Convex Sets and Minimal Sublinear Functions

Amitabh Basu∗  Gérard Cornuéjols †  Giacomo Zambelli‡

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Abstract

We show that, given a closed convex set $K$ with the origin in its interior, the support function of the set \{ $y \in K^\ast \mid \exists x \in K$ such that $xy = 1$ \} is the pointwise smallest sublinear function $\sigma$ such that $K = \{ x \mid \sigma(x) \leq 1 \}$.

1 Introduction

Let $K$ be a closed, convex set with the origin in its interior. A standard concept in convex analysis [1, 2] is that of gauge (sometimes called Minkowski function), which is the function $\gamma_K$ defined by

$$\gamma_K(x) = \inf\{ t > 0 \mid t^{-1}x \in K \}, \quad \text{for all } x \in \mathbb{R}^n.$$  

By definition $\gamma_K$ is nonnegative. It is also sublinear, another classical concept that we define next. A function $\sigma : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous if $\sigma(tx) = t\sigma(x)$ for every $x \in \mathbb{R}^n$ and $t > 0$, and it is sublinear if it is convex and positively homogeneous. One can readily verify that $K = \{ x \mid \gamma_K(x) \leq 1 \}$.

Given any sublinear function $\sigma$ such that $K = \{ x \mid \sigma(x) \leq 1 \}$, it follows from positive homogeneity that $\sigma(x) = \gamma_K(x)$ for every $x$ where $\sigma(x) > 0$. Hence $\sigma(x) \leq \gamma_K(x)$ for all $x \in \mathbb{R}^n$. In this paper we introduce a sublinear function $\rho_K$ such that $K = \{ x \mid \rho_K(x) \leq 1 \}$ and $\rho_K(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$.

The polar of $K$ is the set $K^* = \{ y \in \mathbb{R}^n \mid xy \leq 1 \text{ for all } x \in K \}$. Clearly $K^*$ is closed and convex, and since $0 \in \text{int}(K)$, it is well known that $K^*$ is bounded. In particular, $K^*$ is a compact set. Also, since $0 \in K$, $K^{**} = K$.

∗Carnegie Mellon University, abasu1@andrew.cmu.edu
†Carnegie Mellon University, gc0v@andrew.cmu.edu. Supported by NSF grant CMMI0653419, ONR grant N00014-03-1-0188 and ANR grant BLAN06-1-138894.
‡Università di Padova, giacomo@math.unipd.it
Given any $T \subset \mathbb{R}^n$, the \textit{support function of} $T$ is defined by

$$\sigma_T(x) = \sup_{y \in T} xy, \quad \text{for all} \ x \in \mathbb{R}^n.$$ 

It is straightforward to show that support functions are sublinear [1]. It is well known that $\gamma_K$ is the support function of $K^*$ (see [1] Proposition 3.2.4).

We define our function $\rho_K$ as the support function of the set

$$\hat{K} = \{y \in K^* | \exists x \in K \text{ such that } xy = 1\}.$$ 

Note that $\hat{K}$ is contained in the relative boundary of $K^*$. By definition

$$\rho_K(x) = \sup_{y \in \hat{K}} xy, \quad \text{for all} \ x \in \mathbb{R}^n.$$ 

Note that $\rho_K$ is sublinear. Furthermore we will show that $K = \{x | \rho_K(x) \leq 1\}$. The next theorem shows that $\rho_K$ is the smallest function with these two properties.

\textbf{Theorem 1} Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. For every sublinear function $\sigma$ such that $K = \{x | \sigma(x) \leq 1\}$, we have $\rho_K(x) \leq \sigma(x)$ for every $x \in \mathbb{R}^n$.

Note that the recession cone of $K$, which is the set $\text{rec}(K) = \{x \in K | tx \in K \text{ for all } t > 0\}$, coincides with $\{x \in K | \sigma(x) \leq 0\}$ for every sublinear function $\sigma$ such that $K = \{x | \sigma(x) \leq 1\}$. In particular $\rho_K(x)$ can be negative for $x \in \text{rec}(K)$, so in general it is different from the gauge.

For example, let $K = \{x \in \mathbb{R}^2 | x_1 \leq 1, x_2 \leq 1\}$. Then $K^* = \text{conv}\{(0, 0), (1, 0), (0, 1)\}$ and $\hat{K} = \text{conv}\{(1, 0), (0, 1)\}$. Therefore, for every $x \in \mathbb{R}^2$, $\gamma_K(x) = \max\{0, x_1, x_2\}$ and $\rho_K(x) = \max\{x_1, x_2\}$. In particular, $\rho_K(x) < 0$ for every $x$ such that $x_1 < 0, x_2 < 0$.

2 Proof of Theorem 1

We will need Straszewicz’s theorem [3] (see [2] Theorem 18.6). Given a closed convex set $C$, a point $x \in C$ is \textit{extreme} if it cannot be written as a proper convex combination of two distinct points in $C$. A point $x \in C$ is \textit{exposed} if there exists a supporting hyperplane $H$ for $C$ such that $H \cap C = \{x\}$. Clearly exposed points are extreme. We will denote by $\text{ext}(C)$ the set of extreme points and $\text{exp}(C)$ the set of exposed points of $C$.

\textbf{Theorem 2} Given a closed convex set $C$, the set of exposed points of $C$ is a dense subset of the set of extreme points of $C$. 

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Let $K$ be a closed convex set with the origin in its interior. Let $\sigma$ be a sublinear function such that $K = \{x \mid \sigma(x) \leq 1\}$. The boundary of $K$, denoted by $\text{bd}(K)$, is the set $\{x \in K \mid \sigma(x) = 1\}$.

**Lemma 3** For every $x \notin \text{rec}(K)$, $\sigma(x) = \rho_K(x) = \sup_{y \in K^*} xy$. In particular, $K = \{x \mid \rho_K(x) \leq 1\}$.

**Proof.** Let $x \notin \text{rec}(K)$. Then $t = \sigma(x) > 0$. By positive homogeneity, $\sigma(t^{-1}x) = 1$, hence $t^{-1}x \in \text{bd}(K)$. Since $K$ is closed and convex, there exists a supporting hyperplane for $K$ containing $t^{-1}x$. Since $0 \in \text{int}(K)$, this implies that there exists $\bar{y} \in K^*$ such that $(t^{-1})\bar{y} = 1$. In particular $\bar{y} \in K$, hence by definition $\rho_K(x) = x\bar{y} = t$.

Furthermore, for any $y \in K^*$, $(t^{-1})y \leq 1$, hence $xy \leq t$, which implies $t \geq \sup_{y \in K^*} xy$. Thus

$$\rho_K(x) \geq t \geq \sup_{y \in K^*} xy \geq \sup_{y \in K} xy = \rho_K(x),$$

where the last inequality holds since $K \subset K^*$, hence equality holds throughout. \hfill \Box

**Lemma 4** Given an exposed point $\bar{y}$ of $K^*$ different from the origin, there exists $x \in K$ such that $xy = 1$ and $xy < 1$ for all $y \in K^*$ distinct from $\bar{y}$.

**Proof.** If $\bar{y} \neq 0$ is an exposed point of $K^*$, then there exists a supporting hyperplane $H = \{y \mid ay = \beta\}$ such that $a\bar{y} = \beta$ and $ay < \beta$ for every $y \in K^* \setminus \{\bar{y}\}$. Since $0 \in K^*$ and $\bar{y} \neq 0$, $\beta > 0$. Thus the point $x = \beta^{-1}a \in K^{**} = K$ satisfies the statement. \hfill \Box

**Lemma 5** For every $x \in \mathbb{R}^n$, $\rho_K(x) = \sup_{y \in K \cap \text{exp}(K^*)} xy$.

**Proof.** We first show that $\rho_K(x) = \sup_{y \in K \cap \text{ext}(K^*)} xy$. Given $y \in \hat{K}$ we show that there exists an extreme point $y'$ of $K^*$ in $\hat{K}$ such that $xy \leq xy'$. Since $y \in \hat{K}$, there exists $\bar{x} \in K$ such that $\bar{x}y = 1$. The point $\bar{y}$ is a convex combination of extreme points $y_1, \ldots, y_k$ of $K^*$, and each $y_i$ satisfies $\bar{x}y_i = 1$. Thus $y_1, \ldots, y_k \in \hat{K}$, and $xy_i \geq xy$ for at least one $i$.

By Straszewicz’s theorem (Theorem 2) the set of exposed points in $K^*$ is a dense subset of the extreme points of $K^*$. By Lemma 4, all exposed points of $K^*$ except the origin are in $\hat{K}$, hence $\text{exp}(K^*) \cap \hat{K}$ is dense in $\text{ext}(K^*) \cap \hat{K}$. Therefore $\rho_K(x) = \sup_{y \in K \cap \text{exp}(K^*)} xy$. \hfill \Box

A function $\sigma$ is subadditive if $\sigma(x_1 + x_2) \leq \sigma(x_1) + \sigma(x_2)$ for every $x_1, x_2 \in \mathbb{R}^n$. It is easy to show that $\sigma$ is sublinear if and only if it is subadditive and positively homogeneous.

**Proof of Theorem 1.** By Lemma 3, we only need to show $\sigma(x) \geq \rho_K(x)$ for points $x \in \text{rec}(K)$. By Lemma 5 it is sufficient to show that, for every exposed point $\bar{y}$ of $K^*$ contained in $\hat{K}$, $\sigma(x) \geq x\bar{y}$.
Let $\bar{y}$ be an exposed point of $K^*$ in $\hat{K}$. By Lemma 4 there exists $\bar{x} \in K$ such that $\bar{x}\bar{y} = 1$ and $\bar{x}y < 1$ for all $y \in K^*$ distinct from $\bar{y}$. Note that $\bar{x} \in \text{bd}(K)$.

We observe that for all $\delta > 0$, $\bar{x} - \delta^{-1}x \notin \text{rec}(K)$. Indeed, since $x \in \text{rec}(K)$, $\bar{x} + \delta^{-1}x \in K$. Hence $\bar{x} - \delta^{-1}x \notin \text{int}(K)$ because $\bar{x} \in \text{bd}(K)$. Since $0 \in \text{int}(K)$ and $\bar{x} - \delta^{-1}x \notin \text{rec}(K)$, then $\bar{x} - \delta^{-1}x \notin \text{rec}(K)$. Thus by Lemma 3

$$\sigma(\bar{x} - \delta^{-1}x) = \sup_{y \in K^*} (\bar{x} - \delta^{-1}x)y.$$  \hspace{1cm} (1)

Since $\bar{x} \in \text{bd}(K)$, $\sigma(\bar{x}) = 1$. By subadditivity, $1 = \sigma(\bar{x}) \leq \sigma(\bar{x} - \delta^{-1}x) + \sigma(\delta^{-1}x)$. By positive homogeneity, the latter implies that $\sigma(x) \geq \delta - \delta \sigma(\bar{x} - \delta^{-1}x)$ for all $\delta > 0$. By (1),

$$\sigma(x) \geq \inf_{y \in K^*} [\delta(1 - \bar{x}y) + xy],$$

hence

$$\sigma(x) \geq \sup_{\delta > 0} \inf_{y \in K^*} [\delta(1 - \bar{x}y) + xy].$$

Let $g(\delta) = \inf_{y \in K^*} \delta(1 - \bar{x}y) + xy$. Since $\bar{x} \in K$, $1 - \bar{x}y \geq 0$ for every $y \in K^*$. Hence $\delta(1 - \bar{x}y) + xy$ defines an increasing affine function of $\delta$ for each $y \in K^*$, therefore $g(\delta)$ is increasing and concave. Thus $\sup_{\delta > 0} g(\delta) = \lim_{\delta \to +\infty} g(\delta)$.

Since $K^*$ is compact, for every $\delta > 0$ there exists $y(\delta) \in K^*$ such that $g(\delta) = \delta(1 - \bar{x}y(\delta)) + xy(\delta)$. Furthermore there exists a sequence $(\delta_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to +\infty} \delta_i = +\infty$ and the sequence $(y_i)_{i \in \mathbb{N}}$ defined by $y_i = y(\delta_i)$ converges, because in a compact set every sequence has a convergent subsequence. Let $y^* = \lim_{i \to +\infty} y_i$.

We conclude the proof by showing that $\sigma(x) \geq xy^*$ and $y^* = \bar{y}$.

$$\sigma(x) \geq \sup_{\delta > 0} g(\delta) = \lim_{i \to +\infty} g(\delta_i) = \lim_{i \to +\infty} [\delta_i(1 - \bar{x}y_i) + xy_i] = \lim_{i \to +\infty} \delta_i(1 - \bar{x}y_i) + xy^* \geq xy^*$$

where the last inequality follows from the fact that $\delta_i(1 - \bar{x}y_i) \geq 0$ for all $i \in \mathbb{N}$. Finally, since $\lim_{i \to +\infty} \delta_i(1 - \bar{x}y_i)$ is bounded and $\lim_{i \to +\infty} \delta_i = +\infty$, it follows that $\lim_{i \to +\infty} (1 - \bar{x}y_i) = 0$, hence $\bar{x}y^* = 1$. By our choice of $\bar{x}$, $\bar{x}y < 1$ for every $y \in K^*$ distinct from $\bar{y}$. Hence $y^* = \bar{y}$. \hfill $\Box$

References
