasparyan and Vušković [5] on universal 2-amalgams.

**Theorem 1.13** [5] *A minimally imperfect graph cannot contain an extended strong 2-join.*

**Theorem 1.14** *A WP-free graph is perfect if and only if it contains no odd hole.*

*Proof:* The "if" part is obvious. We prove the "only if" statement. Let $G$ be a minimally imperfect WP-free graph that contains no odd hole. Then $G$ is even-signable. By Theorem 1.12, $G$ does not contain a star cutset and by Theorem 1.13, $G$ does not contain an extended strong 2-join. Furthermore, $\bar{G}$ is connected. Hence, by Corollary 1.11, $G$ is a bipartite graph or the line graph of a bipartite graph. In both cases $G$ is perfect, a contradiction. □

## 1.4 Proof Outline of the Main Theorem

A graph $G$ has an *amalgam* if $V(G)$ can be partitioned into subsets $V_A$, $V_B$ and $U$ ($U$ possibly empty), such that $A_1 \in V_A$, $B_1 \in V_B$ are nonempty sets with the following properties: (i) every node of $A_1$ is adjacent to every node of $B_1$ and these are the only adjacencies between $V_A$ and $V_B$, (ii) $U$ is a clique and every node of $U$ is adjacent to $A_1 \cup B_1$ and possibly to other nodes in $V(G)$, (iii) $|V_A| \geq 2$ and $|V_B| \geq 2$.

The notion of *amalgam* was introduced by Burlet and Fonlupt [2]. The *join* introduced by Cunningham and Edmonds [10] is an amalgam with $U = \emptyset$.

A node $u$ is *universal* for a graph $H$ if $u$ is adjacent to all the nodes in $H$.

Theorem 1.10 is in fact the consequence of the following stronger results.

**Theorem 1.15** *Let $G$ be an even-signable WP-free graph that does not contain an L-wheel nor a $3PC(\Delta, \Delta)$. Then either $G$ is a triangle-free graph plus at most one universal node or $G$ contains a clique cutset or an amalgam.*

This theorem is proved in Section 2.

**Theorem 1.16** *Let $G$ be an even-signable WP-free graph that contains an L-wheel or a $3PC(\Delta, \Delta)$. Then either $G$ is the line graph of a triangle-free graph or $G$ contains a star cutset or an extended strong 2-join or $\bar{G}$ is disconnected.*

This theorem is proved in Section 3.

## 2 GM-graphs

**Definition 2.1** *A graph $G$ is a GM-graph (Generalized Meyniel graph) if $G$ is an even-signable WP-free graph and $G$ does not contain an L-wheel or a $3PC(\Delta, \Delta)$.*

In this section we prove Theorem 1.15 which states that every GM-graph $G$ is a triangle-free graph plus at most one universal node or $G$ contains a clique cutset or an amalgam. This theorem is interesting in its own right. Indeed, when specialized to Meyniel graphs, this result is a famous theorem of Burlet and Fonlupt [2]: every Meyniel graph $G$ is a bipartite graph plus at most one universal node or $G$ contains a clique cutset or an amalgam. In addition, Theorem 1.15 has algorithmic consequences that we do not develop in this paper.

We first introduce some definitions.

For $S \subseteq V(G)$, we let $G(S)$ be the subgraph of $G$ induced by the nodes in $S$. We let $N(S)$ denote the set of nodes with at least one neighbor in $S$. Two nodes $u$, $v$ are *twins with respect to* $S$ if $u$ and $v$ are adjacent and $N(u) \cap (S \setminus \{u, v\}) = N(v) \cap (S \setminus \{u, v\})$. If $u$ and $v$ are twins with respect to $V(G)$, we simply say that $u$ and $v$ are *twins*.

We denote a cap by $(H, x)$ where $H$ is a hole and $x$ is a node adjacent to consecutive node $a$, $b$ in $H$. The nodes $a$, $b$ are called the *attachments* of the cap.

Given three disjoint node sets $A$, $B$ and $C$ such that no node of $A$ is adjacent to a node of $B$, a *direct connection* between $A$ and $B$ is a minimal path $P$ (in terms of its node set) between a node in $A$ and a node in $B$. The direct connection $P$ *avoids* the set $C$ if no node of $P$ is in $C$.

We will need the following technical lemma about caps in GM-graphs.

**Lemma 2.2** *Let $G$ be a GM-graph that contains no clique cutset but contains a cap $(H, x)$ with attachments $a$, $b$. Then $G$ has the following properties:*

*(i) In every direct connection $P = x_1, \ldots, x_n$ from $x$ to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$, node $x_n$ is a universal node for $H$ or is a twin of $a$ or $b$ with respect to $H$.*

*(ii) Let $U$ be the set of universal nodes for $H$ that are endnodes of some such direct connection and let $T$ be the set of twins of $a$ or $b$ that are endnodes of some direct connection. Then $T$ is a clique, every node of $U$ is adjacent to every node of $T$ and $U$ contains two nonadjacent nodes $u$ and $u'$.*

*(iii) There exists a node $x'$ adjacent to $u$ and $u'$ such that $(H, x')$ is a cap with attachments $a$ and $b$.*

*Proof:* Suppose that (i) does not hold. Among all caps $(Q, y)$ with attachments $\{a, b\}$ and direct connection $P = x_1, \ldots, x_n$ from $y$ to $V(Q) \setminus \{a, b\}$ in $G \setminus (V(Q) \cup \{y\})$, such that $x_n$ is neither a universal node for $Q$ nor a twin of $a$ or $b$ with respect to $Q$, choose $(Q, y)$ and $P$ such that $P$ is shortest possible. It follows from this choice of $(Q, y)$ and $P$ that no node $x_j$ with $j \leq n-1$ is adjacent to both $a$ and $b$. Also, at least one of the nodes $a$, $b$ is not adjacent to any of the nodes $x_j$ for $2 \leq j \leq n-1$ (otherwise $Q$ can be modified, $P$ shortened and (i) still does not hold). Assume w.l.o.g. that $b$ is not adjacent to any of the nodes $x_j$ for $2 \leq j \leq n-1$. By construction, $x_n$ has at least one neighbor $z$ in $V(Q) \setminus \{a, b\}$.

Assume first that $x_n$ has one or two neighbors in $Q$. We only sketch the proof since checking the various cases is routine. If $n = 1$, there is a $3PC(\Delta, \cdot)$ or an odd wheel or a T-parachute or a $3PC(\Delta, \Delta)$. So $n \geq 2$. Since $G$ does not contains an L-wheel or a $3PC(\Delta, \Delta)$ or a $3PC(\Delta, \cdot)$, it follows that $a$ is adjacent to some node $x_j$ for $j \leq n-1$ or $b$ is adjacent to $x_1$. Let $S$ be the hole containing $V(P) \cup \{b\}$ and possibly nodes of $(V(Q) \setminus \{a\}) \cup \{y\}$. Since $(S, a)$ is neither a proper wheel nor an L-wheel, either $a$ or $b$ is adjacent to $x_1$. But now, there is a T-parachute or a $3PC(\Delta, \cdot)$ or a proper wheel or a $3PC(\Delta, \Delta)$, a contradiction.

So $x_n$ has at least three neighbors in $Q$. Assume that $x_n$ is adjacent to at most one of the nodes $a$, $b$, and let $S$ denote the hole with nodes in $V(Q) \cup \{x_n\}$ that contains $a$, $b$ and $x_n$. If $n \geq 2$, we have a contradiction to the choice of $(Q, y)$ and $P$. If $n = 1$, we have a T-parachute if $x_1$ is adjacent to $a$ or $b$ and a proper wheel otherwise. So $x_n$ is adjacent to both $a$ and $b$ and at least one other node of $Q$. Since $(Q, x_n)$ is not a proper wheel nor a line wheel, $x_n$ must be universal for $Q$ or a twin of $a$ or $b$ with respect to $Q$. This completes the proof of (i).

Suppose that (ii) does not hold. Let $x_n$ and $x'_m$ be the last nodes of direct connections $P$ and $P'$ where $x_n \in T$ and $x'_m \in T \cup U$ are not adjacent. Assume w.l.o.g. that $x_n$ is a twin of $b$ with respect to $H$. If $x'_m$ is a twin of $a$, then $V(H) \cup \{x_n, x'_m\}$ induces a T-parachute, a contradiction. So we can assume w.l.o.g. that both $x_n$ and $x'_m$ are adjacent to $a$, $b$ and the neighbor $b'$ of $b$ in $V(H) \setminus \{a\}$.

If $P$ and $P'$ have no common node nor adjacent nodes, let $C$ denote the hole induced by $V(P) \cup V(P') \cup \{b', x\}$. Now $(C, b)$ is a proper wheel unless $C$ is of length four, i.e. $x$ is adjacent to $x_n$ and $x'_m$. But then there is a T-parachute induced by $(V(H) \setminus \{b\}) \cup \{x, x_n, x'_m\}$.

So $P$ and $P'$ have a common node or adjacent nodes. Let $Q$ be a shortest path from $x_n$ to $x'_m$ in $P \cup P'$. There is a T- parachute with top node $b'$, side nodes $x_n$ and $x'_m$ and side paths contained in $Q$.

So $x_n$ and $x'_m$ are adjacent. This shows that $T$ is a clique and every node of $T$ is adjacent to every node of $U$. Since $U \cup T$ is not a clique cutset separating $x$ from $V(H) \setminus \{a, b\}$, there must exist two nodes in $U$ that are nonadjacent, say $u$ and $u'$. This completes the proof of (ii).

Now we prove (iii). Let $P$ and $P'$ be direct connections from $x$ to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$ that end in $u$ and $u'$ respectively.

If $P$ and $P'$ have no common node nor adjacent nodes, let $C$ denote the hole induced by $V(P) \cup V(P') \cup \{b', x\}$, where $b'$ is the neighbor of $b$ in $V(H) \setminus \{a\}$. Since $(C, b)$ is not a proper wheel, $b$ must be adjacent to every node of $P$ and $P'$. By symmetry, $a$ is adjacent to every node of $P$ and $P'$. Since $(C, a)$ is not a proper wheel, it follows that $C$ has length four. So (iii) holds in this case.

Now assume that $P$ and $P'$ have a common node or adjacent nodes. Let $Q$ be a shortest path from $u$ to $u'$ in $P \cup P'$, let $C$ be the hole induced by $V(Q) \cup \{u, u', b'\}$ and $C'$ the hole induced by $V(Q) \cup \{u, u', a'\}$ where $a'$ is the neighbor of $a$ in $V(H) \setminus \{b\}$. If $Q$ contains an intermediate node adjacent to $b$, then $Q$ has length two, otherwise $(C, b)$ or $(C', b)$ is a proper wheel. By symmetry, the same holds for $a$. Furthermore, when $Q$ has length two, the claim holds if its intermediate node is adjacent to both $a$ and $b$. So, whether $Q$ has length two or not, we can assume w.l.o.g. that $b$ is not adjacent to any intermediate node of $Q$. Let $M$ be a shortest path from $b$ to $Q$ in $V(P) \cup V(P') \cup \{x\}$. Let $m$ be the node of $M$ adjacent to $Q$. By the choice of $Q$, $m$ has at most three neighbors in $Q$. If $m$ has two adjacent neighbors $q_1, q_2$ in $Q$, there is a $3PC(mq_1q_2, b)$. So we can assume w.l.o.g. that $m$ has only one neighbor $z$ in $Q$ since, otherwise we can modify $Q$ to get the desired property. Now there is a parachute with side nodes $u$ and $u'$, side paths $Q_{uz}$ and $Q_{u'z}$, top node $b'$, center node $b$ and middle path $M$. This completes the proof of (iii). $\square$

### 2.1 D-structures

**Definition 2.3** *A* D-structure $(C_1, C_2, K)$ *of* $G$ *consists of disjoint sets of nodes* $C_1, C_2$ *and* $K$, *where* $|C_1| \geq 2$, $|C_2| \geq 2$ *and the nodes of* $K$ *induce a clique of* $G$ *(possibly* $K$ *is empty). Furthermore, the subgraph* $G(C_1)$ *is connected and every node in* $C_1$ *is universal for* $C_2 \cup K$, *every node in* $C_2$ *is universal for* $C_1 \cup K$ *and there exists no node in* $V(G) \setminus (C_1 \cup C_2 \cup K)$ *adjacent to a node in* $C_1$ *and a node in* $C_2$.

This notion was introduced in [7], where it was shown that, if a cap-free graph $G$ contains a D-structure, then $G$ contains an amalgam. Here, we show the following result.

**Theorem 2.4** *If* $G$ *is a GM-graph that contains a D-structure, then* $G$ *contains a clique cutset or an amalgam.*

*Proof:* Let $U$ be the set of nodes in $V(G) \setminus (C_1 \cup C_2 \cup K)$ that are adjacent to $C_1$ and are connected to a node in $C_2$ by a path with nodes in $V(G) \setminus (C_1 \cup K)$.

**Claim 1:** *If* $G$ *contains no clique cutset, every node in* $U$ *is universal for* $C_1$.

*Proof:* Assume not and choose $u \in U$ contradicting the claim and $c_2 \in C_2$ connected by a shortest possible path with nodes in $V(G) \setminus (C_1 \cup K)$ and among all these paths, let $P = x_0 = u, x_1, \ldots, x_n, x_{n+1} = c_2$ be one with the largest number of nodes adjacent to $C_1$. Since $C_1$ and $C_2$ belong to a D-structure, then $n \geq 1$. By our choice, intermediate nodes of $P$ are either nonadjacent to $C_1$ or universal for $C_1$. Since $u$ is adjacent but not universal to $C_1$ and $G(C_1)$ is connected, $C_1$ contains adjacent nodes $a, b$ such that $u$ is adjacent to $a$ but not to $b$.

We now show that $G(V(P) \cup \{a, b\})$ contains a cap $(H, x)$ where $H = a, x_i, P_{x_ix_j}, x_j, a$. Assume that $P$ contains consecutive nodes that are both adjacent to $a$ and let $x_i$, $x_{i+1}$ be such nodes with highest index. Then $i < n$ by the definition of $D$-structure, so $P_{x_{i+2}x_{n+1}}$ contains a node adjacent to $a$. Let $x_j$ be such a node, of lowest index and let $H = a, x_{i+1}, P_{x_{i+1}x_j}, x_j, a$. Now $(H, x_i)$ is a cap. If $P$ does not contain consecutive nodes that are both adjacent to $a$, let $x_i$ be the node of lowest index $i \geq 1$ adjacent to $a$ (and $b$) and let $H = a, x_0, P_{x_0x_i}, x_i, a$. Now $(H, b)$ is a cap.

Let $(H, x)$ be a cap where $H = a, x_i, P_{x_ix_j}, x_ja$ and $j \geq i + 2$. Since $G$ contains no clique cutset, by Lemma 2.2, $G$ contains nonadjacent nodes $z$, $z'$, universal for $H$ (possibly adjacent to $x$) and, since $K$ is a clique, at least one of these nodes, say $z$, is not in $K$. Now $z \notin C_1$, since otherwise $x_{j-1}$ is adjacent to $z \in C_1$ but not $a \in C_1$ and so, if $j = n + 1$, the definition of $D$-structure is contradicted, and if $j \leq n$, the choice of $u$ is contradicted. Furthermore $z \notin C_2$, since otherwise $x_i$ is adjacent to $a \in C_1$ and $z \in C_2$, a contradiction to the definition of $D$-structure.

So $z \in V(G) \setminus (C_1 \cup C_2 \cup K)$. Now $j = i+2$ and $z$ is universal for $C_1$, else the minimality of $P$ is contradicted. Let $P'$ be obtained from $P$ by removing $x_{i+1}$ and adding $z$. Now $P$ and $P'$ have the same length and $P'$ contradicts our assumption that $P$ has the largest number of neighbors in $C_1$. So this completes the proof of Claim 1.

Let $K'$ contain the nodes in $K$ that are not universal for $U$ and $K'' = K \setminus K'$. Define $A = C_1$, $B = C_2 \cup K' \cup U$. We show that, if $G$ contains no clique cutset, $(A, B, K'')$ is an amalgam of $G$. Claim 1 shows that every node in $B$ is universal for $A$ and by definition of $K''$, every node in $K''$ is universal for $U$. Since $(C_1, C_2, K)$ is a D-structure, every node in $K''$ is universal for $C_1 \cup C_2 \cup K'$.

**Claim 2:** *Let $G'$ be the graph obtained from $G$ by removing all edges with one endnode in $A$ and the other in $K'$. If $G$ contains no clique cutset, in $G'(V(G) \setminus (C_2 \cup K'' \cup U))$ no path connects a node of $K'$ and a node of $C_1 = A$.*

*Proof:* Let $P = x, v_1, \ldots, v_p, k$ be a shortest path connecting $x \in C_1$ and $k \in K'$ and contradicting the claim. No intermediate node of $P$ is adjacent to a node in $C_2$ else, by the definition of $U$, $v_1$ belongs to $U$. If $p \geq 2$, let $c_2$ be any node in $C_2$ and $H = k, x, v_1, \ldots, v_p, k$. Then $(H, c_2)$ is a cap and since $G$ contains no clique cutset, by Lemma 2.2, $G$ contains two nonadjacent nodes universal for $H$ and one of them, say $z$, is not in $K$. Since $v_1$ is adjacent to $x \in C_1$ and $z$, $z$ is not in $C_2$. $z \notin C_1$, else $v_p$ is adjacent to $z \in C_1$ and $k$ and $P' = z, v_p, k$ contradicts the minimality of $P$. Now since $v_1 \notin U$ and $v_1$ is adjacent to $z$, $z$ is also not in $U$. So $z \in V(G) \setminus (C_1 \cup C_2 \cup K \cup U)$ and $P' = x, z, k$ again contradicts the minimality of $P$.

So $P = x, v_1, k$. Since $k$ is not universal for $U$, $U$ contains a node not adjacent to $k$. Let $u$ be such a node, connected in $G \setminus (C_1 \cup K)$ to a node of $C_2$, say $c_2$, by a shortest possible path and among these paths, let $Q = x_1 = u, \ldots, x_m = c_2$ have the largest number of neighbors of $C_1$. Note that $Q$ may contain several nodes that are universal for $C_1$, so let $u_1, \ldots, u_n$ be such nodes of $Q$, with $u_i$ closer to $u$ than $u_{i+1}$ ($u_1 = x_1 = u$ and $u_n = x_m = c_2$). Note that all nodes $u_1, \ldots, u_{n-1}$ belong to $U$.

We now show that no two consecutive nodes of $Q$ are universal for $C_1$. For, let $u_{i-1}$, $u_i$, be consecutive nodes of highest index. Note that $i < n-1$ by the definition of $D$-structure. So let $H = x, u_i, Q_{u_i u_{i+1}}, u_{i+1}, x$, and $(H, u_{i-1})$ is a cap and again since $G$ contains no clique cutset, by Lemma 2.2, there exists a node $z$ not in $K$ universal for $H$. Since $u_i$ is adjacent to $x \in C_1$ and $z$, then $z \notin C_2$. Let $x_j$ be the neighbor of $u_{i+1}$ in $Q_{u_i u_{i+1}}$. Now $z \notin C_1$, else since $x_j$ is adjacent to $z$, then $x_j \in U$ and, since $x_j$ is not adjacent to $x$, Claim 1 is contradicted. So since $z$ is adjacent to $x$ and to $x_j$, then $z$ is in $U$. Now $Q_{u_i u_{i+1}}$ has length 2 and $z$ has no neighbor in $V(Q) \setminus V(Q_{u_i u_{i+1}})$ else the minimality of $P$ is contradicted. Let $P'$ be obtained from $P$ by removing $x_j$ and adding $z$. Now $P$ and $P'$ have the same length and $P'$ contradicts the fact that $P$ has the largest number of neighbors in $C_1$. So no two consecutive nodes of $Q$ are universal for $C_1$.

Let $x_i$ be the node of smallest index adjacent to $k$. Since by our choice, $k$ is not adjacent to $u_1$ but is adjacent to all the nodes $u_2, \ldots, u_n = x_m$, such a node exists and it belongs to $Q_{x_2 u_2}$ (possibly $n = 2$). If $x_i = u_2$, let $H = x, u_1, Q_{u_1 u_2}, u_2, x$, and $(H, k)$ is a cap. So by the same argument as above, there exists a node $z$ not in $K$ universal for $H$. Again, the above argument rules out the existence of such a node $z$ and so $x_i$ is an intermediate node of $Q_{u_1 u_2}$. Let $H = x, u_1, Q_{u_1 x_i}, x_i, k, x$. Since $v_1 \notin U$, $v_1$ is not adjacent to any node in

$Q_{u_1x_i}$. Now $(H, v_1)$ is a cap and so there exists a node $z$ not in $K$ universal for $H$. Since $x_1 = u_1$ is adjacent to $x \in C_1$ and $z$, $z \notin C_2$. Since $x_2$ is adjacent to $z$ but not to $x \in C_1$, by Claim 1, $z \notin C_1$. So the same argument as above rules out the existence of such a node $z$ when $z$ is adjacent to $x_1$, $x_2$ and $x_3$. So $i = 2$ and $z$ is adjacent to $x_1$, $x_2$ but not $x_3$. By Lemma 2.2(i), $G \setminus \{x, k\}$ contains a chordless path $R = v_1 = r_1, \ldots, r_q = z$. Note that intermediate nodes of $R$ may be adjacent to $x$ or $k$ but not to $x_1$ or $x_2$. At least one node of $R$ belongs to $C_1 \cup K$, otherwise there exists a path from $v_1$ to $C_2$ whose intermediate nodes are in $V(R) \cup V(Q)$ and this path contains no node of $C_1 \cup K$, thus proving that $v_1 \in U$, a contradiction. So let $r_i$ be the node of $R$ with lowest index in $C_1 \cup K$. Then $r_i$ is adjacent to $c_2$. So let $S = s_1 = v_1, \ldots, s_{n-1} = x_3, s_n = x_2$ be a shortest $v_1x_2$-path whose nodes are in $R_{r_1r_i} \cup Q_{x_2x_m}$. Since $S_{s_2s_{n-1}}$ is a direct connection from $v_1$ to $H$, avoiding $x$ and $k$, by Lemma 2.2(i), $x_3$ must be a twin of node $k$ with respect to $H$ (indeed, $x_3$ is not adjacent to $x_1$, so it can be neither universal for $H$ nor a twin of $x$). Now, by Lemma 2.2(ii), $z$ is adjacent to $x_3$, a contradiction. This completes the proof of Claim 2.

The following claim shows that $(A, B, K'')$ is an amalgam of $G$.

**Claim 3:** *Let $G''$ be obtained from $G$ by removing all edges with one endnode in $A$ and the other in $B$. Then in $G''(V(G) \setminus K'')$, no path connects a node in $A$ and a node in $B$.*

*Proof:* Let $P = x_1, \ldots, x_n$ be a chordless path between $x_1$ in $A$ and $x_n$ in $B$ and contradicting the claim. Claim 1 shows that if $x_n \in C_2$, then $x_2 \in U$, a contradiction. Claim 2 shows $x_n \notin K'$. So $x_n \in U$ and let $P_{x_n}$ be a path with nodes in $V(G) \setminus (C_1 \cup K)$ connecting $x_n$ and a node in $C_2$. Now there is a path with nodes in $V(G) \setminus (C_1 \cup K)$ between $x_2$ and a node in $C_2$ only using nodes of $V(P_{x_n}) \cup V(P)$. So $x_2$ must belong to $U$, a contradiction. □

## 2.2 M-structures

M-structures were first introduced by Burlet and Fonlupt [2] in their study of Meyniel graphs.

An induced subgraph $G(V_1)$ of $G$ is called an *M-structure* (multipartite structure) if $\bar{G}(V_1)$ contains at least two connected components each with at least two nodes. Let $W_1, \ldots, W_k$ be the node sets of these connected components. The *proper subclasses* of $G(V_1)$ are the sets $W_i$ of cardinality greater than or equal to 2. The *partition* of an M-structure is denoted by $(W_1, \ldots, W_r, K)$ where $K$ is the union of all non-proper subclasses. Note that $K$ induces a clique in $G$.

**Lemma 2.5** *An M-structure $G(V_1)$ of $G$ is maximal with respect to node inclusion, if and only if there exists no node $v \in V(G) \setminus V_1$ such that $v$ is universal for a proper subclass of $G(V_1)$.*

*Proof:* Let $G(V_1 \cup \{u\})$ be an M-structure. Assume node $u$ is not universal for any proper subclass of $G(V_1)$. In $\bar{G}(V_1 \cup \{u\})$ node $u$ is adjacent to at least one node in each of the proper subclasses. Thus there exists only one proper subclass in $G(V_1 \cup \{u\})$, contradicting the assumption.

Conversely let node $u$ be universal for some proper subclass $W_i$ of $G(V_1)$. Then $\bar{G}(V_1 \cup \{u\})$ has at least two components with more than one node, the graph induced by $W_i$ and at least one component with more than one node in $(V_1 \cup \{u\}) \setminus W_i$. □

The above proof yields the following:

**Corollary 2.6** *Let $G(V_1)$ and $G(V_2)$ be M-structures with $V_1 \subseteq V_2$. Let $W_i$ and $Z_j$ be connected components of $\bar{G}(V_1)$ and $\bar{G}(V_2)$ respectively having nonempty intersection. Then $W_i \subseteq Z_j$.*

**Lemma 2.7** *Let $G(V_1)$ be a maximal M-structure of a GM-graph $G$ that has no clique cutset. Then no node in $V(G) \setminus V_1$ can be adjacent to two distinct proper subclasses of $G(V_1)$.*

*Proof:* Assume node $x' \in V(G) \setminus V_1$ is adjacent to two proper subclasses $W_1$ and $W_2$ of $G(V_1)$. Since $G(V_1)$ is maximal, by Lemma 2.5 node $x'$ is not universal for either of the classes. Also since $\bar{G}(W_1)$ is connected, $W_1$ contains a pair of nonadjacent nodes $x_1, y_1$, such that $x'$ is adjacent to $x_1$ but not to $y_1$. Similarly $W_2$ contains a pair of nonadjacent nodes $x_2, y_2$ such that $x'$ is adjacent to $x_2$ but not to $y_2$. Let $H = x_1, x_2, y_1, y_2, x_1$. Then $(H, x')$ is a cap. Since $G$ has no clique cutset, by Lemma 2.2(iii), $G$ contains a node $x$ (possibly $x' = x$) adjacent to $x_1, x_2$ but not to $y_1, y_2$ and two nonadjacent nodes $u, u'$ that are universal for the cap $(H, x)$.

**Claim 1:** *Nodes $u$ and $u'$ are universal for $W_1$ and $W_2$ and neither $u$ nor $u'$ is in $W_1 \cup W_2$.*
*Proof:* Note first that the edges of $\bar{G}(V_1)$ that have their endnodes in $\{x_1, y_1, x_2, y_2, x, u, u'\}$ are $x_1y_1$, $x_2y_2$, $xy_1$, $xy_2$ and $uu'$. If the claim does not hold, then $W_1$ or $W_2$, say $W_1$, has the property that, in $\bar{G}$, $u$ or $u'$ has a neighbor in $W_1$, or $u$ or $u'$ is in $W_1$. In both cases, $\bar{G}(W_1 \cup \{x, u, u'\})$ is connected. Consider a shortest path in this graph between $x$ and $u$, $u'$. W.l.o.g. let $P = u, z_1, \ldots, z_n, x$ be such a path. Now if $n = 1$ and $u'$ is adjacent to $z_1$ in $\bar{G}$, then $\bar{G}$ contains a triangle $z_1, u, u'$ together with a chordless path $z_1, x, y_2, x_2$ and no other edge connects the triangle and the path. This is the complement of a $T$-parachute on six nodes. Otherwise, if $n > 1$ or $u'$ is not adjacent to $z_1$ in $\bar{G}$, then $u, z_1, \ldots, z_n, x, y_2, x_2$ contains a chordless path of length five. Again, this is the complement of a $T$-parachute on six nodes and the proof of Claim 1 is complete.

So by Lemma 2.5, since $u$, $u'$ are nonadjacent, they must belong to the same proper subclass of $G(V_1)$, say $W_3$, which is distinct from $W_1$, $W_2$.

**Claim 2:** *Node $x$ is universal for $W_3$.*
*Proof:* Assume not. Then $\bar{G}(W_3 \cup \{x\})$ is connected. Let $P = u, z_1, \ldots, z_n, x$ be a shortest path in this graph between $u$, $u'$, say $u$, and $x$. Now the same proof as in Claim 1 shows the existence of a parachute.

So, by Lemma 2.5, $x$ belongs to $G(V_1)$. However, in $\bar{G}(V_1)$, $x$ is adjacent to $y_1 \in W_1$ and $y_2 \in W_2$, a contradiction to the fact that $W_1$, $W_2$ are distinct proper subclasses of $G(V_1)$. □

**Theorem 2.8** *If $G$ is a GM-graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then $G$ contains a clique cutset or an amalgam.*

*Proof:* If $G$ contains a D-structure $(C_1, C_2, K)$ then, by Lemma 2.4, $G$ contains an amalgam. So the theorem follows from the proof of the following statement:

*If $G$ is a GM-graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then $G$ contains a D-structure $(C_1, C_2, K)$.*

Let $G(V_1)$ be an M-structure of $G$ satisfying the above property and $G(V_2)$ a maximal M-structure with $V_1 \subseteq V_2$.

**Claim 1:** *The M-structure $G(V_2)$ either contains at least three proper subclasses or contains exactly two proper subclasses not both of which are stable sets.*

*Proof:* If $G(V_1)$ contains a proper subclass, say $W_i$, which is not a stable set, by Corollary 2.6, there exists a proper subclass, say $Z_j$ of $G(V_2)$ such that $W_i \subseteq Z_j$. Then $Z_j$ is not a stable set. If all proper subclasses of $G(V_1)$ are stable sets, then $G(V_1)$ has at least three proper subclasses say $W_1, W_2, \dots, W_k$. If $G(V_2)$ has only two proper subclasses, say $Z_1$, $Z_2$, then by Corollary 2.6, we may assume w.l.o.g. that $W_1 \cup W_2 \subseteq Z_1$. Then $Z_1$ is not a stable set, since every node in $W_1$ is adjacent to a node in $W_2$. This completes the proof of Claim 1.

**Claim 2:** *Suppose that $G(V_2)$ is a maximal M-structure of $G$ with partition $(W_1, W_2, K)$, where $W_1$ is not a stable set. Then $G$ contains a D-structure $(C_1, C_2, K)$.*

*Proof:* Let $C_1$ be a connected component of $G(W_1)$ with more than one node. Let $C_2 = W_2$. Then $(C_1, C_2, K)$ is a D-structure, since by Lemma 2.7 no node of $V(G) \setminus V_2$ is adjacent to a node in $C_1$ and a node in $C_2$, and $|C_2| \geq 2$, since $W_2$ is a proper subclass of $G(V_2)$. This completes the proof of Claim 2.

**Claim 3:** *Suppose that $G(V_2)$ is a maximal M-structure of $G$ with at least three proper subclasses. Then $G$ contains a D-structure $(C_1, C_2, K)$.*

*Proof:* Let $W_1, W_2, \dots, W_l$, $l \geq 3$ be the proper subclasses of $G(V_2)$ and let $K$ be the collection of all non-proper subclasses. Let $C_1$ be the nodes in two proper subclasses of $G(V_2)$ (note that $G(C_1)$ is a connected graph), $C_2$ be the nodes in all the other proper subclasses of $G(V_2)$. Then $(C_1, C_2, K)$ is a D-structure since $|C_1| \geq 2$, $|C_2| \geq 2$ and Lemma 2.7 shows that the only nodes having neighbors in both $C_1$ and $C_2$ belong to $K$. So the proof of Claim 3 is complete. □

**Corollary 2.9** *Let $G$ be a GM-graph that contains a cap. Then $G$ contains a clique cutset or an amalgam.*

*Proof:* Assume $G$ contain a cap but no clique cutset. By Lemma 2.2, $G$ contains a cap $(H, x)$ and nonadjacent nodes $u, u'$ universal for $(H, x)$. Since $\bar{G}(V(H) \cup x)$ is connected, $G$ contains an $M$-structure with proper subclasses $W_1 = \{u, u'\}$ and $W_2 = \{V(H) \cup x\}$ and $W_2$ is not a stable set. By Theorem 2.8, $G$ contains an amalgam. □

In [7], it was shown that, if $G$ is a cap-free graph, then $G$ contains an amalgam or $G$ is triangulated or $G$ is a triangle-free graph plus at most one universal node. Theorem 1.15 follows from this result and Corollary 2.9. Here, for the sake of completeness, we give a direct proof (without using [7]).

## 2.3 Expanded Holes

An *expanded hole* consists of nonempty sets of nodes $S_1, \dots, S_n$, $n \geq 4$, not all singletons, such that, for all $1 \leq i \leq n$, the graphs $G(S_i)$ are connected. Furthermore, every $s_i \in S_i$ is adjacent to $s_j \in S_j$, $i \neq j$, if and only if $j = i + 1$ or $j = i - 1$ (modulo $n$).

**Lemma 2.10** *Let $G$ be a cap-free graph and let $H$ be a hole of $G$. If $s$ is a node having two adjacent neighbors in $H$, then either $s$ is universal for $H$ or $s$ together with $H$ induces an expanded hole.*

*Proof:* Let $s$ be a node with two adjacent neighbors in $H$. If $s$ has no other neighbors on $H$, then $s$ induces a cap with $H$. Let $H = x_1, \ldots, x_n, x_1$ with node $s$ adjacent to $x_1$ and $x_n$. If $s$ is not universal for $H$, and does not induce an expanded hole together with $H$, then let $k$ be the smallest index for which $s$ is not adjacent to $x_k$. Let $l$ be the smallest index such that $l > k$ and $s$ is adjacent to $x_l$. Now node $x_{k-2}$ ($x_n$ if $k = 2$) together with the hole $s, x_{k-1}, \ldots, x_l, s$ forms a cap. $\square$

**Lemma 2.11** *Let $G$ be a cap-free graph and let $S = \cup_{i=1}^{n} S_i$, $n > 4$, be a maximal expanded hole in $G$ with respect to node inclusion. Either $G$ contains an M-structure with a proper subclass that is not a stable set of $G$, or all nodes that are adjacent to a node in $S_i$ and a node in $S_{i+1}$ ($S_{n+1} = S_1$) for some $i$, are universal for $S$ and induce a clique of $G$.*

*Proof:* Let $u$ be a node adjacent to $s_1 \in S_1$ and $s_2 \in S_2$. By applying Lemma 2.10 to any hole that contains $s_1$ and $s_2$ and a node each from the sets $S_j$, $j > 2$, we have that $u$ is adjacent to all nodes in $S \setminus (S_1 \cup S_2)$, else the maximality of $S$ is contradicted. Now since node $u$ is adjacent to $s_1$, $s_2$ and is universal for all sets $S_j$, $j > 2$, Lemma 2.10 shows that $u$ is universal for $S_1$ and $S_2$, hence for $S$.

Let $u$ and $v$ be two nonadjacent nodes that are universal for $S$. Then $u$, $v$ together with $s_1 \in S_1$, $s_2 \in S_2$ and $s_4 \in S_4$ induces an M-structure with proper sets $W_1 = \{u, v\}$ and $W_2 = \{s_1, s_2, s_4\}$. Furthermore $W_2$ is not a stable set of $G$. $\square$

**Theorem 2.12** *A cap-free graph that contains an expanded hole contains a clique cutset or an amalgam.*

*Proof:* Let $S = \cup_{i=1}^{n} S_i$ be a maximal expanded hole in $G$. First assume that $n = 4$. Then the node set $S$ induces an M-structure with proper subclasses $S_1 \cup S_3$ and $S_2 \cup S_4$. $S_2 \cup S_4$ is not a stable set because, say, $|S_2| \geq 2$ and $G(S_2)$ is connected. Hence by Theorem 2.8 we are done. Now assume that $n > 4$. By Theorem 2.8 we may assume that $G$ does not contain an M-structure with a proper subclass that is not a stable set of $G$. By Theorem 2.4, it is sufficient to show that $G$ contains a D-structure $(C_1, C_2, K)$. Assume w.l.o.g. that $|S_2| \geq 2$ and let $K$ be the set of nodes that are universal for $S$. Lemma 2.11 shows that $K$ is a clique of $G$. Let $C_1 = S_2$ and $C_2 = S_1 \cup S_3$. Lemma 2.11 shows that every node that is adjacent to a node of $C_1$ and a node of $C_2$ is universal for $S$ and hence belongs to $K$. Therefore $(C_1, C_2, K)$ is a D-structure. $\square$

## 2.4 A Proof of Theorem 1.15

Now we are ready to prove Theorem 1.15.

*Proof:* If $G$ contains a cap, by Corollary 2.9, $G$ contains a clique cutset or an amalgam.

Assume that $G$ is a connected cap-free graph. If $G$ is a triangulated graph, $G$ is either a clique or it contains a clique cutset. If $G$ is a clique and contains at least four nodes, $G$

contains a join and if $G$ contains less than four nodes, then $G$ is a triangle-free graph plus at most one universal node.

Assume now that $G$ is a connected cap-free graph that contains a hole. Let $F$ be a maximal node set inducing a biconnected triangle-free subgraph of $G$. Assume that $G$ does not have a clique cutset or an amalgam.

**Claim 1:** *Every node in $V(G) \setminus F$ that has at least two neighbors in $F$ is universal for $F$.*

*Proof:* Let $u$ be a node in $V(G) \setminus F$ having at least two neighbors in $F$. The graph induced by $F \cup \{u\}$ contains a triangle $u, x, y$ else the maximality of $F$ is contradicted. Let $H$ be a hole in $G(F)$ containing $x$ and $y$. ($H$ exists since, by biconnectedness, $x$ and $y$ belong to a cycle and since $G(F)$ contains no triangle, a smallest cycle containing $x$ and $y$ is a hole). Lemma 2.10 shows that either $u$ is universal for $H$ or forms an expanded hole with $H$. Theorem 2.12 rules out the latter possibility. Let $F' \subseteq F$ be a maximal set of nodes such that $G(F')$ contains $H$, is biconnected and such that node $u$ is universal for $F'$. If $F \neq F'$, then since $G(F)$ and $G(F')$ are biconnected, some $z \in F \setminus F'$ belongs to a hole that contains an edge of $G(F')$. Let $H'$ be such a hole. By Lemma 2.10 and Theorem 2.12, node $u$ is adjacent to all the nodes of $H'$. Let $F'' = F' \cup V(H')$. $G(F'')$ is biconnected, $u$ is universal for $F''$. Hence $F''$ contradicts the maximality of $F'$. Hence $u$ is universal for $F$ and the proof of Claim 1 is complete.

**Claim 2:** *Let $U$ be the set of universal nodes for $F$. Then the nodes in $U$ induce a clique of $G$.*

*Proof:* Let $w, z \in U$ be two nonadjacent nodes of $U$ and let $v_1, \ldots, v_n, v_1$ be a hole of $G(F)$. Then nodes $w$, $z$ together with $v_1$, $v_2$, $v_3$ and $v_4$ induce an M-structure, either with two proper subclasses not both of which are stable if $v_1$ and $v_4$ are not adjacent, or with three proper subclasses. By Theorem 2.8, $G$ contains an amalgam. This completes the proof of Claim 2.

**Claim 3:** *$V(G) = F \cup U$.*

*Proof:* Let $S = V(G) \setminus (F \cup U)$. By Claim 1, every node in $S$ has at most one neighbor in $F$. Let $C$ be a connected component of $G(S)$. By maximality of $F$, there is at most one node in $F$, say $y$, that has a neighbor in $C$. If such a node $y$ exists, let $C_1, \ldots, C_l$ be the connected components of $G(S)$ adjacent to $y$. Let $V_1 = C_1 \cup \ldots C_l \cup \{y\}$, $A = \{y\}$, $K = U$, $V_2 = V(G) \setminus (V_1 \cup K)$ and $B$ be the set of neighbors of $y$ in $F$. Then $(A, B, K)$ is an amalgam of $G$, separating $V_1$ from $V_2$.

If no component of $G(S)$ is adjacent to a node of $F$, let $V_1 = U \cup S$, $A = U$, $V_2 = B = F$. Then $(A, B, \emptyset)$ is an amalgam of $G$. This completes the proof of Claim 3.

If $U$ contains at least two nodes, then let $V_1 = A = U$, $V_2 = B = F$ and $(A, B, \emptyset)$ is an amalgam of $G$. If $U$ contains at most one node, then $G$ is a triangle-free graph plus at most one universal node. □

# 3 Line graphs of triangle-free graphs and extensions

In this section, we prove Theorem 1.16.

### 3.1 L-graphs

If $G$ is the line graph of a graph $H$, the nodes in a maximal clique of $G$ correspond either to the edges in a triangle of $H$ or to the edges incident with a node of $H$.

A graph $G$ is *$L\Delta$-free* if $G$ is the line graph of a triangle-free simple graph. In this case, there obviously is a one to one correspondence between maximal stars of $H$ and maximal cliques of $G$.

Harary and Holtzmann [11] characterize the line graphs of bipartite simple graphs. In Remark 3.1 below we characterize $L\Delta$-free graphs in a similar way. The proof can be easily deduced from the proof of Remark 3.2.

**Remark 3.1** *The following three conditions are equivalent.*

*1) $G$ is $L\Delta$-free.*

*2) $G$ contains no claw and no diamond.*

*3) Every node $v \in V(G)$ belongs to at most two maximal cliques $C_1$ and $C_2$, and no node of $C_1 \setminus \{v\}$ is adjacent to a node in $C_2 \setminus \{v\}$.*

Maffray and Reed [12] characterize the line graphs of bipartite multigraphs. The following remark has a similar proof and characterizes the line graphs of triangle-free multigraphs. A *gem* is a graph induced by a 5-cycle $a, b, c, d, e, a$ with chords $ac$ and $ad$.

**Remark 3.2** *The following three conditions are equivalent.*

*1) $G$ is the line graph of a triangle-free multigraph $H$.*

*2) $G$ contains no claw, no gem and no universal 4-wheel.*

*3) Every node $v \in V(G)$ belongs to at most two maximal cliques $C_1$ and $C_2$, and $C_1 \cap C_2$ consists of $v$ and all its twins. No node of $C_1 \setminus C_2$ is adjacent to a node in $C_2 \setminus C_1$.*

*Proof:* Assume $G$ is the line graph of a triangle-free multigraph $H$. Since an edge of $H$ has at most two endnodes, $G$ is claw-free. Assume $G$ contains a gem $G'$ with $V(G') = \{a, b, c, d, e\}$ and $E(G') = \{ab, bc, cd, de, ea, ac, ad\}$. Since $\{b, c, a, d\}$ induce a diamond of $G$, the edges $e_a$, $e_c$ of $H$ corresponding to the nodes $a$ and $c$ of $G$ are parallel edges with endnodes $s$, $t$, while $e_b$ has $s$ but not $t$ as endnode and $e_d$ has $t$ but not $s$ as endnode. By the same argument applied to the diamond induced by $\{a, c, d, e\}$, $e_a$, $e_d$ are parallel, a contradiction. So $G$ cannot contain a gem. The same argument shows that $G$ cannot contain a universal 4-wheel and 1) $\rightarrow$ 2).

Assume that $G$ satisfies 2) and suppose first that $v$ belongs to three maximal cliques, $C_1$, $C_2$, $C_3$. Since every pair of cliques contains nonadjacent nodes, $C_1$ contains (possibly coincident) nodes $a_2$, $a_3$, $C_2$ contains (possibly coincident) nodes $b_1$, $b_3$ and $C_3$ contains (possibly coincident) nodes $c_1$, $c_2$ where $b_1$ and $c_1$, $a_2$ and $c_2$, $a_3$ and $b_3$ are nonadjacent. Together with $v$, these nodes induce a graph that contains a claw, a gem or an universal 4-wheel. For, choose $a_2, a_3$, $b_3, b_1$, $c_1, c_2$ so that they form the maximum number of coincident pairs. If all three pairs are concident, there is a claw. If two of the pairs are coincident, there is a gem. Otherwise there is a universal 4-wheel. So every node of $G$ is in at most two maximal cliques. $C_1 \cap C_2$ obviously contains all twins of $v$. If a node in $C_1 \cap C_2$ is not a twin of $v$ then it belongs to three maximal cliques, a contradiction. Finally, if a node of $C_1 \setminus C_2$

is adjacent to a node in $C_2 \setminus C_1$, $v$ is in at least three maximal cliques, again a contradiction and 2) $\rightarrow$ 3).

Assume that $G$ satisfies 3) and construct $H$ as follows: $V(H)$ corresponds to the set of maximal cliques of $G$. For every node belonging to a unique maximal clique $C_1$, add to $H$ a pendant edge attached to the node $v_{C_1}$. For every node belonging to maximal cliques $C_1$, $C_2$, add to $H$ an edge with endnodes $v_{C_1}$ and $v_{C_2}$, so that the nodes in $C_1 \cap C_2$ are associated to parallel edges. Since no node of $C_1 \setminus C_2$ is adjacent to a node of $C_2 \setminus C_1$, $G$ is the line graph of $H$ and $H$ is a triangle-free multigraph. So 3) $\rightarrow$ 1). $\square$

Let $G$ be an $L\Delta$-free graph. Let $G'$ be an induced subgraph of $G$ and $K'$ be a clique of $G'$ with at least two nodes. Since $G$ contains no diamond by Remark 3.1, there exists a unique clique $K$ of $G$ containing $K'$. We say that $K$ is the *extension* of $K'$.

We say that a clique $K$ of $G$ is *big* if $K$ has more than two nodes and $K$ is *flat* if $K$ contains exactly two nodes. Unless otherwise specified, all the cliques will be maximal.

A connected graph $G$ has a *2-node cutset* $\{u, v\}$ if $G \setminus \{u, v\}$ is a disconnected graph.

**Definition 3.3** *A graph $G$ is an* L-graph *if it is an $L\Delta$-free graph and it satisfies the following properties.*

*a) $G$ is connected, contains a big clique and every node of $G$ is in two cliques. (Equivalently, $H$ is connected, contains a node of degree at least 3 and every node has degree at least 2).*

*b) $G$ contains no join. (Equivalently, $H$ contains no cutnode).*

*c) For every 2-node cutset of $G$, one of the components is an induced path. (Equivalently, if $H$ contains two edges whose removal disconnects $H$, then one of the two components is a path).*

It follows from this definition that if $G$ is an L-graph, then $G$ contains at least two big cliques. In fact, every hole of $G$ has at least two edges belonging to big cliques.

A *segment* $S$ of an L-graph $G$ is a maximal induced connected subgraph of $G$ such that no pair of nodes of $S$ belongs to the same big clique of $G$. Note that a segment is a chordless path of $G$ and may have length one or zero. Every node $x$ of $G$ is in exactly one segment, that we call $S_x$, so the segments of $G$ partition $V(G)$. A segment $S$ is *long* if $|V(S)| \geq 3$, *short* if $|V(S)| = 2$ and *atomic* if $|V(S)| = 1$. Furthemore if a segment $S$ is short and $K_x$, $K_y$ are the big cliques containing the endnodes of $S$, no atomic segment is in $K_x \cap K_y$ (i.e. $K_x \cap K_y$ is empty) since $G$ contains no diamond.

Every L-graph has at least three segments. If $G$ is a $3PC(\Delta, \Delta)$ or an L-wheel, then $G$ is an L-graph and $G$ is minimal with this property. These two graphs are called *elementary* L-graphs.

**Lemma 3.4** *Let $S_1$, $S_2$, $S_3$ be three segments in an L-graph $G$. Then $G$ contains an elementary L-graph $B$, such that $S_1$, $S_2$ and $S_3$ are all in $B$ and $S_1$, $S_2$ are contained in distinct segments of $B$.*

*Proof:* By Definition 3.3 b), $H$ is 2-connected and therefore $H$ contains a cycle going through any two given edges. This implies the existence of a hole in $G$ going through nodes $x_1 \in S_1$ and $x_2 \in S_2$. Since all cliques of $S_1$, $S_2$ are atomic or flat in $G$ and $S_1$, $S_2$ are maximal with this property, it follows that, in every hole $C = P_1, S_1, P_2, S_2$ of $G$ going through $x_1$ and $x_2$, at least one edge of $P_1$ and at least one edge of $P_2$ are extendable to big cliques of $G$.

Assume first that $G$ contains a hole $C$ going through $x_1$ and $x_2$ such that $S_3 \in C$. Let $C = x_1, Q_1, x_2, Q_2, x_1$, where $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$. By Definition 3.3 c), $Q_1 \setminus \{x_1, x_2\}$ and $Q_2 \setminus \{x_1, x_2\}$ are in the same connected component of $G \setminus \{x_1, x_2\}$, so they are connected by a shortest path $P$ in $G \setminus \{x_1, x_2\}$. Since every node of $C$ is in two cliques and the cliques of $S_1$, $S_2$ are atomic or flat in $G$, then $P = y_1, \ldots, y_m$ (possibly $m = 1$) , where $y_1$ belongs to an extension of a clique in $P_1$ and $y_m$ belongs to an extension of a clique in $P_2$. If $m = 1$, $C \cup P$ induce an L-wheel and if $m > 1$, $C \cup P$ induce a $3PC(\Delta, \Delta)$.

Assume now that no hole $C$ going through $x_1$ and $x_2$ contains $S_3$. By Definition 3.3 b), c) $S_3$ belongs to a path $P = y_1, \ldots, y_m$ (possibly $m = 1$) , where $y_1$ belongs to an extension of a clique of $P_1$ and $y_m$ belongs to an extension of a clique $P_2$. This shows that $C \cup P$ induce an elementary L-graph of $G$. □

**Lemma 3.5** *Let $C$ be a hole of an L-graph $G$. For every segment $S_3$ of $G \setminus C$, there is a path $P$ in $G$ containing $S_3$ such that $C \cup P$ is an elementary L-graph $G_1$ in $G$. The segments of $G_1$ are $P$ and two subpaths of $C$.*

*Proof:* The proof is identical to the previous one. □

## 3.2 Tripods

A *triad* is a graph consisting of three internally node-disjoint paths $t, \ldots, x$; $t, \ldots, y$ and $t, \ldots, z$ of length greater than one, where $t$, $x$, $y$, $z$ are distinct nodes. Furthermore, the graph induced by the nodes of the triad contains no other edge than those of the three paths. Node $t$ is the *meet* of the triad.

A *fan* is a graph consisting of a path $P = x, \ldots, y$ of length greater that one, together with a node $z$ not in $P$ adjacent to at least one intermediate node in $P$ and not adjacent to $x$ and $y$. Node $z$ is the *center* of the fan and the edges connecting $z$ to $P$ are the *spokes*. Furthermore, the graph induced by the nodes of the fan contains no other edge than those of $P$ and the spokes.

A *stool* consists of a triangle $x'y'z'$ together with three node-disjoint paths $x', \ldots, x$; $y', \ldots, y$ and $z', \ldots, z$ of length at least one. Furthermore, the graph induced by the nodes of the stool contains no other edges than those of the triangle and of the three paths.

A *tripod* is a triad or a stool or a fan. Nodes $x$, $y$ and $z$ are called the *attachments* of the tripod.

**Lemma 3.6** *Let $G$ be a node-minimal graph with the following properties.*

*(i) $G$ contains nodes $x$, $y$, $z$ such that no edge has both endnodes in $\{x, y, z\}$.*

*(ii) $V(G) \setminus \{x, y, z\}$ is nonempty.*

*(iii) $G$ and $G \setminus \{x, y, z\}$ are both connected.*

*Then $G$ is a tripod with attachments $x$, $y$ and $z$.*

*Proof:* Let $G$ be a graph with the above properties and let $P_{xy} = x = y_1, \ldots, y_m = y$ be a shortest $xy$-path in $G \setminus \{z\}$, $P_{xz}$ and $P_{yz}$ similarly defined. Assume w.l.o.g. that $P_{xy}$ is not shorter than any of the other two. If $P_{xy}$ contains an intermediate node that is a neighbor of $z$, then by the minimality of $G$, $V(G) = V(P_{xy}) \cup \{z\}$ and $G$ is a fan.

Otherwise let $P = z_1, \ldots, z_n$ be a direct connection between $z$ and $V(P_{xy}) \setminus \{x, y\}$ ($P$ exists since $G$ and $G \setminus \{x, y, z\}$ are both connected), and let $P_z = z, z_1, \ldots, z_n$. By the minimality of $G$, $V(G) = V(P_{xy}) \cup V(P_z)$ and $z_n$ either has a unique neighbor in $P_{xy}$ or $z_n$ has two neighbors in $P_{xy}$ and these neighbors are adjacent.

By the minimality of $G$, at most one among $x$ and $y$ has a neighbor in $P_z$. Assume $x$ has a neighbor in $P_z$. Then by the minimality of $G$, $z_n$ is adjacent to the neighbor of $x$ in $P_{xy}$, possibly to $x$ and to no other node of $P_{xy}$. Now $P_{zy}$ is longer than $P_{xy}$, contradicting our choice. So by symmetry, neither $x$ nor $y$ have a neighbor in $P_z$ and therefore if $z_n$ has two neighbors in $P_{xy}$, neither of these nodes is $x$ or $y$ and we have a stool in this case. Finally, if $z_n$ has a unique neighbor in $P_{xy}$, say $t$, then $t$ is not adjacent to $x$ or $y$ else our choice of $P_{xy}$ is again contradicted and in this case we have a triad. □

### 3.3 Links

Let $G$ be a graph that contains an L-wheel or a $3PC(\Delta, \Delta)$.

Let $G'$ be an L-graph that is an induced subgraph of $G$. A *link* of $G'$ is a chordless path $P = x_1, \ldots, x_n$ in $G \setminus G'$ (possibly $n = 1$) such that $x_1$ has a neighbor $x_0$ in $G'$, $x_n$ has a neighbor $x_{n+1}$ in $G'$, and $x_0$, $x_{n+1}$ are *nonadjacent* nodes in *distinct* segments of $G'$. Furthermore $P$ is minimal with the above property.

**Lemma 3.7** *Let $G'$ be an L-graph that is an induced subgraph of a graph $G$. Let $U$ be a connected component of $G \setminus G'$ such that $N(U) \cap G'$ is not contained in a clique of $G'$ and is not contained in a segment of $G'$. Then*

- *a) either $U$ contains a link, or*
- *b) $N(U) \cap V(G') = \{x,\ y,\ z\}$ where $x$ and $y$ are the distinct endnodes of a long segment and $z$ is an atomic segment such that $z = K_x \cap K_y$, where $K_x$ and $K_y$ are the big cliques containing $x$ and $y$.*

*If $G$ is a WP-free graph, only a) can occur.*

*Proof:* If $N(U) \cap G'$ contains two nodes that are nonadjacent and in distinct segments, then a) holds. So we may assume that this is not the case.

Since $N(U) \cap G'$ contains two nodes, say $x$ and $z$, that are in distinct segments, then $x$ and $z$ belong to some big clique $K_x$ of $G'$. Since $N(U) \cap G'$ contains a node $y$ not in $K_x$, by Remark 3.1 3) we can assume that $y$ is not adjacent to $x$. Since a) does not hold, the segments $S_x$, $S_y$ coincide, so $S_z$, $S_y$ are distinct. This implies that $z$ and $y$ belong to some big clique $K_y$ and $z = K_x \cap K_y$ is an atomic segment of $G'$. Now all the other nodes of $G'$ are readily seen to be nonadjacent to $U$. So b) holds.

Now, we show that if $G$ is a WP-free graph, only a) can occur. Assume now that b) holds and let $S_x$ be the long segment of $G'$ containing $x$ and $y$ and $S_z$ the atomic segment containing $z$. Since $G'$ is an L-graph, by Lemma 3.4, $G'$ contains an elementary L-graph $G$"

containing $S_x$ and $S_z$ in distinct segments and $G$" must be an L-wheel with atomic segment $S_z$.

Let $U'$ be the subgraph of $G$ induced by $V(U) \cup \{x, y, z\}$ with edges $xz$, $yz$ removed. By Lemma 3.6 $U'$ contains an induced subgraph $T$ which is a tripod with attachments $x$, $y$, $z$. We show that the graph $G" \cup T$ contains a proper wheel or a parachute or a $3PC(\Delta, \cdot)$. Therefore if $G$ is a WP-free graph, b) cannot occur and a) is the only possibility.

If $T$ is a triad, then $G" \cup T$ contains an L-parachute $LP(x'x, y'y, z, t)$, where $t$ is the meet of the triad.

If $T$ is a stool with triangle $abc$, then $G" \cup T$ contains a $3PC(abc, z)$.

If $T$ is a fan with center $z$, then $G" \cup T$ contains a proper wheel with center $z$.

If $T$ is a fan with center $x$ or $y$ but not $z$, say $x$, let $P_{yz}$ be a shortest $yz$-path in $T$ and $C_{yz}$ be the chordless cycle closed by edge $yz$ with $P_{yz}$. If $x$ has two neighbors in $C_{yz}$, then these neighbors are nonadjacent (since $T$ is not a fan with center $z$). So, in this case, we have an L-parachute of type a) $LP(x'x, y'y, z, t)$, where $t$ is the neighbor of $x$ distinct from $z$. Now assume that $x$ has at least three neighbors on $P_{yz}$. Since $(C_{yz}, x)$ is a wheel which is not proper, $(C_{yz}, x)$ is either a $\Delta$-free wheel or a T-wheel or an L-wheel. We then have an L-parachute $LP(x'x, y'y, z, t)$ where $t$ is the neighbor of $x$ closest to $y$ in $P_{yz}$. This L-parachute is of types b), c) or d). (Remark that in this proof we have not used the fact that $G$ contains no T-parachute). □

So it is important to study the links of an L-graph $G'$ of $G$. The following lemma gives a list of all possible links, when $G'$ is an elementary L-graph.

**Lemma 3.8** *Let $G$ be an even-signable WP-free graph, $G'$ an elementary L-graph in $G$ and $P = x_1, \ldots, x_n$ be a link of $G'$. Then*

*a) either $G' \cup P$ is an L-graph, or*

*b) $n = 1$ and $x_1$ is either universal for $G'$ or the twin of an endnode of a segment of $G'$.*

*Proof:* We use the following notation: The two big cliques of $G'$ are $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. The segments of $G'$ are $P_1 = a_1, \ldots, b_1$, $P_2 = a_2, \ldots, b_2$ and $P_3 = a_3, \ldots, b_3$. If $G'$ is an L-wheel, then $a_3 = b_3$ and the segment $P_3$ is atomic, while $P_1$ and $P_2$ are long segments. Otherwise $G'$ is a $3PC(\Delta, \Delta)$ and its segments are either long or short. The nodes in distinct segments $P_i$ and $P_j$ induce a hole of $G'$, that we denote with $H_{ij}$.

**Case 1** $n = 1$.

Let $x_0$, $x_2$ be neighbors of $x_1$ that are nonadjacent and in distinct segments, say $P_i$ and $P_j$, of $G'$. Then either $x_0$, $x_2$ are the unique neighbors of $x_1$ in $H_{ij}$ or $(H_{ij}, x_1)$ is a wheel.

**Case 1.1** $x_0$, $x_2$ are the unique neighbors of $x_1$ in $H_{ij}$, or $(H_{ij}, x_1)$ is a $\Delta$-free wheel.

**Case 1.1.1** $G'$ is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $i = 1$ and $j = 3$. Now $x_1$ has more that two neighbors in $G'$, else we have a $3PC(A, x_0)$ or a $3PC(B, x_0)$. If $x_1$ has at most one neighbor in $A$ and at most one neighbor in $B$, we have a $3PC(A, x_1)$ or a $3PC(B, x_1)$. Since no two neighbors of $x_1$ in $H_{13}$ are adjacent, then $x_1$ cannot be adjacent to both $a_1$ and $a_3$ or both $b_1$ and $b_3$, so by symmetry we may assume that $x_1$ is adjacent to $a_2$, $a_3$ but not to $a_1$. Let $x_0$ be the neighbor of $x_1$, closest to $a_1$ in $P_1$. Then we have a T-parachute $TP(a_2, a_3, x_1, a_1, x_0)$ of type a or b.

**Case 1.1.2** $G'$ is an L-wheel.

Assume first that $i = 1$ and $j = 2$. If $x_1$ has at most one neighbor in $A$ and at most one neighbor in $B$, we have a $3PC(A, x_1)$ when $x_1$ has at least three neighbors in $G'$ and when $x_1$ has two neighbors in $G'$ we have a $3PC(A, x_0)$ or a $3PC(B, x_0)$. So by symmetry, we may assume that $x_1$ is adjacent to $b_3 = a_3$, $b_2$ but not $b_1$. Let $x_0$ be the neighbor of $x_1$, closest to $b_1$ in $P_1$. Now $x_0$ is distinct from $a_1$, else let $C = a_1, P_1, b_1, b_2, x_1, a_1$ and $(C, b_3)$ is an odd wheel. Now we have a T-parachute $TP(b_2, b_3, x_1, b_1, x_0)$ of type a or b.

Assume now that $i = 1$ and $j = 3$. If $(H_{13}, x_1)$ is a $\Delta$-free wheel and $x_1$ has no neighbor in $P_2$, then there is a proper wheel with center $a_3$ and if $x_1$ is adjacent to $a_3$, $x_0$ and to no node in $P_2$, then we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, x_0)$. So $x_1$ has at least one neighbor in $P_2$. Now $x_1$ is adjacent to $a_2$ or $b_2$, say $b_2$, else we have a $3PC(A, x_1)$. Then we have a T-parachute $TP(b_2, b_3, x_1, b_1, x_0)$, where $x_0$ is the neighbor of $x_1$, closest to $b_1$ in $P_1$.

**Case 1.2** $(H_{ij}, x_1)$ is a universal wheel.

We may assume that $x_1$ is not universal for $G'$, else b) holds.

**Case 1.2.1** $G'$ is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $(H_{13}, x_1)$ is a universal wheel. Then both $P_1$ and $P_3$ have length less than three, otherwise $(H_{12}, x_1)$ or $(H_{23}, x_1)$ is a proper wheel. Furthermore, since $(H_{13}, x_1)$ is an even wheel, either both $P_1$ and $P_3$ have length one or both $P_1$ and $P_3$ have length two. Assume $P_1 = a_1, t_1, b_1$ and $P_3 = a_3, t_3, b_3$. Then $x_1$ has no neighbor in $P_2$, else $(H_{23}, x_1)$ is a proper wheel. Now the graph induced by $V(G') \setminus \{a_3, t_3\} \cup \{x_1\}$ is a T-parachute $TP(b_3, b_1, x_1, b_2, a_1)$ of type c.

Assume $P_1 = a_1, b_1$ and $P_3 = a_3, b_3$. Then $x_1$ has neighbors in $P_2$, else $V(G') \setminus \{a_1\} \cup \{x_1\}$ induces an odd wheel with center $b_3$. So $(H_{12}, x_1)$ is an L-wheel or a T-wheel. If $(H_{12}, x_1)$ is a L-wheel, let $x_2$, $x_3$ be the neighbors of $x_1$, where $x_2$ is closest to $b_2$ in $P_2$. Then $V(G') \setminus \{a_1\} \cup \{x_1\}$ induces an L-parachute of type c $LP(x_3x_2, a_3b_3, x_1, b_2)$. If $(H_{12}, x_1)$ is a T-wheel, with $x_1$ adjacent to, say, $b_2$ we have a T-parachute $TP(b_3, x_1, a_3, b_2, a_2)$ of type c.

**Case 1.2.2** $G'$ is an L-wheel.

If $(H_{12}, x_1)$ is a universal wheel and $x_1$ is not adjacent to $a_3 = b_3$, then $P_1$ and $P_2$ have both length 2, else $(H_{13}, x_1)$ or $(H_{23}, x_1)$ is a proper wheel. But then, we have a T-parachute $TP(a_2, a_1, x_1, a_3, b_1)$ of type c.

If $(H_{13}, x_1)$ is a universal wheel, then $x_1$ has at least one neighbor in $P_2$ since otherwise $(H, a_3)$ is a proper wheel where $H = a_2, P_2, b_2, b_1, x_1, a_1, a_2$. But now, since $x_1$ is not universal for $G'$, $(H_{12}, x_1)$ is a proper wheel.

**Case 1.3** $(H_{ij}, x_1)$ is an L-wheel or a T-wheel.

If $x_1$ has at most one neighbor in $A$ and at most one neighbor in $B$, then $(H_{ij}, x_1)$ is an L-wheel and $x_1$ has no neighbor in $P_k$, $k \neq i, j$, for otherwise we have a $3PC(A, x_1)$, and in this case a) holds. So by symmetry, we assume that $x_1$ has at least two neighbors in $B$. Furthermore $x_1$ is adjacent to both $b_i$ and $b_j$, since otherwise, if $x_1$ is adjacent to $b_i$ but not $b_j$, then $(H_{ij}, x_1)$, $(H_{ik}, x_1)$ or $(H_{jk}, x_1)$, $k \neq i, j$, is a proper wheel or $H_{jk}$ together with $b_i$ and $x_1$ induces a T-parachute of type c.

**Case 1.3.1** $G'$ is an L-wheel.

Assume $i = 1$ and $j = 3$. Then $x_1$ is adjacent to $b_1$ and $b_3$.

If $(H_{13}, x_1)$ is an L-wheel, then $b_2$ is the only neighbor of $x_1$ in $P_2$, else $(H_{12}, x_1)$ is a proper wheel, and a) holds.

If $(H_{13}, x_1)$ is a T-wheel, then either $x_1$ is adjacent to the neighbor $b_1'$ of $b_1$ in $P_1$, or $x_1$ is adjacent to $a_1$.

If $x_1$ is adjacent to $b_1'$, then $x_1$ has a neighbor in $P_2$, else we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, b_1')$ of type c. Since $x_1$ is not adjacent to $a_1$, $(H_{12}, x_1)$ is an L-wheel or a T-wheel. If $(H_{12}, x_1)$ is a T-wheel, then $b_2$ is the only neighbor of $x_1$ in $P_2$, so $x_1$ is a twin of $b_1$ and b) holds. If $(H_{12}, x_1)$ is an L-wheel, then the neighbors of $x_1$ in $P_2$ are $a_2$ and its neighbor in $P_2$, else $(H_{23}, x_1)$ is a proper wheel. But now we have a T-parachute $TP(b_1, x_1, b_3, b_1', a_1)$ of type c.

If $x_1$ is adjacent to $a_1$, then $x_1$ has a neighbor in $P_2$, else we have a proper wheel with center $b_3$, so $(H_{12}, x_1)$ must be an L-wheel, $x_1$ is a twin of $a_3$ and b) holds.

Assume $i = 1$ and $j = 2$. Then $x_1$ is adjacent to $b_1$ and $b_2$.

If $(H_{12}, x_1)$ is an L-wheel then $x_1$ is adjacent to $a_1$, $a_2$, $b_1$, $b_2$ and no other node of $H_{12}$, else there is a proper wheel with center $a_3$. If $x_1$ is adjacent to $b_3$, it is a twin of $b_3$ and b) holds, and if $x_1$ is not adjacent to $b_3$, we have a T-parachute $TP(b_2, b_1, a_3, x_1, a_1)$ of type a. If $(H_{12}, x_1)$ is a T-wheel then $x_1$ is w.l.o.g. adjacent to $b_1$, $b_2$, $b_1'$ and no other node of $H_{12}$. If $x_1$ is adjacent to $b_3$, it is a twin of $b_1$ and b) holds, and if $x_1$ is not adjacent to $b_3$, we have an odd wheel with center $b_3$.

**Case 1.3.2** $G'$ is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $i = 1$ and $j = 3$. Then $x_1$ is adjacent to $b_1$ and $b_3$. If $(H_{13}, x_1)$ is an L-wheel and $x_1$ has two neighbors in $P_1 \setminus \{b_1\}$ or $P_3 \setminus \{b_3\}$, say $P_1 \setminus \{b_1\}$, then $b_2$ is the only neighbor of $x_1$ in $P_2$, else $(H_{12}, x_1)$ is a proper wheel, and a) holds. If $x_1$ is adjacent to $a_1$ and $a_3$, then $x_1$ has a neighbor in $P_2$, else we have a T-parachute $TP(a_1, a_3, a_2, x_1, b_3)$. If $(H_{23}, x_1)$ is a $\Delta$-free wheel, there is a T-parachute. So $(H_{23}, x_1)$ must be an L-wheel, $x_1$ is adjacent to both $a_2$ and $b_2$ and a) holds.

If $(H_{13}, x_1)$ is a T-wheel, then assume w.l.o.g. that $x_1$ is adjacent to $b_1'$. Node $x_1$ has a neighbor in $P_2$ since, otherwise, there is an odd wheel with center $b_1$. Now suppose that $(H_{12}, x_1)$ is a universal wheel. Then, there is a T-parachute $TP(a_1, a_2, a_3, x_1, b_2)$. So $(H_{12}, x_1)$ is an L-wheel or a T-wheel. $(H_{12}, x_1)$ is a T-wheel, else $(H_{23}, x_1)$ is a proper wheel. If $P_1$ has length one and $x_1$ is adjacent to $a_1$ and $a_2$, there is a T-parachute $TP(a_1, a_2, a_3, x_1, b_2)$. So $b_2$ is the only neighbor of $x_1$ in $P_2$ and b) holds.

**Case 2** $n > 1$.

By Lemma 3.7, since $P$ is a link, the neighbors of $x_1$ in $G'$ are either contained in a big clique $A$ or $B$ or in a segment of $G'$ and the same holds for $x_n$.

**Claim 1** No intermediate node of $P$ has a neighbor in $G'$.

Assume node $y$ of $G'$ is adjacent to an intermediate node $x_j$ of $P$. Since $P$ is a link, by Lemma 3.7, $(N(x_1) \cap G') \cup \{y\}$ is contained in a big clique or a segment of $G'$ and the same holds for $(N(x_n) \cap G') \cup \{y\}$. So either $y = a_3 = b_3$, and the neighbors of one endnode of $P$, say $x_1$, are contained in $A$ while the neighbors of $x_n$ are contained in $B$ or $y$ is the endnode of a nonatomic segment say $P_1$, the neighbors of one endnode of $P$, say $x_1$, are contained in $P_1$ and the neighbors of $x_n$ are contained in a big clique, say $A$. This shows that such a node $y$ is unique.

Assume first that $y = a_3 = b_3$, so $G'$ is an L-wheel. We also assume w.l.o.g. that $x_1$ is adjacent to $a_1$, while $x_n$ is adjacent to $b_2$. Now $x_1$ is adjacent to $a_2$ but not to $a_3$, since otherwise $(H, y)$ is a proper wheel where $H = a_2, a_1, x_1, P, x_n, b_2, P_2, a_2$. By symmetry, $x_n$ is adjacent to $b_1$ and not to $a_3$. Let $x_j$ be the node of lowest index adjacent to $a_3$. Now we have a T-parachute $TP(a_2, a_1, x_1, a_3, x_j)$.

Assume now that $y = a_1$, the neighbors of $x_1$ are contained in $P_1$ and the neighbors of $x_n$ are contained in $A$, so $x_n$ is adjacent to $a_2$ or $a_3$.

If $x_n$ is adjacent to $a_2$, let $Q$ be a shortest path between $x_1$ and $b_1$, whose intermediate nodes (if any) are in $P_1$. Let $H_1 = x_n, a_2, P_2, b_2, b_1, Q, x_1, P, x_n$. Now either $(H_1, a_1)$ is a wheel or $a_1$ has two neighbors in $H_1$ and they are nonadjacent. Now $x_n$ is also adjacent to $a_3$, else let $H_2 = x_n, a_2, a_3, P_3, b_3, b_1, Q, x_1, P, x_n$, then $(H_2, a_1)$ is a proper wheel.

Finally $x_n$ is also adjacent to $a_1$, else we have a T-parachute $TP(a_2, a_3, a_1, x_n, x_j)$, where $x_j$ is the node of highest index adjacent to $a_1$. Let $H_2' = x_n, a_3, P_3, b_3, b_1, Q, x_1, P, x_n$. Now $(H_1, a_1)$ and $(H_2', a_1)$ are either both L-wheels or both T-wheels. If $x_1$ has a unique neighbor $x_0$ in $P_1$, we have either an L-parachute $LP(x_j x_{j-1}, x_n a_3, a_1, x_0)$ or a T-parachute $TP(x_n, a_1, x_{n-1}, a_3, x_0)$. If $x_1$ has several neighbors in $P_1$, we have a $3PC(\Delta, a_1)$ or an L-parachute $LP(x_j x_{j-1}, x_n a_3, a_1, x_1)$ or a T-parachute $TP(x_n, a_1, x_{n-1}, a_3, x_1)$.

If $x_n$ is adjacent to $a_3$ and $a_3 \neq b_3$, the proof is identical. Finally, consider the case where $x_n$ is adjacent to $a_3$ and $a_3 = b_3$. If $x_1$ has exactly two neighbors on $P_1$ and they are adjacent, there is a $3PC(\Delta, a_1)$ or a proper wheel with center $a_1$. Otherwise, there is an L-parachute $LP(a_2 a_1, b_2 b_1, a_3, x_0)$ or $LP(a_2 a_1, b_2 b_1, a_3, x_i)$ where $x_i$ is the node of lowest index adjacent to $a_1$. This completes the proof of Claim 1.

**Case 2.1** All the neighbors of $x_1$ in $G'$ are contained in a big clique, say $A$, and all the neighbors of $x_n$ are contained in $B$.

We assume w.l.o.g. that $x_1$ is adjacent to $a_1$ and $x_n$ is adjacent to $b_2$.

**Case 2.1.1** $G'$ is an L-wheel.

Assume $x_1$ is adjacent to $a_1$ only. Then $x_n$ is adjacent to $a_3 = b_3$, else we have an odd wheel with center $a_3$. Let $H = a_1, x_1, P, x_n, b_2, b_1, P_1, a_1$ if $x_n$ is not adjacent to $b_1$, and $H = a_1, x_1, P, x_n, b_1, P_1, a_1$ if $x_n$ is adjacent to $b_1$. Then $(H, a_3)$ is a proper wheel.

Assume $x_1$ is adjacent to $a_1$, $a_2$ but not to $a_3$. If $x_n$ is adjacent to $a_3$, we have an odd wheel with center $a_3$ and if $x_n$ is not adjacent to $a_3$ we have a T-parachute $TP(a_1, a_2, x_1, a_3, b_2)$.

Assume $x_1$ is adjacent to $a_1$, $a_3$ but not to $a_2$. Let $H = a_1, x_1, P, x_n, b_2, P_2, a_2, a_1$. Then $(H, a_3)$ is a proper wheel.

So $x_1$ is adjacent to $a_1$, $a_2$, $a_3$ and by symmetry, $x_n$ is adjacent to $b_1$, $b_2$, and $b_3$ and a) holds in this case.

**Case 2.1.2** $G'$ is a $3PC(\Delta, \Delta)$.

Assume $x_1$ is adjacent to $a_1$ only. If $x_n$ is adjacent to $b_2$ only, we have a $3PC(B, a_1)$. If $x_n$ is adjacent to $b_1$, we have a $3PC(b_1 b_2 x_n, a_1)$. If $x_n$ is adjacent to $b_3$ but not $b_1$, we have a T-parachute $TP(b_3, b_2, x_n, b_1, a_1)$.

Assume $x_1$ is adjacent to $a_1$, $a_2$ but not to $a_3$. If $x_n$ is not adjacent to $b_1$, we have a $3PC(a_1 a_2 x_1, b_2)$. If $x_n$ is adjacent to $b_1$, we have a T-parachute $TP(a_2, a_1, x_1, a_3, b_1)$ if $x_n$ is not adjacent to $b_3$ and a $3PC(b_1 b_3 x_n, a_1)$ if $x_n$ is adjacent to $b_3$.

Assume $x_1$ is adjacent to $a_1$, $a_3$ but not to $a_2$. By symmetry, we may assume that $x_n$ is not adjacent to $b_3$ and we have a T-parachute $TP(a_1, a_3, x_1, a_2, b_2)$.

So $x_1$, is adjacent to $a_1$, $a_2$ and $a_3$. Again by symmetry, $x_n$ is adjacent to $b_1$, $b_2$, and $b_3$ and a) holds in this case.

**Case 2.2** All the neighbors of $x_1$ in $G'$ are contained in a big clique, say $A$, and all the neighbors of $x_n$ are contained in a segment, say $P_1$.

Then $x_1$ is adjacent to $a_2$ or $a_3$.

**Case 2.2.1** $G'$ is an L-wheel.

We first show that $x_n$ has two neighbors in $P_1$ and these neighbors are adjacent. If not, either $P_1$ contains two neighbors of $x_n$ and these neighbors are nonadjacent or $P_1$ has a unique neighbor of $x_n$. Assume the first possibility holds: If $x_1$ is adjacent to $a_2$ only, we have a $3PC(A, x_n)$. If $x_1$ is adjacent to $a_3$ only, we have an L-parachute $LP(a_2a_1, b_2b_1, a_3, x_n)$. If $x_1$ is adjacent to $a_1$, $a_3$ and possibly $a_2$ we have a $3PC(x_1a_1a_3, x_n)$ if $x_n$ is not adjacent to both $a_1$ and $x_1$, and a T-parachute $TP(x_1, a_1, a_2, x_n, t)$, where $t$ is the neighbor of $x_n$ in $P_1$ that is closest to $b_1$, if $x_n$ is adjacent to both $a_1$ and $x_1$. If $x_1$ is adjacent to $a_1$, $a_2$ but not $a_3$, there is a proper wheel with center $a_3$. Finally if $x_1$ is adjacent to $a_2$, $a_3$ but not $a_1$, we have a T-parachute $TP(a_2, a_3, a_1, x_1, x_n)$. So $P_1$ cannot contain two nonadjacent neighbors of $x_n$

The same proof rules out the case where $P_1$ has a unique neighbor of $x_n$, so $x_n$ has two adjacent neighbors, say $y$ and $z$, in $P_1$ and $y$ is closer than $z$ to $a_1$ in $P_1$.

Assume $x_1$ is adjacent to $a_3$. Then $x_1$ is adjacent to $a_1$, else we have a $3PC(x_nyz, a_3)$. Now $x_1$ is also adjacent to $a_2$, else we have a $3PC(x_nyz, a_1)$ when $a_1 \neq y$ and an L-parachute $LP(x_nz, x_1a_3, a_1, b_1)$ of type d when $y = a_1$ and $n > 2$. When $y = a_1$ and $n = 2$, we have an odd wheel with center $a_1$. So a) holds in this case.

Assume finally $x_1$ is adjacent to $a_2$ but not $a_3$. Then $x_1$ adjacent to $a_1$, else we have a $3PC(x_nyz, a_2)$. Now we have a $3PC(x_nyz, a_1)$ when $a_1 \neq y$ and when $y = a_1$, we have a proper wheel with center $a_1$.

**Case 2.2.2** $G'$ is a $3PC(\Delta, \Delta)$.

We assume w.l.o.g. that $x_1$ is adjacent to $a_2$. $x_1$ is adjacent to $a_3$ since, otherwise, there is a $3PC(B, a_2)$.

Assume that $x_1$ is not adjacent to $a_1$. If $x_n$ has two nonadjacent neighbors in $P_1$, there is a T-parachute $TP(a_2, a_3, a_1, x_1, x_n)$. If $x_n$ has a unique neighbor $x_{n+1}$ in $P_1$, there is a T-parachute $TP(a_2, a_3, a_1, x_1, x_{n+1})$. If $x_n$ has exactly two neighbors in $P_1$, say $y$, $z$, and they are adjacent, there is a $3PC(x_nyz, a_2)$.

So $x_1$ is adjacent to $a_1$, $a_2$ and $a_3$. If $x_n$ has a unique neighbor $x_{n+1}$ in $P_1$, there is a $3PC(x_1a_1a_2, x_{n+1})$. If $x_n$ has two nonadjacent neighbors in $P_1$, there is a $3PC(x_1a_1a_2, x_n)$ if $x_n$ is not adjacent to both $x_1$ and $a_1$ and a T-parachute $TP(x_1, a_1, a_2, x_n, t)$ otherwise, where $t$ is the neighbor of $x_n$ closest to $b_1$. So, $x_n$ has exactly two neighbors in $P_1$, say $y$, $z$, and they are adjacent. Let $y$ be the one that is closest to $a_1$ in $P_1$. If $y = a_1$, there is a proper wheel with center $a_1$. So $y \neq a_1$ and a) holds in this case.

**Case 2.3** All the neighbors of $x_1$ are contained in a segment, say $P_1$ and all the neighbors of $x_n$ are contained in a segment, say $P_2$.

Note that the choice of $P_1$ and $P_2$ is done w.l.o.g. by assuming that we are not in Case 2.2. We show that $x_1$ has exactly two neighbors and these two neighbors are adjacent. Assume $x_1$ has exactly one neighbor $y$ in $P_1$. Since $x_n$ has a neighbor in $P_2 \setminus \{b_2\}$, there is a $3PC(A, y)$.

If $x_1$ has two nonadjacent neighbors in $P_1$, replace $P_1$ by the chordless $a_1b_1$-path containing $x_1$ and nodes of $P_1$ and let $P' = x_2, \ldots, x_n$. The proof of Case 1 shows that $P'$ has length bigger than 2 and $x_2$ now has $x_1$ as unique neighbor in $P_1'$ and by the above argument, this is impossible. So $x_1$ has exactly two neighbors in $P_1$ and they are adjacent. By symmetry, $x_n$ has exactly two neighbors in $P_2$ and they are adjacent. So a) holds in this case. □

**Lemma 3.9** *Let $G$ be an even-signable WP-free graph, $G'$ be an L-graph in $G$ and $P = x_1, \ldots, x_n$ be a link of $G'$. Then*

*a) either $G' \cup P$ is an L-graph, or*

*b) $n = 1$ and $x_1$ is either universal for $G'$ or the twin of an endnode of a segment of $G'$.*

*Proof:* Since $P$ is a link of $G'$, $x_1$, $x_n$ have neighbors $x_0$, $x_{n+1}$ that are nonadjacent and in distinct segments $S_{x_0}$ and $S_{x_{n+1}}$ of $G'$. Let $S_3$ be any other segment of $G'$. By Lemma 3.4, $G'$ contains an elementary L-graph $G_1$, with $S_{x_0}$ and $S_{x_{n+1}}$ in distinct segments of $G_1$ and containing $S_3$. So $P$ is a link of $G_1$, and by Lemma 3.8, the statement holds when $G' = G_1$.

**Case 1** $n = 1$.

Assume $x_1$ is a universal node for $G_1$ and $x_1$ is not adjacent to node $y$ of $G'$. Let $C$ be any hole of $G_1$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_y$. Since at least two nodes of $C$ are nonadjacent and in distinct segments of $G_2$, $x_1$ is a link of $G_2$. Since $(C, x_1)$ is a universal wheel but $x_1$ is not universal for $G_2$, Lemma 3.8 is contradicted.

Assume $x_1$ is a link of $G_1$ and is adjacent to all nodes in distinct cliques $K_1'$, $K_2'$ of $G_1$, not in the same segment of $G_1$ and to no other node of $G_1$. (This happens both when $x_1$ is a twin of an endnode of a segment of $G_1$ and when $G_1 \cup \{x_1\}$ is an L-graph). Let $K_1$, $K_2$ be the cliques of $G'$, that extend $K_1'$, $K_2'$.

Assume $x_1$ is adjacent to node $y$ in $G' \setminus (K_1 \cup K_2)$ and let $C$ be a hole of $G_1$ containing two nodes of $K_1'$ and two nodes of $K_2'$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_y$. Now $x_1$ is a link of $G_2$, for $x_1$ is adjacent to $y$, and no node of $C$ is in the same segment as $y$ and at least one neighbor of $x_1$ in $C$ is nonadjacent to $y$. Since $(C, x_1)$ is an L-wheel or a T-wheel and $x_1$ is adjacent to $y$, Lemma 3.8 is contradicted in $G_2$.

Assume $x_1$ is not adjacent to node $z$ in $K_1$. Let $C$ be a hole containing two nodes of $K_1'$ and two nodes of $K_2'$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_z$. Since $G_2$ contains at least three nodes of $K_1$, its restriction $K_1^*$ to $G_2$ is a big clique of $G_2$ and since $x_1$ is adjacent to two nodes in $K_1^*$, two other nodes of $C$ and no other node of $G_2$, $x_1$ is a link of $G_2$, violating Lemma 3.8.

So $x_1$ is adjacent to all nodes in $K_1 \cup K_2$ and is adjacent to no other node of $G'$. If $K_1$, $K_2$ have a common node in $G'$, then $x_1$ is a twin of such a node and b) holds. Otherwise $G' \cup \{x_1\}$ is an L-graph and a) holds.

**Case 2** $n > 1$.

Assume that node $y$ of $G'$ is adjacent to an intermediate node $x_j$ of $P$. By Lemma 3.4, $G'$ contains an elementary graph $G_2$ containing $S_y$, where $S_{x_0}$ and $S_{x_{n+1}}$ are in distinct segments of $G_2$, So $P$ is a link of $G_2$ (the minimality of $P$ follows from the fact that $P$ is a link of

$G'$). Since an intermediate node of $P$ is adjacent to $y$, Lemma 3.8 is violated in $G_2$. So no intermediate node of $P$ has a neighbor in $G'$.

Since $P$ is a link of $G_1$, by Lemma 3.8, $x_1$ and $x_n$ are adjacent to all the nodes in cliques $K_1'$, $K_2'$, not in the same segment of $G_1$. Let $K_1$, $K_2$ be the cliques of $G'$ that extend $K_1'$, $K_2'$.

Assume $x_1$ is adjacent to node $y$ in $G' \setminus K_1$. By Lemma 3.4, $G'$ contains an elementary L-graph $G_2$, containing $S_y$, where $S_{x_0}$ and $S_{x_{n+1}}$ are in distinct segments of $G_2$. So $P$ is a link of $G_2$, contradicting Lemma 3.8. So all the neighbors of $x_1$ in $G'$ are contained in $K_1$ and by symmetry, all the neighbors of $x_n$ in $G'$ are contained in $K_2$.

Assume $x_1$ is not adjacent to node $z$ in $K_1$. By Lemma 3.4, $G'$ contains an elementary L-graph $G_2$, containing $S_z$, where $S_{x_0}$ and $S_{x_{n+1}}$ are in distinct segments of $G_2$. So $P$ is a link of $G_2$, contradicting Lemma 3.8. So $x_1$ belongs to the extension of $K_1$ and by symmetry, $x_n$ belongs to the extension of $K_2$ and a) holds in this case. $\square$

### 3.4 A Proof of Theorem 1.16

A *twin class* of a graph $G$ is a maximal subset of $V(G)$ with the property that every pair of nodes in it are twins. The twin classes of $G$ partition $V(G)$ into cliques. A *restriction* of $G$ is an induced subgraph $H$ of $G$ obtained by keeping exactly one node in each twin class. All the restrictions of $G$ are obviously isomorphic graphs.

Let $G$ be a graph. An induced subgraph $G'$ of $G$ is an *extended L-graph* if any restriction $H$ of $G'$ is an L-graph and every twin class of $G'$ containing an intermediate node in a segment of $H$ contains no other node. A *segment* of $G'$ is a segment of one of its restrictions $H$ together with all the nodes in the twin classes of its endnodes. A path $P$ in $G \setminus G'$ is a *link* of $G'$ if $P$ is a link of some restriction $H$ of $G'$. By Lemma 3.9, it follows that if $P$ is a link of $G'$, then $P$ is a link of all the restrictions of $G'$.

**Theorem 3.10** *Let $G$ be an even-signable WP-free graph, $G'$ a node-maximal extended L-graph in $G$ and $P = x_1, \ldots, x_n$ a link of $G'$. Then $n = 1$ and $x_1$ is a universal node for $G'$.*

*Proof:* Follows by applying Lemma 3.9 to all the possible restrictions of $G'$ and the maximality of $G'$. $\square$

We can now prove Theorem 1.16.

*Proof:* Let $G$ be an even-signable WP-free graph that contains an L-wheel or a $3PC(\Delta, \Delta)$ as induced subgraph. Then $G$ contains a node-maximal extended L-graph $G'$ and let $U$ be the set of nodes that are universal for $G'$.

Assume first that $V(G) = V(G') \cup U$. If $U \neq \emptyset$, then $\bar{G}$ is disconnected. If $U = \emptyset$ and at least one twin class of $G$ contains at least two nodes, then $G$ contains a star cutset. Finally, if $U = \emptyset$ and every twin class of $G$ contains a single node, $G$ is the line graph of a triangle-free graph.

Assume now $G \setminus (V(G') \cup U)$ is nonempty and let $C_1 \ldots, C_n$ be the connected component of $G \setminus (V(G') \cup U)$. By Theorem 3.10 and Lemma 3.7, for every connected component $C_i$, $N(C_i) \cap G'$ is either contained in a clique of $G'$ or in a segment of $G'$ which is not atomic.

Assume first that a component, say $C_1$, has its neighbors in $G'$ contained in a clique $K$. Then the removal of the nodes in $K \cup U$ separates $C_1$ from $G' \setminus K$ and we have a star cutset.

Assume now that no component has its neighbors in $G'$ contained in a clique of $G'$ and let $K_1$, $K_2$ be the two big cliques of $G'$ that contain the endnodes of the nonatomic segment $S$ containing the neighbors of $C_1$. Let $A_1$, $A_2$ be the subsets of $K_1$, $K_2$ that are the twin classes of the endnodes of $S$ and let $B_1 = K_1 \setminus A_1$, $B_2 = K_2 \setminus A_2$. Since $K_1$, $K_2$ are cliques of an extended L-graph, $A_1$, $A_2$ are nonempty and disjoint, while $B_1 \setminus B_2$, $B_2 \setminus B_1$ are both nonempty. If $B_1 \cap B_2$ is empty, we have an extended strong 2-join separating $S \cup C_1$ from $G' \setminus S$, and if $B_1 \cap B_2$ is nonempty, we have a star cutset separating the same sets. $\square$

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