Perfect Graphs, Partitionable Graphs and Cutsets

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Abstract

We prove a theorem about cutsets in partitionable graphs that generalizes earlier results on amalgams, 2-amalgams and homogeneous pairs.

1 Introduction

A graph G is *perfect* if, for all induced subgraphs of G, the size of a largest clique is equal to the chromatic number. A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are. A *hole* is a chordless cycle of length at least four. The strong perfect graph conjecture of Berge [1] states that G is minimally imperfect if and only if G or its complement is an odd hole. (The *complement* $\bar{G}$ of G is a graph with same node set as G and two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in G).

A graph G is *partitionable* if there exist integers α and ω greater than one such that G has exactly $\alpha\omega + 1$ nodes and, for each node $v \in V$, $G \setminus v$ can be partitioned into both α cliques of size ω and ω stable sets of size α. Lovász [13] has proven that every minimally imperfect graph is partitionable.

A graph G has a *universal 2-amalgam* if $V(G)$ can be partitioned into subsets V_A, V_B and U (U possibly empty), such that:

- V_A contains sets A_1, A_2 such that $A_1 \cup A_2$ is nonempty, V_B contains sets B_1, B_2 such that $B_1 \cup B_2$ is nonempty, every node of A_1 is adjacent to every node of B_1, every node of A_2 is adjacent to every node of B_2 and these are the only adjacencies between V_A and V_B.

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- Every node of U is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$.
- $|U \cup V_A| \geq 2$ and if $(V_A \cup U) \setminus (A_1 \cup A_2) = \emptyset$ then at least one of the sets A_1 or A_2 contains at least two nodes.

  $|U \cup V_B| \geq 2$ and if $(V_B \cup U) \setminus (B_1 \cup B_2) = \emptyset$ then at least one of the sets B_1 or B_2 contains at least two nodes.

In this paper we prove the following result.

Theorem 1.1 *If G is a partitionable graph that contains a universal 2-amalgam, then G or $\bar{G}$ is an odd hole.*

The notion of universal 2-amalgam is related to concepts that have appeared in the literature.

The *join* introduced by Cunningham and Edmonds [11] is a universal 2-amalgam where $U = \emptyset$, $A_1 = A_2$ and $B_1 = B_2$.

The *amalgam* introduced by Burlet and Fonlupt [5] is a universal 2-amalgam where U induces a clique, $A_1 = A_2$, $B_1 = B_2$ and $|V_A| \geq 2$, $|V_B| \geq 2$.

The *2-amalgam* introduced by Cornuéjols and Cunningham [10] is a universal 2-amalgam where U induces a clique, A_1, A_2, B_1, B_2 are all disjoint, V_A and V_B induce connected graphs and, if $|A_1| = |A_2| = 1$ then V_A does not induce a chordless path, if $|B_1| = |B_2| = 1$ then V_B does not induce a chordless path. The special case of a 2-amalgam where $U = \emptyset$, is known as a *2-join*.

Homogeneous pairs introduced by Chvátal and Sbihi [8] are very closely related to universal 2-amalgams where $U = \emptyset$, $V_A \setminus (A_1 \cup A_2) = \emptyset$ and $A_1 \cap A_2 = \emptyset$, as we discuss in Section 5.

A graph G has a *skew partition* if its node set $V(G)$ can be partitioned into nonempty sets A, B, C, D such that there are all possible edges between A and B and no edge between C and D. Chvátal [7] conjectured that a minimally imperfect graph cannot have a skew partition. When all three sets U, $V_A \setminus (A_1 \cup A_2)$ and $V_B \setminus (B_1 \cup B_2)$ are nonempty, a universal 2-amalgam is a special case of a skew partition.

Partitionable graphs are discussed in Section 3, where we give short proofs of several classical results. Using these results, we prove Theorem 1.1 in Section 4. This theorem is used by Conforti and Cornuéjols [9] to show the validity of the strong perfect graph conjecture for wheel-and-parachute-free graphs, a class of graphs that contains Meyniel graphs [5] as well as line graphs of bipartite graphs. Sections 5 and 6 discuss the relation between universal 2-amalgams and homogeneous pairs.

2 Notations and Terminology

We use lower case boldface letters for vectors and upper case boldface letters for matrices. In particular, $\mathbf{1}$ and $\mathbf{0}$ denote the vector of all one and the zero vector, respectively, and $\mathbf{I}$ and $\mathbf{J}$ denote the identity matrix and the all one matrix of suitable size. $\mathbf{e}_i$ denotes the i th unit vector. For a vector $\mathbf{x}$, x_i is its ith coordinate. Similarly, if $\mathbf{A}$ is a matrix, then a_{ij} denotes its (i,j)th element.

If $\mathcal{A} = \{A_1, \ldots, A_m\}$ is a set system on a finite ground set $V(\mathcal{A}) = \{v_1, \ldots, v_n\}$, then $\mathbf{A}$ denotes the point-set incidence matrix of $\mathcal{A}$, i.e. $\mathbf{A}$ is an $n \times m$ $0-1$ matrix such that $a_{ij} = 1$ if and only if $v_i \in A_j$.

If $G = (V, E)$ is a graph, then $n(G)$ denotes its number of nodes, $\omega(G)$ its clique number, $\alpha(G)$ its stability number, $\chi(G)$ its chromatic number and $\theta(G)$ its clique cover number (i.e. the chromatic number of $\bar{G}$). A k-clique or k-stable set will mean a clique or stable set of size k. We refer to an $\omega(G)$-clique or an $\alpha(G)$-stable set as a *big* clique and a *big* stable set.

We denote by $\mathcal{C}_k = \mathcal{C}_k(G)$ and $\mathcal{S}_k = \mathcal{S}_k(G)$ respectively the set systems of k-cliques and k-stable sets of G. $\mathbf{S}_k$ and $\mathbf{C}_k$ are respectively the incidence matrices of nodes versus k-stable sets and nodes versus k-cliques of G.

3 (α, ω)-Graphs.

Definition 3.1 *Let α and ω be integers greater than one. A graph G is called an (α, ω)-graph (or partitionable graph) if G has exactly $n = \alpha\omega + 1$ nodes and, for each node $v \in V$, $G \setminus v$ can be partitioned into both α cliques of size ω and ω stable sets of size α.*

It follows from this definition that the complement of a partitionable graph is also partitionable.

Remark 3.2 *It is easy to see that if G is an (α, ω)-graph, then α and ω are the stability number and clique number of G.*

An example of an (α, ω)-graph is the (α, ω)-web $W_{\alpha\omega}$ constructed as follows: $V(W_{\alpha\omega}) = \{v_0, \cdots, v_{\alpha\omega}\}$ and nodes v_i, v_j are adjacent if and only if $i - j \in \{-\omega + 1, \ldots, -1, 1, \ldots, \omega - 1\} \pmod{\alpha\omega + 1}$. Partitionable graphs are one of the central objects in the theory of perfect graphs due to the following theorem of Lovász [13].

Theorem 3.3 *A graph G is perfect if and only if $\alpha(G')\omega(G') \geq n(G')$ for every induced subgraph G' of G.*

Let G be a minimally imperfect graph. By applying Lovasz's inequality to G and $G \setminus v$, we get $n(G) = \alpha(G)\omega(G) + 1$. Since the graph $G \setminus v$ is perfect, it can be colored with $\omega(G \setminus v) = \omega(G)$ colors and therefore its nodes can be partitioned into $\omega(G)$ stable sets of size $\alpha(G)$. Since the inequalities of Theorem 3.3 are unchanged in the complement graph, the nodes of $G \setminus v$ can also be partitioned into $\alpha(G)$ cliques of size $\omega(G)$. Thus we have the following result.

Theorem 3.4 *If G is a minimally imperfect graph, then G is an (α, ω)-graph.*

Using Lovász's theorem, Padberg [17] proved that a minimally imperfect graph G has exactly n big cliques and n big stable sets, and $\mathbf{J} - \mathbf{S}_\alpha^T \mathbf{C}_\omega$ is a permutation matrix. Bland et al. [4] proved that the statement remains true for (α, ω)-graphs and afterwards Chvátal et al. [6] observed that the converse is also true.

Theorem 3.5 *G is an (α, ω)-graph if and only if $\mathbf{J} - \mathbf{S}_\alpha^T \mathbf{C}_\omega$ is a permutation matrix of size n.*

Further structural properties of (α, ω)-graphs follow from the following special case of a result of Bridges and Ryser [3]:

Theorem 3.6 *Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ $0-1$ matrices such that $\mathbf{A}^T\mathbf{B} = \mathbf{J} - \mathbf{I}$. Then there exist integers r and s such that $rs = n-1$, $\mathbf{JA} = \mathbf{AJ} = r\mathbf{J}$, $\mathbf{JB} = \mathbf{BJ} = s\mathbf{J}$ and $\mathbf{A}^T\mathbf{B} = \mathbf{B}\mathbf{A}^T$.*

Proof: It is straightforward to check that $(\mathbf{J} - \mathbf{I})^{-1} = \frac{1}{n-1}\mathbf{J} - \mathbf{I}$. Hence

$$\mathbf{A}^T\mathbf{B} = \mathbf{J} - \mathbf{I} \Rightarrow \mathbf{A}^T\mathbf{B}(\frac{1}{n-1}\mathbf{J} - \mathbf{I}) = \mathbf{I} \Rightarrow \mathbf{B}(\frac{1}{n-1}\mathbf{J} - \mathbf{I})\mathbf{A}^T = \mathbf{I}$$

$$\text{i.}e. \qquad \mathbf{B}\mathbf{A}^T = \frac{1}{n-1}\mathbf{BJA}^T - \mathbf{I} = \frac{1}{n-1}\mathbf{s}\mathbf{r}^T - \mathbf{I}$$

where $\mathbf{s} \equiv \mathbf{B1}$ and $\mathbf{r} \equiv \mathbf{A1}$.

It follows that, for each i and j, $n-1$ divides $r_i s_j$. On the other hand, the trace of the matrix $\mathbf{BA}^T$ is equal to the trace of $\mathbf{A}^T\mathbf{B}$, which is zero. As $\mathbf{BA}^T$ is a non-negative matrix, it follows that it has zeros in its main diagonal. Hence, for each i, $r_i s_i = n-1$. Now consider distinct i, j. Since $n-1$ divides $r_i s_j$ and $r_j s_i$ and $r_i s_i = r_j s_j = n-1$, it follows that $r_i = r_j$. Therefore, for some r and s, $\mathbf{r} = r\mathbf{1}$, $\mathbf{s} = s\mathbf{1}$ and $\mathbf{BA}^T = \mathbf{J} - \mathbf{I}$. By similar arguments we get that $\mathbf{B}^T\mathbf{1} = s\mathbf{1}$ and $\mathbf{A}^T\mathbf{1} = r\mathbf{1}$. □

Theorem 3.7 *Let G be a graph with n nodes and let $\alpha > 1$, $\omega > 1$ be integers. The following are equivalent:*

1) *G is an (α, ω)-graph.*

2) *$\alpha = \alpha(G)$ and, for every node $v \in V$ and stable set $S \subseteq V$, $\omega = \omega(G \backslash S) = \chi(G \backslash v)$.*

3) *$\mathbf{J} - \mathbf{S}_\alpha^T \mathbf{C}_\omega$ is a permutation matrix of size n.*

Proof: 1)⇒2) : Let G be an (α, ω)-graph. It follows from the definition that, for every node $v \in V$, $\chi(G \backslash v) = \omega$. By Remark 3.2, $\alpha(G) = \alpha$ and $\omega(G) = \omega$, so 2) holds when $S = \emptyset$. Now assume $S \neq \emptyset$ and let $x \in S$. Consider a partition of $G \backslash x$ into α cliques of size ω. As $|S| \leq \alpha$, it follows that one of the cliques of size ω of this partition is disjoint from S. Hence, $\omega(G \backslash S) = \omega$.

2)⇒3) : Let G satisfy 2) and let A_1 be a big stable set of G. Fix an ω-coloration of each of the α graphs $G \backslash s$ ($s \in A_1$), let $A_2, \ldots, A_{\alpha\omega+1}$ be the stable sets occurring as a color-class in one of these colorations and let $\mathcal{A} := \{A_1, A_2, \ldots, A_{\alpha\omega+1}\}$. Define $\mathcal{B} := \{B_1, B_2, \ldots, B_{\alpha\omega+1}\}$ where B_i is a big clique of $G \backslash A_i$.

Let C be a big clique of G and let $\mathcal{P}_v$ be a set of ω stable sets partitioning $G \setminus v$. If C does not contain v, then C intersects all the stable sets of $\mathcal{P}_v$. If C contains v, then there is exactly one stable set of $\mathcal{P}_v$ that is disjoint from C. So any big clique of G is disjoint from exactly one $A_i \in \mathcal{A}$. Hence the incidence vector of C satisfies the equation $\mathbf{A}^T\mathbf{x} = \mathbf{1} - \mathbf{e}_i$ for some i. In particular, it follows that $\mathbf{A}^T\mathbf{B} = \mathbf{J} - \mathbf{I}$. Since $G\backslash s$ $(s \in V)$ is ω-colorable, $n \leq \alpha\omega + 1$ is obvious. On the other hand, since $\mathbf{J} - \mathbf{I}$ is nonsingular, $\mathbf{A}$ has full column rank, and $n \geq \alpha\omega + 1$ follows. Thus $n = \alpha\omega + 1$ and therefore every stable set in $\mathcal{A}$ is big. As $\mathbf{A}$ is nonsingular, it follows that for each i, $\mathbf{A}^T\mathbf{x} = \mathbf{1} - \mathbf{e}_i$ has a unique solution. Hence $\mathcal{B} = \mathcal{C}_\omega$. As A_1 is an arbitrary big stable set and $\mathbf{B}$ is nonsingular, it follows that $\mathcal{A} = \mathcal{S}_\alpha$. Thus $\mathbf{J} - \mathbf{S}_\alpha^T\mathbf{C}_\omega$ is a permutation matrix of size n.

3)⇒1) : We may assume that $\mathbf{S}_\alpha^T\mathbf{C}_\omega = \mathbf{J} - \mathbf{I}$. So, by Theorem 3.6, $\mathbf{C}_\omega\mathbf{S}_\alpha^T = \mathbf{J} - \mathbf{I}$, $\mathbf{J}\mathbf{S}_\alpha = \mathbf{S}_\alpha\mathbf{J} = \alpha\mathbf{J}$, $\mathbf{J}\mathbf{C}_\omega = \mathbf{C}_\omega\mathbf{J} = \omega\mathbf{J}$ and $\alpha\omega = n - 1$. As, for each i, $\mathbf{C}_\omega\mathbf{x}_i = \mathbf{1} - \mathbf{e}_i$, where $\mathbf{x}_i$ is the i^{th} column of $\mathbf{S}_\alpha^T$, it follows that there are α cliques of size ω that partition the nodes of $G\backslash v_i$. Similarly, as $\mathbf{S}_\alpha\mathbf{y}_i = \mathbf{1} - \mathbf{e}_i$, where $\mathbf{y}_i$ is the i^{th} column of $\mathbf{C}_\omega^T$, it follows that there are ω stable sets of size α that partition the nodes of $G\backslash v_i$. □

The implication 2)⇒3) is from [12]. The implication 3)⇒1) was originally observed in [6].

As minimally imperfect graphs satisfy Condition 2) of the above theorem, Theorem 3.7 implies both Theorem 3.4 and Theorem 3.5.

Next, we show some known properties of partitionable graphs.

Theorem 3.8 *An (α, ω)-graph G with n nodes has the following properties:*

1) *[17][4] G has exactly n big cliques and n big stable sets, which can be indexed as $C_1, \ldots, C_n$ and $S_1, \ldots, S_n$, so that $C_i \cap S_j$ is empty if and only if $i = j$. We say that S_i and C_i are mates.*

2) *[17][4] Every $v \in V$ belongs to exactly α big stable sets and their intersection contains no other node.*

3) *[17][4] For every $v_i \in V$, $G \setminus v_i$ has a unique ω-coloration whose color classes are the big stable sets that are mates of the big cliques containing v_i.*

4) *[6] If $e \in E(G)$ does not belong to any big clique, then $G \setminus e$ is an (α, ω)-graph.*

5) *[4] Let S_1, S_2 be two big stable sets. Then the graph induced by $S_1 \Delta S_2$ is connected.*

6) *[7] G contains no star cutset (a node cutset with a node adjacent to all the other nodes of the cutset).*

7) *[18] Any proper induced subgraph H of G with $\omega(H) = \omega$ has at most $n(H) - \omega + 1$ big cliques.*

8) *[18] Let (V_1, V_2) be a partition of $V(G)$ such that both V_1 and V_2 contain at least one big clique. Then the number of big cliques that intersect both V_1 and V_2 is at least $2\omega - 2$.*

9) *[18] G is $(2\omega - 2)$-node connected.*

Proof: 1) follows from Theorem 3.7 3).

2) follows Theorem 3.6 and Theorem 3.7 3).

3) From Theorem 3.6 and Theorem 3.7 3), we have that $\mathbf{S}_\alpha \mathbf{C}_\omega^T$ is a 0,1 matrix. Hence if two big cliques intersect, their mates are disjoint and vice versa. It follows that the mates of big cliques containing v partition $G \setminus v$. To show that this partition is unique, just notice that there is a one-to-one correspondence between ω-colorations of $G \setminus v_i$ and $0-1$ solutions of $\mathbf{S}_\alpha \mathbf{x} = \mathbf{1} - \mathbf{e}_i$. As $\mathbf{S}_\alpha$ is non-singular, it follows that $G \setminus v_i$ has a unique ω-coloration.

4) follows from the definition of (α, ω)-graphs.

5) Suppose $G(S_1 \Delta S_2)$ is disconnected and let $S \subset S_1 \Delta S_2$ induce one of its connected components. For $i = 1, 2$, let $S_i' = S \cap S_i$ and $S_i'' = S_i - S_i'$. Then $S' = S_1' \cup S_2''$ and $S'' = S_2' \cup S_1''$ are stable sets of G. As $|S'| + |S''| = 2\alpha$, it follows that S' and S'' are big stable sets. Let C_1 be the mate of S_1. Then C_1 meets both S' and S''. As C_1 is disjoint from S_1, it follows that C_1 meets both S_2' and S_2''. But this is a contradiction, as $S_2' \cup S_2'' = S_2$ is a stable set.

6) Let U, V_1, V_2 be a partition of V such that U is a star cutset of G and V_1 induces a connected component of $G \setminus U$. Let G_1 and G_2 be the graphs induced by $U \cup V_1$ and $U \cup V_2$ respectively, and let $u \in U$ be adjacent to all the nodes in U. Finally, let S_i be the color class of an ω-coloration of G_i containing u, where $i \in \{1, 2\}$. Then S_i meets all the big cliques of G_i, i.e. $\omega(G \setminus \{S_1 \cup S_2\}) < \omega$. On the other hand, $S_1 \cup S_2$ is a stable set, a contradiction to Theorem 3.7 2).

7) Let $\mathcal{S} = \{S_1, \ldots, S_\omega\}$ be the color classes of an ω-coloration of H. Then $\mathbf{C}^T \mathbf{S} = \mathbf{J}$, where $\mathbf{C}$ denotes the incidence matrix of nodes versus big cliques of H. Since $\mathbf{C}_\omega$ has full column rank, $\mathbf{C}$ also has full column rank. As $rk(\mathbf{S}) = \omega$ and $rk(\mathbf{J}) = 1$, it follows from linear algebra that $|\mathcal{C}| = rk(\mathbf{C}) \leq n(H) - \omega + 1$.

7) easily implies 8) and 2) plus 8) imply 9). □

4 Universal 2-Amalgams

We recall the definition of universal 2-amalgam:

Definition 4.1 *A graph G has a* universal 2-amalgam *if $V(G)$ can be partitioned into subsets V_A, V_B and U (U possibly empty), such that:*

- *V_A contains sets A_1, A_2 such that $A_1 \cup A_2$ is nonempty, V_B contains sets B_1, B_2 such that $B_1 \cup B_2$ is nonempty, every node of A_1 is adjacent to every node of B_1, every node of A_2 is adjacent to every node of B_2 and these are the only adjacencies between V_A and V_B.*

- *Every node of U is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$.*

- *$|U \cup V_A| \geq 2$ and if $(V_A \cup U) \setminus (A_1 \cup A_2) = \emptyset$ then at least one of the sets A_1 or A_2 contains at least two nodes.*

  $|U \cup V_B| \geq 2$ and if $(V_B \cup U) \setminus (B_1 \cup B_2) = \emptyset$ then at least one of the sets B_1 or B_2 contains at least two nodes.

Remark 4.2 *If we omit the last condition in the above definition, then every graph with at least 2 nodes has a universal 2-amalgam.*

Holes with at least 6 nodes and their complements have a universal 2-amalgam where $U = \emptyset$.

Proof of Theorem 1.1: Let G be a partitionable graph that contains a universal 2-amalgam.

Claim 1: The four sets $A_1 \setminus A_2$, $B_1 \setminus B_2$, $A_2 \setminus A_1$, $B_2 \setminus B_1$ are all nonempty.

Proof of Claim 1: We first show that A_1, A_2, B_1, B_2 are all nonempty.

Assume first A_1 is empty. Then it follows from the definition that A_2 is nonempty. Assume B_2 is empty. Then B_1 is nonempty. Let $a_2 \in A_2$ and $b_1 \in B_1$. If neither $\{a_2\} \cup U$ nor $\{b_1\} \cup U$ is a star cutset, then $V_A = \{a_2\}$ and $V_B = \{b_1\}$. Now it follows from the definition that U is nonempty. Therefore $\bar{G}$ is disconnected, a contradiction to Theorem 3.8 9). So A_2 and B_2 are both nonempty and, since A_1 is empty, we can assume that B_1 is also empty. If $V_A \setminus A_2$ and $V_B \setminus B_2$ are both empty, $\bar{G}$ is disconnected. If, say $V_A \setminus A_2$ is nonempty and V_B contains at least two nodes, then any node b_2 of B_2 is the center of a star cutset, separating $V_A \setminus A_2$ from $V_B \setminus \{b_2\}$. So $V_B = \{b_2\}$ and therefore U is nonempty. Now in $\bar{G}$, $\{b_2\} \cup (V_A \setminus A_2)$ is a star cutset, separating A_2 and U.

So A_1 is nonempty and by symmetry A_2, B_1, B_2 are also nonempty. Suppose now that $A_2 \setminus A_1 = \emptyset$. Then $A_1 \setminus A_2$ and $B_2 \setminus B_1$ are both nonempty, else we can assume $B_2 = \emptyset$, a contradiction. Now $V_B \setminus (B_1 \cup B_2)$ and $V_A \setminus (A_1 \cup A_2)$ are both empty, else let $a_2 \in A_2$ and $b_1 \in B_1$: either $\{a_2\} \cup B_1 \cup B_2 \cup U$ is a star cutset, separating $A_1 \setminus A_2$ and $V_B \setminus (B_1 \cup B_2)$ or $\{b_1\} \cup A_1 \cup U$ is a star cutset, separating $B_2 \setminus B_1$ and $V_A \setminus (A_1 \cup A_2)$. Let $b_2 \in B_2 \setminus B_1$. Now $\{b_2\} \cup (A_1 \setminus A_2)$ is a star cutset of $\bar{G}$, separating A_2 and B_1 and the proof of Claim 1 is complete.

The family of big cliques of G can be partitioned into three families :

- $\mathcal{K}_A$ the big cliques contained in $V_A \cup U$;
- $\mathcal{K}_B$ the big cliques contained in $V_B \cup U$;
- $\mathcal{K}_{AB}$ the big cliques intersecting both V_A and V_B.

Claim 2: At least one of the two sets $V_A \setminus (A_1 \cup A_2)$, $B_1 \cap B_2$ is empty and at least one of the two sets $V_B \setminus (B_1 \cup B_2)$, $A_1 \cap A_2$ is empty.

Proof of Claim 2: Assume both $V_A \setminus (A_1 \cup A_2)$ and $B_1 \cap B_2$ are nonempty and let $b \in B_1 \cap B_2$. Now $\{b\} \cup A_1 \cup A_2 \cup U$ is a star cutset, separating $V_A \setminus (A_1 \cup A_2)$ and $B_1 \Delta B_2$.

Claim 3: We may assume that $A_1 \cap A_2$ is empty.

Proof of Claim 3: If both $A_1 \cap A_2$ and $B_1 \cap B_2$ are nonempty, by Claim 2, $V_A \setminus (A_1 \cup A_2)$ and $V_B \setminus (B_1 \cup B_2)$ are both empty. Now U is also empty, else $\bar{G}$ is disconnected. So in $\bar{G}$ every node of $A_1 \setminus A_2$ is adjacent to every node of $B_2 \setminus B_1$, every node of $A_2 \setminus A_1$ is adjacent to every node of $B_1 \setminus B_2$ and these are the only adjacencies between V_A and V_B. So by symmetry and possibly looking at $\bar{G}$, the claim holds.

From now on, we assume that $A_1 \cap A_2$ is empty.

For $V' \subseteq V(G)$, let $\omega(V')$ be the size of a largest clique contained in V'.

Claim 4: By possibly removing edges not belonging to a big clique, we may assume that $\omega(A_1) + \omega(B_1) + \omega(U) = \omega(A_2) + \omega(B_2) + \omega(U) = \omega$, every node of $B_1 \setminus B_2$ belongs to an $\omega(B_1)$-clique contained in B_1 and every node of $B_2 \setminus B_1$ belongs to an $\omega(B_2)$-clique contained in B_2.

Proof of Claim 4: Clearly $\omega(A_1) + \omega(B_1) + \omega(U) \leq \omega$ and $\omega(A_2) + \omega(B_2) + \omega(U) \leq \omega$. Assume the first inequality is strict and let $b_1 \in B_1 \setminus B_2$. Since $A_1 \cap A_2$ is empty, no edge connecting b_1 to a node of A_1 is in a big clique of G. So by Theorem 3.8 4), we can remove all these edges and obtain a partitionable graph with a universal 2-amalgam and having the same values of α and ω. This argument also shows the second statement.

Claim 5: Every node of A_1 belongs to an $\omega(A_1)$-clique included in A_1 and every node of A_2 belongs to an $\omega(A_2)$-clique included in A_2.

Proof of Claim 5: If $B_1 \cap B_2$ is empty, the claim follows from Claim 4 and symmetry. So $B_1 \cap B_2$ is nonempty and by Claim 2, $V_A \setminus (A_1 \cup A_2)$ is empty. Let $a_1 \in A_1$ be a node that is not in an $\omega(A_1)$-clique included in A_1. Then each big clique containing a_1 belongs to $U \cup A_1 \cup A_2 \cup (B_1 \cap B_2)$ and intersects both A_1 and A_2. Now $A_1 \cup A_2$ contains a unique clique of size $\omega(A_1 \cup A_2)$ or $B_1 \cap B_2$ contains a unique clique of size $\omega(B_1 \cap B_2)$, otherwise G contains big cliques K_1 and K_2 that intersect $A_1 \cup A_2$ and $B_1 \cap B_2$ in distinct cliques. But then in $\bar{G}$, K_1, K_2 correspond to stable sets that contradict Theorem 3.8 5). Since $\omega(A_1 \cup A_2) \geq 2$, the intersection of all the big cliques containing a_1 contains at least one other node in $A_1 \cup A_2 \cup (B_1 \cap B_2)$, a contradiction to Theorem 3.8 2).

By Claims 4 and 5, we can assume that every edge between A_1 and $B_1 \setminus B_2$ and every edge between A_2 and $B_2 \setminus B_1$ belongs to a big clique of G.

Claim 6: If U is nonempty, $G(U)$ has a unique maximum clique.

Proof of Claim 6: If the claim is false, by the above assumption $\mathcal{K}_{AB}$ contains two cliques, say K_1 and K_2 that contain an edge between A_1 and $B_1 \setminus B_2$ and an edge between A_2 and $B_2 \setminus B_1$ respectively and intersect U in two distinct $\omega(U)$-cliques. So in $\bar{G}$, K_1, K_2 correspond to stable sets contradicting Theorem 3.8 5).

Claim 7: At least one of A_1, $B_1 \setminus B_2$ is a clique and at least one of A_2 , $B_2 \setminus B_1$ is a clique.

Proof of Claim 7: Assume both A_1, $B_1 \setminus B_2$ contain nonadjacent nodes a, a' and b, b' respectively. By Claim 5, a, a' are in distinct $\omega(A_1)$-cliques and by Claim 4, b, b' are in distinct $\omega(B_1)$-cliques. So $\mathcal{K}_{AB}$ contains two big cliques included in $A_1 \cup B_1 \cup U$ that intersect A_1 in distinct $\omega(A_1)$-cliques and intersect B_1 in distinct $\omega(B_1)$-cliques. In $\bar{G}$, these cliques correspond to stable sets contradicting Theorem 3.8 5).

Claim 8: If the stability number of $G(V_A)$ is less than α then both A_1 and A_2 are cliques.

Proof of Claim 8: Assume A_1 is not a clique. By Claim 7, $B_1 \setminus B_2$, is a clique: By Claims 4 and 5, $\mathcal{K}_{AB}$ contains two cliques, say K_1, K' that intersect A_1 in distinct $\omega(A_1)$-cliques and intersect B_1 in the same $\omega(B_1)$-clique.

Let S_1, S' be the big stable sets that are mates of K_1, K'. Since S_1, S' intersect K' and K_1 respectively, they each contain a node of A_1, so they contain no node in $B_1 \cup U$.

Let $S_{A1} = S_1 \cap V_A$ and define S_{B1}, $S_{A'}$,$S_{B'}$ accordingly. Since S_1 and S' are disjoint (as K_1 and K' intersect) and the stability number of $G(V_A)$ is less than α, these four sets are all disjoint and nonempty. Let K_2 be a clique in $\mathcal{K}_{AB}$ containing an edge between A_2 and $B_2 \setminus B_1$: Now both $S_1 \cap K_2$ and $S' \cap K_2$ consist of a single node, say t_1 and t' respectively. If $\{t_1, t'\} \subseteq A_2$ or $\{t_1, t'\} \subseteq B_2 \setminus B_1$ then $G(S_1 \Delta S')$ is disconnected, a contradiction to Theorem 3.8 5). So by symmetry, we can assume w.l.o.g. that $t' \in A_2$ and $t_1 \in B_2 \setminus B_1$. By Claim 7, at least one of A_2, $B_2 \setminus B_1$ is a clique. Assume first that A_2 is a clique.

Now $S_{A1} \cup S_{B'}$ is a stable set that is different from S_1 and avoids K_1, so it cannot be a big stable set. Similarly, $S_{A'} \cup S_{B1} \setminus \{t'\}$ is a stable set that is different from S' and avoids K', so it cannot be a big stable set. Since the union of these two stable sets has $2\alpha - 1$ nodes, we have a contradiction. If $B_2 \setminus B_1$ is a clique, the proof is identical.

Claim 9: Either G is an odd hole or at least one of $V_A \setminus (A_1 \cup A_2)$ and $V_B \setminus (B_1 \cup B_2)$ is empty.

Proof of Claim 9: Assume both the above sets are nonempty: Then by Claim 2 both $A_1 \cap A_2$ and $B_1 \cap B_2$ are empty. Since the stability numbers of both $G(V_A)$ and $G(V_B)$ are less than α, by Claim 8 both $A_1 \cup B_1$ and $A_2 \cup B_2$ are cliques. So by Claim 6 $\mathcal{K}_{AB}$ contains only two big cliques, say K_1 containing $A_1 \cup B_1$ and K_2 containing $A_2 \cup B_2$.

Assume U is nonempty and let S_1, S_2 be the big stable sets that are mates of K_1, K_2. Then S_1, S_2 are disjoint (as K_1, K_2 have the same intersection in U). Since S_2 intersects K_1 in a node of $A_1 \cup B_1$ and S_1 intersects K_2 in a node of $A_2 \cup B_2$, and stability numbers of both $G(V_A)$ and $G(V_B)$ are less than α, $S_1 \Delta S_2$ is disconnected, a contradiction.

So U is empty. Since both $A_1 \cup B_1$ and $A_2 \cup B_2$ are cliques, by Theorem 3.8 8) applied to the partition (V_A, V_B), we have that $\omega = 2$ and G is an odd hole. This completes the proof of Claim 9.

By Claims 2 and 9, from now on, we can assume that both $A_1 \cap A_2$ and $V_A \setminus (A_1 \cup A_2)$ are empty.

Claim 10: The set U is empty.

Proof of Claim 10: Let $a_1 \in A_1$. Since $V_A \setminus (A_1 \cup A_2)$ is empty every big clique that contains a_1 is in $U \cup A_1 \cup A_2 \cup B_1$ and by Claim 6, intersects U in the same $\omega(U)$-clique. So if U is nonempty, we have a contradiction to Theorem 3.8 2).

In $\bar{G}$, every node of A_1 is adjacent to every node of $B_2 \setminus B_1$ and every node of A_2 is adjacent to every node of $B_1 \setminus B_2$. If both $V_B \setminus (B_1 \cup B_2)$ and $B_1 \cap B_2$ are nonempty, then by Claim 8 applied to both G and $\bar{G}$, we have that A_1, A_2 are cliques in both G and $\bar{G}$, so A_1, A_2 consist of single nodes, a contradiction to the definition of universal 2-amalgam. Therefore by possibly looking at $\bar{G}$ where all our assumptions still hold, we can assume that also $B_1 \cap B_2$ is empty. By Claim 7, at least one of A_1, B_1 and at least one of A_2, B_2 is a clique. By Claim 7 applied to $\bar{G}$, we have that at least one of A_1, B_2 and at least one of A_2, B_1 is a stable set.

Assume first that none of A_1, A_2, B_1, B_2 contains a single node. Then we can also assume that A_1 is a clique: For if A_1 is a stable set, by Claim 8, $V_B \setminus (B_1 \cup B_2)$ is empty and we can again look at the complement graph where all our assumptions still hold. Now it follows that A_2 is also a clique and B_1, B_2 are stable sets. So since every edge between A_1 and B_1 and every edge between A_2 and B_2 belongs to a big clique of G, we have that $|A_1| = |A_2|$.

We now claim that both B_1, B_2 contain a single node: Assume not and consider the big cliques $K_1 = A_1 \cup \{b_1\}$, $K' = A_1 \cup \{b'\}$, where b_1, b' are two nodes in B_1. There is a big clique, say K_2, of G contained in $A_1 \cup A_2$, otherwise all the big cliques that contain a node $a_1 \in A_1$ contain all the nodes in A_1, a contradiction to Theorem 3.8 2). Let S_1, S' be the disjoint big stable sets that are the mates of K_1, K'. Since S_1 and S' intersect K_2, then $S_1 \Delta S_2$ is disconnected, a contradiction to Theorem 3.8 5). So both B_1, B_2 contain a single node and by the definition of universal 2-amalgam, $V_B \setminus (B_1 \cup B_2)$ is nonempty. Therefore, $B_1 \cup B_2$ is a 2-node cutset of G and by Theorem 3.8 9) $\omega(G) = 2$ a contradiction to $|A_1| \geq 2$.

Assume now that one of A_1, A_2, say A_1, contains a single node. By Claim 4, we have that $\omega(B_1) = \omega - 1$. So either G is an odd hole or $\omega \geq 3$ and $\omega(B_1) \geq 2$. So B_1 is not a stable set and by Claim 7 applied to $\bar{G}$, A_2 is a stable set. Since A_2 contains at least two nodes (by definition of universal 2-amalgam), A_2 is not a clique and therefore B_2 is a clique and by Claim 4, we have that $\omega(B_2) = \omega - 1$. Now $A_1 \cup B_2$ is a cutset of G of size ω and by Theorem 3.8 9), $\omega = 2$, a contradiction.

If one of B_1, B_2 contains a single node the proof is identical. □

5 Homogeneous Pairs and Universal 2-Amalgams

Two nodes x, y are *antitwins* if $N(x) \setminus y$ and $N(y) \setminus x$ partition $V \setminus \{x, y\}$. Partitionable graphs can have antitwins. This is the case, for example, for a (3,3)-web. Indeed, Maffray [15] shows that for every α, ω at least 5, there are (α, ω)-graphs containing antitwins. Olariu [16] proved that minimally imperfect graphs cannot have antitwins. In fact, Olariu [16] shows the following stronger result. Here we give a short proof.

Lemma 5.1 *If a partitionable graph has antitwins, then it contains a 5-hole.*

Proof: Let x, y be antitwins, $A_1 = N(x) \setminus y$ and $A_2 = N(y) \setminus x$. Denote by S the unique big stable set that contains x and participates as a color class of an $\omega-$coloration of $G \setminus y$. Then each node in A_1, has a neighbor in $S \setminus \{x\}$: Otherwise the mate of S (that is contained in $A_2 \cup \{y\}$ by Theorem 3.8 3)) avoids both S and $S \setminus \{x\} \cup \{u\}$, a contradiction to Theorem 3.8 1). Similarly, denote by C the unique big clique that contains x and participates in the unique partition of $G \setminus y$ into big cliques. Then each node in A_2 is nonadjacent to at least one node in $C \setminus \{x\}$.

Among the nodes of $S \setminus \{x\}$, let w be the one with the largest number of neighbors in C and let $u \in C \setminus \{x\}$ be nonadjacent to w. Let $v \in S \setminus \{x\}$ be a node adjacent to u. As w is adjacent to a maximum number of nodes in C among the nodes of S and $uw \notin E$, there is a node $z \in C \setminus \{x\}$ adjacent to w but not v. Therefore y, w, z, u, v induces a 5-hole. □

The notion of antitwins is generalized to the notion of homogeneous pairs by Chvátal and Sbihi [8].

Definition 5.2 *A graph G has a* homogeneous pair *if $V(G)$ can be partitioned into subsets A_1, A_2 and B, such that:*

- *$|A_1| \geq 2$ and $|B| \geq 2$.*
- *If a node of B is adjacent to a node of A_1 (A_2) then it is adjacent to all the nodes of A_1 (A_2).*

Note that in the special case where $B = \{x, y\}$, $A_1 = N(x) \setminus y$ and $A_2 = N(y) \setminus x$, the nodes x and y are antitwins. Chvátal and Sbihi [8] show that no minimally imperfect graph has a homogeneous pair. The following theorem implies the result of Chvátal and Sbihi that minimally imperfect graphs do not have homogeneous pairs.

Theorem 5.3 *If a partitionable graph has a homogeneous pair, then it contains antitwins.*

Proof: Consider a homogeneous pair A_1, A_2, B in a graph G that contains no antitwins. Since G cannot be disconnected, a node of B is adjacent to a node of $A_1 \cup A_2$. Since B does not consist of exactly two nodes that are antitwins, $V_A = A_1 \cup A_2$ and $V_B = B$ defines a universal 2-amalgam, and by Theorem 1.1, G or $\bar{G}$ is an odd hole. It is easy to verify that an odd hole or its complement does not have a homogeneous pair. □

Theorem 5.3 together with Lemma 5.1 imply the following corollary.

Corollary 5.4 *If a partitionable graph has a homogeneous pair, then it contains a 5-hole.*

6 Universal 2-Joins

A graph G has a *universal 2-join* if $V(G)$ can be partitioned into subsets V_A, V_B and U (U possibly empty), such that:

- V_A contains nonempty disjoint sets A_1 and A_2, V_B contains nonempty disjoint sets B_1 and B_2, every node of A_1 is adjacent to every node of B_1, every node of A_2 is adjacent to every node of B_2 and these are the only adjacencies between V_A and V_B.
- Every node of U is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$.
- $|U \cup V_A| \geq 3$ and $|U \cup V_B| \geq 3$.

The relation between universal 2-amalgams and homogeneous pairs, universal 2-joins and star cutsets is clarified in the following result.

Theorem 6.1 *If a graph G has a universal 2-amalgam, then one of the following holds:*

(i) G or $\bar{G}$ has a star cutset or is disconnected.

(ii) G or $\bar{G}$ has a universal 2-join.

(iii) G has a homogeneous pair A_1, A_2 such that $V(G) \setminus (A_1 \cup A_2)$ does not consist of exactly two nodes that are antitwins.

Proof: Suppose G has a universal 2-amalgam and (i) does not hold. Then, using the same argument as in Claim 1 in the proof of Theorem 1.1, $A_1 \setminus A_2$, $B_1 \setminus B_2$, $A_2 \setminus A_1$ and $B_2 \setminus B_1$ are all nonempty. Using the argument in Claim 2 of Theorem 1.1, at least one of $V_A \setminus (A_1 \cup A_2)$, $B_1 \cap B_2$ is empty and at least one of $V_B \setminus (B_1 \cup B_2)$, $A_1 \cap A_2$ is empty. If both $A_1 \cap A_2$ and $B_1 \cap B_2$ are empty then the universal 2-amalgam is a universal 2-join. Suppose both $A_1 \cap A_2$ and $B_1 \cap B_2$ are nonempty. Then $V_A \setminus (A_1 \cup A_2)$ and $V_B \setminus (B_1 \cup B_2)$ are both empty. U is empty, else $\bar{G}$ is disconnected. But then $\bar{G}$ has a universal 2-join. So we may assume w.l.o.g. that $A_1 \cap A_2$ is empty and $B_1 \cap B_2$ is nonempty. Then $V_A \setminus (A_1 \cup A_2)$ is empty. Suppose A_1, A_2 is not a homogeneous pair. Then $|A_1| = |A_2| = 1$. By the definition of universal 2-amalgam, U is nonempty. But then $A_1 \cup (V_B \setminus (B_1 \cup B_2))$ is a star cutset in $\bar{G}$. □

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