Structure theorems for two classes of resistant sets

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Abstract

A subset of the unit hypercube $\{0,1\}^n$ is *cube-ideal* if its convex hull is described by hypercube and generalized set covering inequalities. In this paper, we provide structure theorems for two classes of cube-ideal sets that remain cube-ideal even after making local changes. We will also discuss applications.

1 Introduction

Take an integer $n \geq 1$. Denote by $\{0,1\}^n$ the extreme points of the n-dimensional unit hypercube $[0,1]^n$. For a coordinate $i \in [n] := \{1,\dots,n\}$, we refer to $x_i \geq 0$ and $x_i \leq 1$ as *hypercube* inequalities. *Generalized set covering* inequalities are ones of the form

$$\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1 \qquad I,J \subseteq [n], I\cap J = \emptyset,$$

which are precisely the inequalities that cut off (sub-)hypercubes of $\{0,1\}^n$. Interpreted as clause satisfaction inequalities for the Boolean satisfiability problem, generalized set covering inequalities are prevalent in the literature. Also referred to as *cropping* inequalities [8, 13], these inequalities have surfaced as *cocircuit* inequalities valid for cycle polytopes of binary matroids [6], as *set covering* inequalities ($J = \emptyset$) for various set covering problems [7, 11, 9], and as *cover* inequalities ($I = \emptyset$) for the knapsack problem [5, 12, 15].

Take a set $S \subseteq \{0,1\}^n$. We say that S is *cube-ideal* if its convex hull, denoted $\mathrm{conv}(S)$, can be described by hypercube and generalized set covering inequalities. This notion was introduced and studied in [2]. What is the structure of cube-ideal sets? This question lays the underpinning theme of our paper.

Among basic classes of cube-ideal sets are: the cycle space of a graph [14]; the up-monotone set associated with an ideal clutter (see [2]); and a set where each infeasible component is a hypercube or has maximum degree at most two [3]. Given such basic classes, there are three binary operations that preserve cube-idealness and can be used to generate more cube-ideal sets: the product of two cube-ideal sets is cube-ideal; the coproduct of two cube-ideal sets is cube-ideal; given two cube-ideal sets whose complements are also cube-ideal, their reflective product is cube-ideal [2]. Subsequently, cube-ideal sets form a rich class and have a complex structure, to say the least. Nonetheless, we conjecture the following:

Conjecture 1.1. *There exists an algorithm that given an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ determines in time polynomial in n and $|S|$ whether or not S is cube-ideal.*

(It is worth mentioning that testing the idealness of an explicitly given clutter is co-NP-complete, as was shown by Ding, Feng and Zang [10].)

In this paper, we provide structure theorems for cube-ideal sets that remain cube-ideal even after making local changes.

1.1 Definitions and notation

Given points $a, b \in \{0,1\}^n$, the (Hadamard) *distance* between a, b, denoted $\mathrm{dist}(a,b)$, is the number of coordinates a and b differ on. Denote by G_n the *skeleton graph* of $\{0,1\}^n$, whose vertices are the points in $\{0,1\}^n$, where two vertices $a, b \in \{0,1\}^n$ are adjacent if $\mathrm{dist}(a,b) = 1$. We refer to the points in S as *feasible* and to the points in $\overline{S} := \{0,1\}^n - S$ as *infeasible*. The (connected) components of $G_n[S]$ are *feasible components*, while the components of $G_n[\overline{S}]$ are *infeasible components*.

For $i \in [n]$, denote by e_i the i^{th} unit vector. To *twist coordinate* $i \in [n]$ is to replace S by

$$S \triangle e_i := \{x \triangle e_i : x \in S\}.$$

We say that $S' \subseteq \{0,1\}^n$ is *isomorphic* to S, and write $S' \cong S$, if S' is obtained from S after relabeling and twisting some coordinates.

The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-*restriction of S over coordinate i*, and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-*restriction of S over coordinate i*. A *restriction* of S is a set obtained after a series of 0- and 1-restrictions. The *projection of S over coordinate i* is the set obtained from S after dropping coordinate i. A *minor of S* is what is obtained after a series of restrictions and projections. A minor is *proper* if at least one operation is applied.

Remark 1.2 ([2]). *If a set is cube-ideal, then so is every isomorphic minor of it.*[1]

1.2 Resistance and structure theorems

Let $P_3 := \{110, 101, 011\} \subseteq \{0,1\}^3$ and $S_3 := \{110, 101, 011, 111\} \subseteq \{0,1\}^3$, as displayed in Figure 1. Then

$$\mathrm{conv}(P_3) = \left\{x \in [0,1]^3 : x_1 + x_2 + x_3 = 2\right\} \quad \text{and} \quad \mathrm{conv}(S_3) = \left\{x \in [0,1]^3 : x_1 + x_2 + x_3 \geq 2\right\},$$

implying in turn that P_3, S_3 are not cube-ideal. In particular, a cube-ideal set has no P_3, S_3 minor by Remark 1.2.

We say that S is 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor. The notion of 1-resistance was introduced and studied by Abdi, Cornuéjols and Lee [3], though the prefix 1- was omitted there. There the authors showed that 1-resistance is a multifaceted property, and they demonstrate that the class of 1-resistant sets is quite rich. In particular, they show that,

[1] Going forward, the prefix "isomorphic" will be omitted from "isomorphic restriction" and "isomorphic minor".

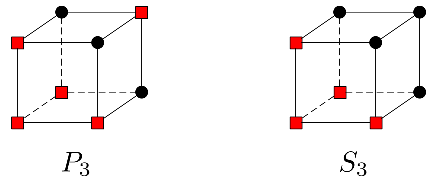


Figure 1: An illustration of P_3 and S_3. Round points are feasible while square points are infeasible.

Theorem 1.3 ([3]). *A 1-resistant set is cube-ideal.*

Although a structure theorem remains elusive even for this special class of cube-ideal sets, we are able to provide structure theorems for two natural classes.

We say that S is 2-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ has no P_3, S_3 minor. We will prove the following theorem, part (iii) of which explains the structure of 2-resistant sets:

Theorem 1.4. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 2-resistant,

(ii) S has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,

(iii) every infeasible component is a hypercube or has maximum degree at most two,

(iv) S has no minor $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$.

This theorem is proved in §2. There we will also show that 2-resistance of a set $S \subseteq \{0,1\}^n$ can be tested in time $O(n^3|S|)$.

We say that S is ± 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \triangle X$ has no P_3, S_3 minor. In §3, we provide excluded minor and excluded restriction characterizations for ± 1-resistant sets. There we will also show that ± 1-resistance of a set $S \subseteq \{0,1\}^n$ can be tested in time $O(n^2|S|^2)$.

Given integers $n_1, n_2 \geq 0$ and $S_1 \subseteq \{0,1\}^{n_1}, S_2 \subseteq \{0,1\}^{n_2}$, the *product of* S_1, S_2 is

$$S_1 \times S_2 := \{(x,y) : x \in S_1, y \in S_2\} \subseteq \{0,1\}^{n_1+n_2}.$$

We will prove the following structure theorem:

Theorem 1.5. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is ± 1-resistant if, and only if, one of the following statements holds:*

(i) $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \dots, n\}$, where $A_k = \{\mathbf{0}^k, \mathbf{1}^k\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$, where $B_k = \{\mathbf{0}^k, e_1, \mathbf{1}^k\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$, where $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$, where $D_k = \{\mathbf{0}^k, e_2, \mathbf{1}^k - e_2, \mathbf{1}^k - e_2 - e_3\}$,

(v) *S is a hypercube, or*

(vi) *every infeasible component of S is a hypercube, and every feasible point has at most two infeasible neighbors.*

Here, $\mathbf{0}^k, \mathbf{1}^k$ denote the k-dimensional vectors of all-zeros and all-ones, respectively. (The superscripts will be dropped when there is no ambiguity.) A proof outline of this theorem is given in §4; the proof spans §5, §6, §7 and §8.[2]

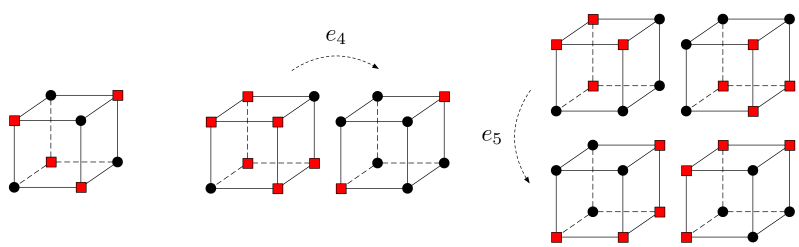
1.3 Strict polarity and applications

We say that S is *polar* if either there are antipodal feasible points, or the feasible points agree on a coordinate:

$$\{x, \mathbf{1}-x\} \subseteq S \quad \text{for some } x \in \{0,1\}^n \qquad \text{or} \qquad S \subseteq \{x : x_i = a\} \quad \text{for some } i \in [n] \text{ and } a \in \{0,1\}.$$

We say that S is *strictly non-polar* if it is not polar, but every proper restriction is polar. It is shown in [3] that if a set is strictly non-polar, and 1-resistant, then it gives rise to an ideal minimally non-packing clutter. Motivated by this fact,

Question 1.6. *What are the* 1*-resistant strictly non-polar sets?*



$R_{1,1}$ $\qquad$ $R_{2,1}$ $\qquad$ R_5

Figure 2: The 2-resistant strictly non-polar sets.

Even though the 1-resistant strictly non-polar sets $S \subseteq \{0,1\}^n$ satisfying $|S| = 2^{n-1}$ are completely found [3], Question 1.6 is still open. Using the structure theorems obtained, we answer this question for 2- and ± 1-resistant sets. To this end, let

$$\begin{aligned} R_{1,1} &:= \{000, 110, 101, 011\} \subseteq \{0,1\}^3 \\ R_{2,1} &:= \{0000, 1110, 1001, 0101, 0011, 1101, 1011, 0111\} \subseteq \{0,1\}^4 \\ R_5 &:= \{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\} \\ &\quad \cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0,1\}^5, \end{aligned}$$

as displayed in Figure 2. We prove the following theorem in §2:

[2] For an explanation of the origin of resistance, see [1], Chapter 6.

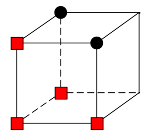


Figure 3: An illustration of a fragile set.

Theorem 1.7. *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only 2-resistant strictly non-polar sets.*

We say that S is *strictly polar* if every restriction of it, including S itself, is polar. Notice that a set is strictly polar if, and only if, it has no strictly non-polar restriction.

Theorem 1.8. *A ± 1-resistant set is strictly polar.*

This theorem is proved in §9.

1.4 Preliminaries

Throughout the paper, we will make use of several results from [3]. Let us state them all here.

Remark 1.9 ([3]). *If a set is 1-resistant, then so is every minor of it.*

Take a set $F \subseteq \{0,1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}.$$

We refer to F, and any set isomorphic to it, as *fragile*. (See Figure 3.)

Theorem 1.10 ([3]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 1-resistant,

(ii) S has no fragile restriction and no $\left\{\mathbf{0}^k, \mathbf{1}^k - e_1\right\}, k \geq 4$ restriction,

(iii) S has no fragile minor.

Testing 1-resistance can be done efficiently:

Theorem 1.11 ([3]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^4|S|^3)$, one can test whether or not S is 1-resistant.*

We will also need the following lemma for 1-resistant sets:

Lemma 1.12 ([3]). *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then S is a hypercube.*

Lastly, we will need the following lemma for general sets:

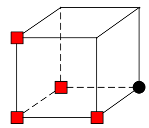


Figure 4: The excluded minor, and restriction, defining 2-resistance.

Lemma 1.13 ([3]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$, where for all $x \in \{0,1\}^n$ and distinct $i, j \in [n]$, the following statement holds:*

> *if $x, x\triangle e_i, x\triangle e_j \in S$ then $x\triangle e_i \triangle e_j \in S$.*

Then every feasible component of S is a hypercube.

2 The structure of 2-resistant sets and consequences

In this section, we prove Theorem 1.4 on the structure of 2-resistant sets. We will then prove three applications, including Theorem 1.7 characterizing the 2-resistant strictly non-polar sets. Let us start with the following remark:

Remark 2.1. *If a set is 2-resistant, then so is every minor of it.*

Proof. Being 2-resistant is clearly closed under restrictions; it remains to show that it is also closed under projections. To this end, take an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$. Let $S' \subseteq \{0,1\}^{n-1}$ be the projection of S over coordinate n. Suppose for a contradiction that S' is not 2-resistant. Then for some $X' \subseteq \{0,1\}^{n-1}$ of cardinality at most two, $S' \cup X'$ has a P_3, S_3 minor. Let $X := \{(x', 0) : x' \in X'\}$. Then $S \cup X$ has $S' \cup X'$ as a projection, and has a P_3, S_3 minor as a consequence, implying in turn that S is not 2-resistant, a contradiction. $\square$

We are now ready to prove Theorem 1.4, stating the following:

> Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:
>
> (i) S is 2-resistant,
>
> (ii) S has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,
>
> (iii) every infeasible component is a hypercube or has maximum degree at most two,
>
> (iv) S has no minor $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$.

Proof. **(i) $\Rightarrow$ (ii)** Observe that F is not 2-resistant, because $F \cup \{101, 011\}$ is either P_3 or S_3. Thus, a 2-resistant set has no F restriction by Remark 2.1.

(ii) $\Rightarrow$ (iii): Assume that S has no F restriction.

Claim 1. *Let x be an infeasible point with at least three infeasible neighbors. If $x\triangle e_i, x\triangle e_j$ are infeasible for some distinct $i, j \in [n]$, then $x\triangle e_i\triangle e_j$ is also infeasible.*

Proof of Claim. Suppose for a contradiction that $x\triangle e_i\triangle e_j$ is feasible. Since x has at least three infeasible neighbors, there is a coordinate $k \in [n]-\{i,j\}$ such that $x\triangle e_k$ is infeasible. Then the 3-dimensional restriction of S containing $x\triangle e_i, x\triangle e_j, x\triangle e_k$ is a set $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$, a contradiction. $\Diamond$

Claim 2. *Let x be an infeasible point with at least three infeasible neighbors. Let $k \geq 3$ be the number of infeasible neighbors of x. Then the k-dimensional hypercube containing x and its infeasible neighbors is infeasible.*

Proof of Claim. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and its infeasible neighbors are $e_1, \ldots, e_k$. We need to show that for all subsets $I \subseteq [k]$, $\sum_{i\in I} e_i \in \overline{S}$. We will proceed by induction on $|I| \geq 0$. The base cases $|I| \in \{0,1\}$ hold by assumption, and the case $|I| = 2$ follows from Claim 1. For the induction step, assume that $|I| \geq 3$. After a possible relabeling, if necessary, we may assume that $I = [\ell]$. Let $y := \sum_{i=1}^{\ell-2} e_i$. By the induction hypothesis, y and its three neighbors $y\triangle e_{\ell-2}, y\triangle e_{\ell-1}, y\triangle e_\ell$ are all infeasible. It therefore follows from Claim 1 that $y\triangle e_{\ell-1}\triangle e_\ell = \sum_{i=1}^{\ell} e_i$ is infeasible, thereby completing the induction step. $\Diamond$

Let $K \subseteq \overline{S}$ be an infeasible component, and let k be the maximum number of infeasible neighbors of a point in K. If $k \leq 2$, then K has maximum degree at most two. Otherwise, $k \geq 3$. It then follows from Claim 2 that K contains a k-dimensional hypercube. Our maximal choice of k in turn implies that K is in fact the k-dimensional hypercube. Thus, every infeasible component is a hypercube or has maximum degree at most two.

(iii) $\Rightarrow$ (iv): Assume that every infeasible component of S is a hypercube or has maximum degree at most two.

Claim 3. *If S' is a minor of S, then every infeasible component of S' is a hypercube or has maximum degree at most two.*

Proof of Claim. It suffices to prove this for restrictions and projections. The claim clearly holds for restrictions. As for projections, assume that S' is obtained from S after projecting away coordinate n. Let $K' \subseteq \{0,1\}^{n-1}$ be an infeasible component of S'. Clearly, $\{(x,0),(x,1) : x \in K'\} \subseteq \{0,1\}^n$ is connected and infeasible, so it is contained in an infeasible component K of S. If K has maximum degree at most two, then so does $\{(x,0),(x,1) : x \in K'\}$, implying in turn that K' has maximum degree at most two. Otherwise, K is a hypercube. In this case, as K' is an infeasible component of S', it must be that $K = \{(x,0),(x,1) : x \in K'\}$, implying in turn that K' is a hypercube. Thus, K' is a hypercube or has maximum degree at most two, as claimed. $\Diamond$

Thus, since the infeasible component of F containing 000 is neither a hypercube or of maximum degree at most two, S does not have an F minor.

(iv) $\Rightarrow$ (i): Assume that S is not 2-resistant. Then there is a subset $X \subseteq \{0,1\}^n$ of cardinality at most two such that $S \cup X$ has a P_3, S_3 minor. Thus there is a subset $Y \subseteq \{0,1\}^3$ of cardinality at most two such that S has a $P_3 - Y, S_3 - Y$ minor. After relabeling the coordinates, if necessary, we see that both $P_3 - Y, S_3 - Y$ are the desired minor. $\square$

2.1 Applications of Theorem 1.4

The first application is that testing 2-resistance can be done efficiently:

Corollary 2.2. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^4|S|^2)$, one can test whether or not S is 2-resistant.*

Proof. By Theorem 1.4 (ii), testing whether S is 2-resistant is equivalent to testing whether S has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$. Such a restriction can be found according to the following simple algorithm:

> Pick a feasible point x and distinct coordinates $i, j, k \in [n]$. Test whether or not the 3-dimensional restriction containing $x\triangle e_i, x\triangle e_j, x\triangle e_k$ is the desired restriction. If so, then S is not 2-resistant. If not, change x, i, j, k.

As a result, testing 2-resistance takes time $n^3|S| \times n|S| = n^4|S|^2$. $\square$

The second application is yet another characterization of 2-resistance:

Corollary 2.3. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is 2-resistant if, and only if, $S \cup X$ has no P_3, S_3 restriction for all subsets $X \subseteq \{0,1\}^n$ of cardinality at most two.*

Proof. ($\Rightarrow$) follows from the definition of 2-resistance. ($\Leftarrow$) Assume that $S \cup X$ has no P_3, S_3 restriction for all subsets $X \subseteq \{0,1\}^n$ of cardinality at most two. Then S has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$. It therefore follows from Theorem 1.4 (ii) that S is 2-resistant, as required. $\square$

The third application of Theorem 1.4 is the characterization of the 2-resistant strictly non-polar sets. We need the following lemma:

Lemma 2.4. *Take an integer $n \geq 5$ and a set $S \subseteq \{0,1\}^n$, where every infeasible point has at most two infeasible neighbors. Then $|S| \geq 2^{n-1}$.*

Proof. It suffices to prove this for $n = 5$, as the general case follows from a simple inductive argument. For $i, j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. We may assume that $|S_{00}| + |S_{10}| \leq 7$ and $|S_{00}| \leq 3$. After a possible twisting of coordinates $1, 2, 3$, we may assume that $\{000, 111\} \subseteq S_{00} \subseteq \{000, 111, 110\}$. This implies that $\{001, 101, 011\} \subseteq S_{10}$. Since $|S_{00}| + |S_{10}| \leq 7$, we get that $S_{00} = \{000, 111, 110\}$ and therefore $S_{10} = \{001, 101, 011, 110\}$. Since every infeasible point of S has at most two infeasible neighbors, it follows that $\{100, 010, 001, 101, 011\} \subseteq S_{01}$ and $\{000, 100, 010\} \subseteq S_{11}$, implying in turn that $|S_{01}| + |S_{11}| \geq 8$. In fact, as every infeasible point of S has at most two infeasible neighbors, $|S_{01}| + |S_{11}| > 8$, so $|S| \geq 7 + 9 = 16$, as required. $\square$

Recall that $R_{1,1} = \{000, 110, 101, 011\}$. Using Lemma 2.4, we prove the following:

Lemma 2.5. *Take an integer $n \geq 5$ and a nonempty set $S \subseteq \{0,1\}^n$, where every infeasible component is a hypercube or has maximum degree at most two. If S has no $R_{1,1}$ restriction and one of its infeasible components is a hypercube of dimension at least 3, then*

- $|S| \geq 2^{n-1}$, *and*
- *if $|S| = 2^{n-1}$, then S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.*

Proof. We will prove this by induction on $n \geq 5$. The base case $n = 5$ is clear. For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a hypercube of dimension $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty.

Assume in the first case that S has an infeasible hypercube of dimension ≥ 4 active in, say, direction e_n. Then both S_0, S_1 have infeasible hypercubes of dimension ≥ 3. Thus by the induction hypothesis, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. By the induction hypothesis, one of the following cases holds:

- S_0 is a hypercube of dimension $n-2 \geq 4$: In this case, we may assume that $S \cap \{x : x_n = 0\} = \{x : x_{n-1} = x_n = 0\}$. Since every infeasible component of S is a hypercube or has maximum degree at most two, the hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is either (all) feasible or infeasible. Since $|S_1| = 2^{n-2}$, it follows that $S \cap \{x : x_{n-1} = 0, x_n = 1\}$ is either

  $$\{x : x_{n-1} = 0, x_n = 1\} \quad \text{or} \quad \{x : x_{n-1} = x_n = 1\}.$$

  Thus, S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.

- S_1 is the union of two antipodal hypercubes of dimension $n-3 \geq 3$: In this case, we may assume that $S \cap \{x : x_n = 0\} = \{x : x_{n-2} = x_{n-1}, x_n = 0\}$. Since every infeasible component of S is a hypercube or has maximum degree at most two, and $|S_1| = 2^{n-2}$, it follows that $S \cap \{x : x_n = 1\}$ is either

  $$\{x : x_{n-2} = x_{n-1}, x_n = 1\} \quad \text{or} \quad \{x : x_{n-2} + x_{n-1} = 1, x_n = 1\}.$$

  However, since S has no $R_{1,1}$ restriction, the latter is not possible. Thus, $S = \{x : x_{n-2} = x_{n-1}\}$, so S is the union of antipodal hypercubes of dimension $n-2$, thereby completing the induction step.

Assume in the remaining case that every infeasible component of S has maximum degree at most two or is a (3-dimensional) cube. By assumption, one of the infeasible components is a cube, which we may assume is contained in S_0. By the induction hypothesis, $|S_0| \geq 2^{n-2}$ and if equality holds, then S_0 is either a hypercube of dimension $n-2$ or the union of antipodal hypercubes of dimension $n-3$. If S_1 has an infeasible component that is a cube, then the induction hypothesis implies that $|S_1| \geq 2^{n-2}$, and if not, S_1 has maximum degree at most two, so by Lemma 2.4, $|S_1| \geq 2^{n-2}$. Either way, $|S_1| \geq 2^{n-2}$, so $|S| = |S_0| + |S_1| \geq 2^{n-1}$. We claim that equality does not hold. Suppose for a contradiction that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. Then S_0 is either a hypercube of dimension $n-2 \geq 4$ or the union of antipodal hypercubes of dimension $n-3 \geq 3$. As S has no infeasible hypercube of dimension ≥ 4, it follows that $n = 6$ and S_0 is the union of antipodal cubes, say

$$S \cap \{x : x_6 = 0\} = \{x : x_4 = x_5, x_6 = 0\},$$

and so

$$S \cap \{x : x_6 = 1\} = \{x : x_4 + x_5 = 1, x_6 = 1\}$$

as $|S_1| = 2^{n-2} = 16$. But then S has an $R_{1,1}$ restriction, a contradiction to our assumption. This completes the induction step. □

Recall the sets $R_{2,1}, R_5$ displayed in Figure 2. We need the following last ingredient:

Theorem 2.6 ([4]). *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only strictly non-polar sets where every infeasible point has at most two infeasible neighbors.*

We are now ready to prove Theorem 1.7, stating that up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only 2-resistant strictly non-polar sets:

Proof of Theorem 1.7. We know that $R_{1,1}, R_{2,1}, R_5$ are strictly non-polar sets, and since their infeasible components have maximum degree at most two, they are 2-resistant by Theorem 1.4 (iii). To prove that they are, up to isomorphism, the only 2-resistant strictly non-polar sets, pick an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, R_{2,1}, R_5$ restriction. It suffices to show that S is polar. By Theorem 1.4 (iii), every infeasible component is a hypercube or has maximum degree at most two. If S has maximum degree at most two, then by Theorem 2.6, S is polar. Otherwise, S has an infeasible hypercube of dimension at least 3. If $n = 4$ or $S = \emptyset$, then S is clearly polar. Otherwise, $n \geq 5$ and $S \neq \emptyset$. By Lemma 2.5, $|S| \geq 2^{n-1}$; if equality holds, then S is either a hypercube or the union of antipodal hypercubes, so S is clearly polar. Otherwise, $|S| > 2^{n-1}$, implying in particular that there are antipodal feasible points, so S is polar, as required. □

3 A co-NP characterization of ± 1-resistant sets

Let us start with the following obvious remark:

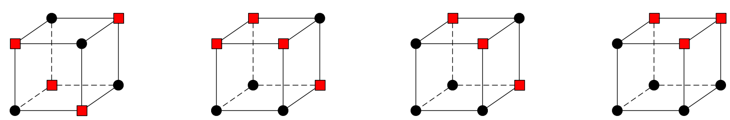
Remark 3.1. *If a set is ± 1-resistant, then so is every restriction of it.*

The class of ± 1-resistant sets turns out to be closed under projections as well, but the reason is not as straightforward as it was for 2-resistant sets. In fact, there is a subtle difference between ± 1- and 2-resistance, which becomes manifest by the following example, showing that there is no ± 1-resistant analogue of Corollary 2.3:

Example. *Let $S := \{111011, 000001, 001110, 110100\} \subseteq \{0,1\}^6$. The feasible points are at pairwise distance 4, so for every subset $X \subseteq \{0,1\}^6$, $S \triangle X$ has no P_3, S_3 restriction. However, as $S - \{110100\}$ has a P_3 projection, S is not ± 1-resistant.*

Nevertheless, in this section, we find the excluded minors and restrictions defining ± 1-resistance. We will then see two applications, that ± 1-resistance is a minor-closed property, and that it can be tested efficiently.

To start, consider the 3-dimensional sets displayed below,



$R_{1,1}$ F_1 F_2 F_3

written as

$$\begin{aligned} R_{1,1} &= \{000, 110, 101, 011\} \\ F_1 &= \{000, 100, 010, 111\} \\ F_2 &= \{000, 100, 010, 001, 111\} \\ F_3 &= \{000, 100, 010, 001, 110\}. \end{aligned}$$

Moreover, for each $k \geq 4$, let $F_k := \{\mathbf{0}, e_1, e_2, e_1 + e_2, \mathbf{1} - e_1 - e_2\} \subseteq \{0,1\}^k$, which has an F_3 projection obtained after projecting away coordinates $4, \ldots, n$.

Remark 3.2. *The sets $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ± 1-resistant.*

Proof. Notice that $R_{1,1} - \{000\} = P_3$, $F_1 - \{000\} \cong P_3$, $F_2 - \{111\} \cong S_3$, and that for each $k \geq 3$, $F_k - \{e_1 + e_2\}$ has an S_3 projection obtained after projecting away coordinates $[k] - \{1, 2, 3\}$. As a result, $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ± 1-resistant. $\square$

We are now ready to prove the following:

Theorem 3.3. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is ± 1-resistant,

(ii) S has no $\{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}$ restriction,

(iii) S has no $\{R_{1,1}, F_1, F_2, F_3\}$ minor.

Proof. It follows from Remark 1.9 that every minor of S is 1-resistant. We will use this throughout the proof. **(i)** $\Rightarrow$ **(ii)**: It follows from Remarks 3.1 and 3.2 that S has no $\{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}$ restriction. **(ii)** $\Rightarrow$ **(iii)**: We will need the following three claims:

Claim 1. *Let $R \subseteq \{0,1\}^4$ be 1-resistant. Let $N \subseteq \{0,1\}^3$ be the projection of R over coordinate 4. Then the following statements hold:*

(1) if $N = R_{1,1}$, then R has an $R_{1,1}$ restriction,

(2) if $N = F_1$, then R has an F_1 restriction,

(3) if $N = F_2$, then R has one of F_1, F_2 as a restriction.

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the i-restriction of R over coordinate 4. Notice that $R_0 \cup R_1 = N$. **(1)** Since R_0 and R_1 are 1-resistant, it follows that $|R_0| \in \{0,1,4\}$ and $|R_1| \in \{0,1,4\}$. Since $|R_0| + |R_1| \geq 4$, it follows that one of R_0, R_1 is $R_{1,1}$, so R has an $R_{1,1}$ restriction. **(2)** We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since the 0-restriction of R over coordinate 1 is 1-resistant, it follows that $010 \in R_0$. Similarly, as the 0-restriction of R over coordinate 2 is 1-resistant, $100 \in R_0$. Because R_0 is 1-resistant, we have that $R_0 = F_1$, so R has an F_1 restriction. **(3)** We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since R has no P_3, S_3 restriction, at least two of $100, 010, 001$ must belong to R_0. Without loss of generality, $100, 010 \in R_0$. As R_0 is 1-resistant, $111 \in R_0$, so R_0 is either F_1 or F_2, implying in turn that R has one F_1, F_2 as a restriction. $\Diamond$

Claim 2. *Let $R \subseteq \{0,1\}^4$ be 1-resistant and have no F_1, F_3 restriction. If the projection of R over coordinate 4 is F_3, then $R \cong F_4$.*

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the i-restriction of R over coordinate 4. Notice that $R_0 \cup R_1 = F_3$. Assume in the first case that $110 \in R_0 \cap R_1$. Since $100 \in R_0 \cup R_1$ and the 1-restriction of R over coordinate 1 is 1-resistant, it follows that $100 \in R_0 \cap R_1$. Similarly, $010 \in R_0 \cap R_1$. After possibly twisting coordinate 4 of R, we may assume that $001 \in R_0$. This implies that R_0 is isomorphic to either F_1 or F_3, which is not the case as R has no F_1, F_3 restriction. Assume in the remaining case that $110 \notin R_0 \cap R_1$. After possibly twisting coordinate 4 of R, we may assume that $110 \in R_0$ and $110 \notin R_1$. As $100 \in R_0 \cup R_1$ and the 1-restriction of R over coordinate 1 is 1-resistant, it follows that $100 \in R_0$ and $100 \notin R_1$. Similarly, $010 \in R_0$ and $010 \notin R_1$. Since $R_0 \not\cong F_1, F_3$, it follows that $001 \notin R_0$ and so $001 \in R_1$. As R_0 is 1-resistant, $000 \in R_0$. Since R has no F_3 restriction, it follows that $000 \notin R_1$, implying in turn that $R \cong F_4$, as required. $\Diamond$

Claim 3. *Take an integer $k \geq 4$ and a 1-resistant set $R \subseteq \{0,1\}^{k+1}$ that has no F_3, F_k restriction. If the projection of R over coordinate $k+1$ is F_k, then $R \cong F_{k+1}$.*

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^k$ be the i-restriction of R over coordinate $k+1$. Then $R_0 \cup R_1 = F_k$. For $i \in \{0,1\}$, since R_i is 1-resistant, it follows that $|R_i \cap \{\mathbf{0}, e_1, e_2, e_1 + e_2\}| \neq 3$, and if

$|R_i \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 2$ then the two points in $R_i \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}$ are adjacent. Since the restriction of R obtained after 0-restricting coordinates $3, \dots, k$ is not isomorphic to F_3, one of the following holds:

- $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 2$: in this case, the 0-restriction of R over coordinates $[k+1] - \{1, 2, 3, k+1\}$ is not 1-resistant,
- $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 4$: in this case, one of R_0, R_1 is F_k,
- one of $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|, |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|$ is 2 and the other one is 4: in this case, the 0-restriction of R over coordinates $[k+1] - \{1, 2, 3, k+1\}$ is not 1-resistant,
- one of $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|, |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|$ is 0 and the other one is 4.

Thus, the last case is the only possibility. In this case, since R has no F_k restriction, it follows that $R \cong F_{k+1}$, as required. $\Diamond$

Assume that S has an $N \in \{R_{1,1}, F_1, F_2, F_3\}$ minor, obtained after applying ℓ single projections and $n-3-\ell$ single restrictions, for some $\ell \in \{0, \dots, n-3\}$. We need to show that S has one of $R_{1,1}, \{F_k : 1 \leq k \leq n\}$ as a restriction. A repeated application of Claim 1 implies that if $N \in \{R_{1,1}, F_1, F_2\}$, then S has one of $\{R_{1,1}, F_1, F_2\}$ as a restriction. We may therefore assume that $N = F_3$, and that S has no $\{R_{1,1}, F_1, F_2\}$ restriction. If $\ell = 0$, then S has an F_3 restriction, so we are done. We may therefore assume that $\ell \geq 1$ and S has no F_3 restriction. If $\ell = 1$, then by Claim 2, S has an F_4 restriction and we are done. We may therefore assume that $\ell \geq 2$ and S has no F_3, F_4 restriction. By repeatedly applying Claim 3, we see that S has one of $F_5, \dots, F_n$ as a restriction, as required.

(iii) $\Rightarrow$ **(i)**: Assume that S is not ± 1-resistant. Since S is 1-resistant, there exists an $x \in S$ such that $S - \{x\}$ has an $N \in \{P_3, S_3\}$ minor. Thus, for some point $y \in \{0,1\}^3$, S has an $N \cup \{y\}$ minor. Since $N \cup \{y\}$ is 1-resistant, it must be isomorphic to one of $R_{1,1}, F_1, F_2, F_3$. Thus, S has one of $\{R_{1,1}, F_1, F_2, F_3\}$ as a minor. $\square$

Thus,

Corollary 3.4. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) *S is ± 1-resistant,*

(ii) *S has none of the following restrictions:*

$$\{F : F \text{ is fragile}\} \cup \left\{\left\{\mathbf{0}^k, \mathbf{1}^k - e_1\right\} : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\},$$

(iii) *S has none of the following minors:*

$$\{F : F \text{ is fragile}\} \cup \{R_{1,1}, F_1, F_2, F_3\}.$$

In particular, ± 1-resistance is a minor-closed property.

Proof. This is an immediate consequence of Theorems 3.3 and 1.10. $\square$

Corollary 3.5. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^4|S|^3)$, one can test whether or not S is ± 1-resistant.*

Proof. We will appeal to Theorem 3.3 (ii). By Theorem 1.11, we can test in time $O(n^4|S|^3)$ whether or not S is 1-resistant. Finding an $R_{1,1}$ restriction can be found as follows:

1. For every pair of points x, y of S at distance 2:
   (a) let $I := \{i \in [n] : x_i = y_i\}$,
   (b) for every coordinate $i \in I$,
      i. let $S' \subseteq \{0,1\}^3$ be the restriction of S over coordinates $I - \{i\}$ containing (the images of) x and y,
      ii. if $S' \cong R_{1,1}$, then output "S has an $R_{1,1}$ restriction",
   (c) if (ii) fails for every $i \in I$, then change the pair x, y.
2. If (ii) fails for every pair x, y, then output "S has no $R_{1,1}$ restriction".

The correctness of this algorithm is clear, and its running time is $n\binom{|S|}{2} \times (n-2) \times n|S|$. Finding an $\{F_k : 1 \leq k \leq n\}$ can be found as follows:

1. For $k \in \{3, 4, \ldots, n\}$:
   (a) for every pair of points x, y of S at distance k,
      i. let $S' \subseteq \{0,1\}^k$ be the smallest restriction of S containing x and y,
      ii. if $k = 3$ and $S' \cong F_1, F_2, F_3$, then output "S has an F_1, F_2, F_3 restriction",
      iii. if $k \geq 4$ and $S' \cong F_k$, then output "S has an F_k restriction",
   (b) if (ii)-(iii) fail for every pair x, y and $k < n$, then increment k.
2. If (ii)-(iii) fail for every pair x, y and $k = n$, then output "S has no $\{F_k : 1 \leq k \leq n\}$ restriction".

The correctness of this algorithm follows from the fact that each one of $\{F_k : 1 \leq k \leq n\}$ has antipodal feasible points; its running time is $\sum_{k=3}^{n} n\binom{|S|}{2} \times n|S| = n\binom{|S|}{2} \times n|S| \times (n-2)$. Thus, by Theorem 3.3 (ii), testing whether or not S is ± 1-resistant can be done in time $O(n^4|S|^3) + n\binom{|S|}{2} \times (n-2) \times n|S| + n\binom{|S|}{2} \times n|S| \times (n-2) = O(n^4|S|^3)$, as required. $\square$

4 An outline of the proof of Theorem 1.5

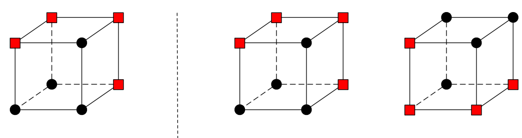
Theorem 1.5 on the structure of ± 1-resistant sets is a consequence of three results, which we summarize here. Assuming the correctness of these results, we then prove Theorem 1.5.

For an integer $k \geq 2$ recall that $A_k = \{\mathbf{0}, \mathbf{1}\} \subseteq \{0,1\}^k$, and for an integer $k \geq 3$ recall that $B_k = \{\mathbf{0}, e_1, \mathbf{1}\} \subseteq \{0,1\}^k$.

Theorem 4.1. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected, then either*

- *$S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \dots, n\}$,*
- *$S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$, or*
- *S has a D_3 minor.*

Here, $D_3 = \{000, 100, 010, 101\} \subseteq \{0,1\}^3$. Recall that $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\} \subseteq \{0,1\}^4$.

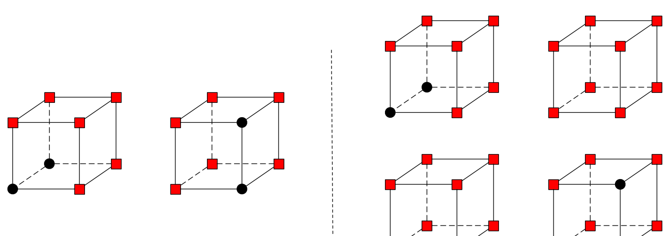


D_3 C_8

Theorem 4.2. *Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S has a D_3 minor, then either*

- *$S \cong C_8 \times \{0,1\}^{n-4}$, or*
- *$S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$.*

Here, for each integer $k \geq 4$, $D_k = \{\mathbf{0}, e_2, \mathbf{1}-e_2, \mathbf{1}-e_2-e_3\} \subseteq \{0,1\}^k$. See the figure below for an illustration of D_4 and D_5:



D_4 D_5

Theorem 4.3. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is connected and has no D_3 minor, then either*

- *S is a hypercube, or*
- *every infeasible component of S is a hypercube.*

As a consequence of these three results, let us prove Theorem 1.5, stating the following:

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is ± 1-resistant if, and only if, one of the following statements holds:

(i) $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2,\ldots,n\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3,\ldots,n\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3,\ldots,n\}$,

(v) S is a hypercube, or

(vi) every infeasible component of S is a hypercube, and every feasible point has at most two infeasible neighbors.

Proof of Theorem 1.5, assuming Theorems 4.1, 4.2 and 4.3. ($\Rightarrow$) Clearly, S is 1-resistant, so by Theorem 3.3 (iii), S has no $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected and has no D_3 minor, then (i) or (ii) holds by Theorem 4.1. If S has a D_3 minor, then (iii) or (iv) holds by Theorem 4.2. Otherwise, S is connected and has no D_3 minor. If (v) holds, then we are done. Otherwise, by Theorem 4.3, then every infeasible component of S is a hypercube. We claim that (vi) holds. Suppose otherwise. Then there is a feasible point x with three infeasible neighbors $x\triangle e_i, x\triangle e_j, x\triangle e_k$, for distinct $i,j,k \in [n]$. Since every infeasible component is a hypercube, it follows that $x\triangle e_i\triangle e_j, x\triangle e_j\triangle e_k, x\triangle e_k\triangle e_i$ are feasible. But then the 3-dimensional restriction of S containing $x\triangle e_i, x\triangle e_j, x\triangle e_k$ is isomorphic to either $R_{1,1}$ or F_2, a contradiction. Hence, (vi) holds, as required.

($\Leftarrow$) We will need the following claim:

Claim. *If S is ± 1-resistant, then so is $S \times \{0,1\}$.*

Proof of Claim. By Corollary 3.4 (ii), the excluded restrictions defining ± 1-resistance are

$$\{F : F \text{ is fragile}\} \cup \left\{\{\mathbf{0}^k, \mathbf{1}^k - e_1\} : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.$$

In particular, every excluded restriction of ± 1-resistance is not isomorphic to $F \times \{0,1\}$ for any set F. This proves the claim. $\diamondsuit$

It can be readily checked that the sets $\{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\}$ and C_8 are ± 1-resistant. Thus, after repeatedly applying the claim above, we see that the four classes (i)-(iv) are ± 1-resistant. It can also be readily checked that (v) is a ± 1-resistant class.

It remains to show that the restriction-closed class (vi) is ± 1-resistant. To this end, pick a set S from (vi). Suppose for a contradiction that S is not ± 1-resistant. By Corollary 3.4 (ii), S has one of the following restrictions:

$$\{F : F \text{ is fragile}\} \cup \left\{\{\mathbf{0}^k, \mathbf{1}^k - e_1\} : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.$$

Out of these sets, $R_{1,1}$ is the only set whose infeasible components are hypercubes. Thus, S has an $R_{1,1}$ restriction. However, $R_{1,1}$ has a feasible point with three infeasible neighbors, implying in turn that S has a feasible point with three infeasible neighbors, a contradiction. $\square$

It remains to prove Theorems 4.1, 4.2 and 4.3; they are proved in §6, §7 and §8.1, respectively.

5 Bridges

Take an integer $n \geq 2$. For a point $x \in \{0,1\}^n$ and distinct coordinates $i, j \in [n]$ such that $x_i = x_j = 0$, we refer to $\{x, x+e_i, x+e_j, x+e_i+e_j\}$ as a *square* that *initiates at* x and is *active in directions* e_i, e_j. Two squares are *parallel* if they are active in the same pair of directions. Two parallel squares are *neighbors* if the points they initiate from are neighbors.

Take a set $S \subseteq \{0,1\}^n$. A *bridge* is a square that contains feasible points from different feasible components. Notice that a bridge contains exactly two feasible points, which are non-adjacent and belong to different feasible components. In this section, we will prove the following statement:

> Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.

We will need three lemmas to prove this statement.

Lemma 5.1. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$, where direction e_n is not active in any bridge. If S' is obtained from S after projecting away coordinate n, then the feasible components of S project onto different feasible components of S'.*

Proof. For a point $x \in \{0,1\}^n$, denote by $x' \subseteq \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. To prove the lemma, it suffices to show that if K is a feasible component of S and $x \in S - K$, then $\operatorname{dist}(x', y') \geq 2$ for all $y \in K$. Well, since x does not belong to the component K, $\operatorname{dist}(x, y) \geq 2$ for all $y \in K$, implying in turn that

$$\operatorname{dist}(x', y') \geq \operatorname{dist}(x, y) - 1 \geq 1 \quad \forall\, y \in K.$$

In particular, $x' \notin \{y' : y \in K\}$. Suppose for a contradiction that $\operatorname{dist}(x', y') = 1$ for some $y \in K$. As the inequalities above are held at equality, there must be a coordinate $i \in [n-1]$ such that $y = x \triangle e_i \triangle e_n$. But then $\{x, x\triangle e_i, x\triangle e_n, x\triangle e_i\triangle e_n\}$ would be a bridge that is active in direction e_n, contrary to our assumption. Hence,

$$\operatorname{dist}(x', y') \geq 2 \quad \forall y \in K,$$

as required. $\square$

Lemma 5.2. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2$ restriction. Take a point $x \in \{0,1\}^n$ and distinct coordinates $i, j, k \in [n]$. Then the following statements hold:*

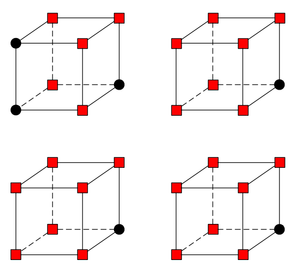
(i) If $x\triangle e_i, x\triangle e_j, x\triangle e_k \in \overline{S}$, then $\left|\{x\triangle e_i\triangle e_j, x\triangle e_j\triangle e_k, x\triangle e_k\triangle e_i\} \cap S\right| \leq 1$.

(ii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\{x\triangle e_i\triangle e_k, x\triangle e_j\triangle e_k\} \cap S = \emptyset$.

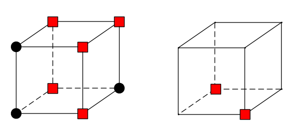
(iii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\left|\{x\triangle e_k, x\triangle e_i\triangle e_j\triangle e_k\} \cap S\right| \geq 1$.

Proof. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and $i = 1, j = 2, k = 3$. Let $S' \subseteq \{0,1\}^3$ be the restriction of S obtained after 0-restricting coordinates $4, \dots, n$. **(i)**: Suppose that $e_1, e_2, e_3 \in \overline{S}$. Assume for a contradiction that two of $e_1 + e_2, e_2 + e_3, e_3 + e_1$, say $e_1 + e_2, e_2 + e_3$ belong to S. If $e_1 + e_3 \in S$, then S' is isomorphic to one of $P_3, S_3, R_{1,1}, F_2$, which cannot occur as S is 1-resistant and has no $R_{1,1}, F_2$ restriction. Otherwise, $e_1 + e_3 \in \overline{S}$. Since $S' \ncong P_3$ and S is 1-resistant, it follows that $\mathbf{0}, e_1 + e_2 + e_3 \in S$, implying in turn that $S' \cong F_1$, a contradiction as S has no F_1 restriction. **(ii)**, **(iii)**: Suppose that $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1 + e_2\}$ is a bridge. Then $e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. Let us first prove (ii), that $\{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset$. Suppose otherwise. After possibly relabeling coordinates $1, 2$, we may assume that $e_1 + e_3 \in S$. Since $\mathbf{0}, e_1 + e_2$ are in different feasible components, it follows that $|\{e_3, e_1 + e_2 + e_3\}| \cap S| \leq 1$. After possibly twisting coordinates $1, 2$, we may assume that $e_3 \in \overline{S}$. Since $e_1, e_2, e_3 \in \overline{S}$, we get from (i) that $|\{e_1 + e_2, e_2 + e_3, e_3 + e_1\} \cap S| \leq 1$, a contradiction. Thus, $\{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset$, so (ii) holds. Since S is 1-resistant, it follows immediately that $\{e_3, e_1 + e_2 + e_3\} \cap S \neq \emptyset$, so (iii) holds. $\square$

Lemma 5.3. *Take a set $S \subseteq \{0,1\}^5$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor, and in every minor, including S itself, every pair of bridges are parallel. If $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1 + e_2\}$ is a bridge without neighboring bridges, then after possibly twisting coordinates 1 and 2, we have that $S = \{\mathbf{0}, e_3, e_1 + e_2, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_2 + e_4 + e_5\}$:*



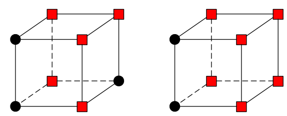
Proof. Let $B := \{\mathbf{0}, e_1, e_2, e_1 + e_2\}$. As B is a bridge and $\mathbf{0} \in S$, $e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. It follows from Lemma 5.2 (ii) that $e_1 + e_3, e_2 + e_3 \in \overline{S}$. By Lemma 5.2 (iii) and the fact that B has no neighboring bridge, we get that exactly one of $e_3, e_1 + e_2 + e_3$ belongs to S. After twisting coordinates 1 and 2, if necessary, we may assume that $e_3 \in S$ and $e_1 + e_2 + e_3 \in \overline{S}$. Moreover, by Lemma 5.2 (ii), we have that $\{e_1 + e_4, e_2 + e_4\} \subseteq \overline{S}$. Let S' be the 0-restriction of S over coordinate 5, which looks as follows:



Claim 1. *$e_4 \in \overline{S}$ and $e_1 + e_2 + e_4 \in S$.*

Proof of Claim. Suppose otherwise. Since B has no neighboring bridge in S, it follows from Lemma 5.2 (iii) that $e_4 \in S$ and $e_1 + e_2 + e_4 \in \overline{S}$. If $e_2 + e_3 + e_4 \in S$, then the 0-restriction of S' over coordinate 1 is either F_1 or F_3, which is not the case. Thus, $e_2 + e_3 + e_4 \in \overline{S}$. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in S$. As the 0-restriction of S' over coordinate 2 is not F_3, we have $e_1 + e_3 + e_4 \in \overline{S}$.

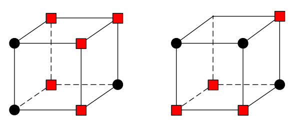
Since the 1-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:



Observe however now that F_3 is obtained from S' after projecting away coordinate 1, a contradiction. $\Diamond$

Claim 2. $\{e_1 + e_3 + e_4, e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Suppose otherwise. After interchanging the roles of $1, 2$, if necessary, we may assume that $e_1 + e_3 + e_4 \in S$. If $e_3 + e_4 \in \overline{S}$, then $\{\mathbf{0}, e_3\}$ is a feasible component of S', so the square initiating from e_3 and active in directions e_1, e_4 is a bridge of S' that is not parallel to B, which is contrary to our assumption. Thus, $e_3 + e_4 \in S$. Since $\mathbf{0}, e_1 + e_2$ belong to different feasible components of S, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:

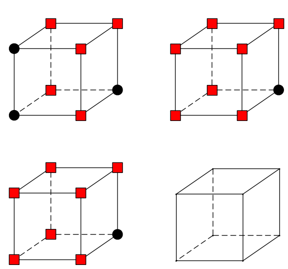
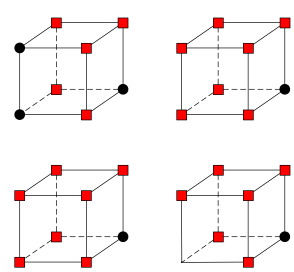


Observe however that S' has two non-parallel bridges, namely B and the square that initiates from $e_1 + e_4$ and is active in directions e_2, e_3, a contradiction. $\Diamond$

Claim 3. $\{e_3 + e_4, e_1 + e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in \overline{S}$. Since the 1-restriction of S' over coordinate 1 is also 1-resistant, we see that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, as required. $\Diamond$

We just determined the status of all the points in $\{x : x_5 = 0\}$. A similar argument applied to $\{x : x_4 = 0\}$ gives us the left figure below:

Consider the set obtained from S after 1-restricting over coordinate 1 and 0-restricting over coordinate 3; since this set is 1-resistant and not isomorphic to F_1, F_3, we get that $e_1 + e_4 + e_5 \in \overline{S}$ and $e_1 + e_2 + e_4 + e_5 \in S$. As the 1-restriction of S over coordinates $1, 2$ is not F_3, we get that $\mathbf{1} \in \overline{S}$. Now consider the set obtained from S after 1-restricting coordinate 2 and 0-restricting over coordinate 3; since this set is not F_3, we get that $e_2 + e_4 + e_5 \in \overline{S}$. Note that $\{e_1 + e_2, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_2 + e_4 + e_5\}$ forms a feasible component of S. Hence, as S does not have non-parallel bridges, it follows that $e_2+e_3+e_4+e_5, e_1+e_3+e_4+e_5 \in \overline{S}$, and also that $e_3+e_4+e_5 \in \overline{S}$. (See the right figure above.) Once again, as S does not have non-parallel bridges, it follows that $e_4+e_5 \in \overline{S}$, thereby finishing the proof. $\square$

We are now ready to prove the main result of this section:

Proposition 5.4. *Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.*

Proof. Suppose for a contradiction that S has a pair of non-parallel bridges. (In particular, S is not connected.) We may assume that in every proper minor of S, every pair of bridges, if any, are parallel.

Claim 1. *Every direction is active in a bridge.*

Proof of Claim. Suppose for a contradiction that direction e_n is not active in any bridge. For a point $x \in \{0,1\}^n$, denote by $x' \subseteq \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. Notice first that by Lemma 5.1, the feasible components of S project onto different feasible components of S', the subset of $\{0,1\}^{n-1}$ obtained from S after projecting away coordinate n. We will derive a contradiction to the minimality of S by showing that S' has non-parallel bridges.

We will show that if B is a bridge of S, then $B' := \{x' : x \in B\}$ is still a bridge of S' that is active in the same directions as before. Since e_n is not active in any bridge of S, we may assume that $n \geq 3$ and $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$ where $\mathbf{0}, e_1+e_2$ belong to different feasible components of S, and $e_1, e_2 \in \overline{S}$. It follows from Lemma 5.2 (ii) that $\mathbf{0}, e_1+e_2 \in S'$ and $e_1, e_2 \in \overline{S'}$. Moreover, since the feasible components of S project onto different feasible components of S', we see that $\mathbf{0}, e_1+e_2$ belong to different feasible components of S'. Thus, B' is still a bridge of S' that is active in the same directions as before.

As a corollary, S' still has non-parallel bridges, thereby contradicting the minimality of S. $\diamondsuit$

Claim 2. *The following statements hold:*

(i) *if B, B' are non-parallel bridges that are not active in direction e_i, then $\{x : x_i = 0\}$ contains one of the bridges and $\{x : x_i = 1\}$ contains the other one,*

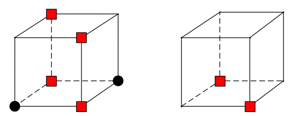
(ii) *if B, B', B'' are pairwise non-parallel bridges, then every direction is active in one of the bridges, and*

(iii) $n \in \{4,5,6\}$.

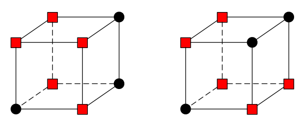
Proof of Claim. **(i)** For if not, then one of the restrictions of S over coordinate i contains B and B', thereby contradicting the minimality of S. **(ii)** Suppose for a contradiction that e_i is not active in any of B, B', B''. Then one of the hyperplanes $\{x : x_i = 0\}, \{x : x_i = 1\}$ contains at least two of B, B', B'', thereby contradicting (i). **(iii)** Let B, B' be non-parallel bridges. It follows from Lemma 5.2 (ii) that $n \geq 4$. If every direction is active in one of B, B', we get that $n = 4$. Otherwise, there is a direction e_i inactive in both B, B'. By Claim 1, there is a bridge B'' active in e_i. Clearly, B, B', B'' are pairwise non-parallel bridges. It now follows from (ii) that $n \leq 6$, as required. $\diamondsuit$

Claim 3. $n \neq 4$.

Proof of Claim. Suppose for a contradiction that $n = 4$. Let B, B' be non-parallel bridges of S. We may assume that $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$, $\mathbf{0}, e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. By Lemma 5.2 (ii), $e_1+e_3, e_2+e_3, e_1+e_4, e_2+e_4 \in \overline{S}$:

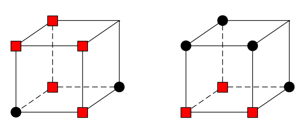


Assume in the first case that B' shares an active direction with B. After possibly relabeling coordinates $1, 2$, we may assume that B' is active in directions e_1, e_3. It follows from Claim 2 (i) that B' is contained in $\{x : x_4 = 1\}$. After possibly twisting coordinates $1, 2$, we may assume that $B' = \{e_4, e_1+e_4, e_3+e_4, e_1+e_3+e_4\}$. Since $e_1 + e_4 \in \overline{S}$, it follows that $e_4, e_1 + e_3 + e_4 \in S$ and $e_3 + e_4 \in \overline{S}$. Applying Lemma 5.2 (ii), we get that $e_3, e_2 + e_3 + e_4, e_1 + e_2 + e_4 \in \overline{S}$. Since the two restrictions of S over coordinate 4 are 1-resistant, it follows that $e_1 + e_2 + e_3, \mathbf{1} \in S$:



Observe, however, that 1-restricting S over coordinate 3 yields a set that is not 1-resistant, a contradiction.

Assume in the remaining case that B' is active in directions e_3, e_4. Observe that B' is not contained in $\{x : x_1 + x_2 = 1\}$. After possibly twisting coordinates $1, 2$, we may assume that B' initiates from $\mathbf{0}$. This means that $e_3, e_4 \in \overline{S}$ and $e_3 + e_4 \in S$. Applying Lemma 5.2 (iii), we get that $e_1 + e_2 + e_4 \in S$ and $e_1 + e_3 + e_4, e_2 + e_3 + e_4 \in S$:



The 1-restriction of S over coordinate 4, however, is isomorphic to either F_1 or F_3, a contradiction. $\Diamond$

Thus, we have that $n \in \{5, 6\}$. It follows from Claim 1 that there are $\lceil \frac{n}{2} \rceil = 3$ pairwise non-parallel bridges B_1, B_2, B_3. We get from Claim 2 (ii) that, after a possible relabeling, B_1 is active in e_1, e_2, B_2 is active in e_3, e_4, and

- if $n = 5$, then B_3 is active in e_3, e_5,
- if $n = 6$, then B_3 is active in e_5, e_6.

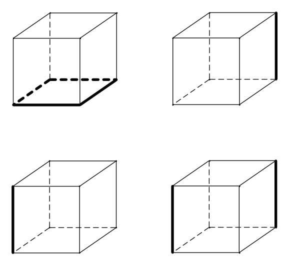
We can further say that,

Claim 4. *If B is a bridge different from B_1, B_2, B_3, then $n = 5$.*

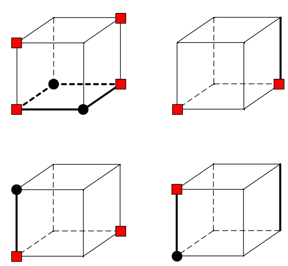
Proof of Claim. Suppose for a contradiction that $n = 6$. It follows from Claim 2 (ii) that B is parallel to one of B_1, B_2, B_3. Consider the bridge B_2. Since B_2, B_3 are inactive in e_1, e_2, it follows from Claim 2 (i) that the hyperplanes $\{x : x_1 = 0\}, \{x : x_2 = 0\}$ split B_2, B_3. Moreover, since B_2, B_1 are inactive in e_5, the hyperplane $\{x : x_5 = 0\}$ splits B_2, B_1. Hence, the residing square of B_2 – and any bridge parallel to it – is uniquely determined once B_1 and B_3 are given, implying that B is not parallel to B_2. By the symmetry between B_1, B_2, B_3, we get that B is not parallel to B_1, B_3 either, a contradiction. $\Diamond$

Claim 5. $n \neq 5$.

Proof of Claim. Suppose for a contradiction that $n = 5$. After twisting coordinates $3, 4, 5$, if necessary, we may assume that B_1 initiates at $\mathbf{0}$. By Claim 2 (i), and after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates at e_5. Another application of Claim 2 (i) tells us that B_3 initiates at $e_1 + e_2 + e_4$:

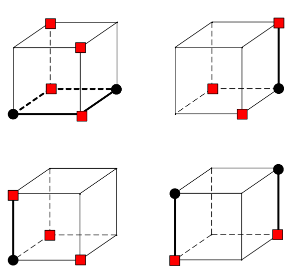
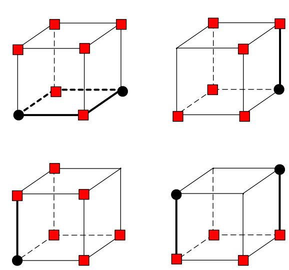


Assume in the first case that $\mathbf{0}, e_1 + e_2 \in \overline{S}$ and $e_1, e_2 \in S$. Then a repeated application of Lemma 5.2 (ii) tells us that $e_3, e_1 + e_2 + e_3, e_5, e_1 + e_2 + e_5, e_4, e_1 + e_2 + e_4 \in \overline{S}$. As a result, in the bridge B_2, we have that $e_3 + e_5, e_4 + e_5 \in S$:



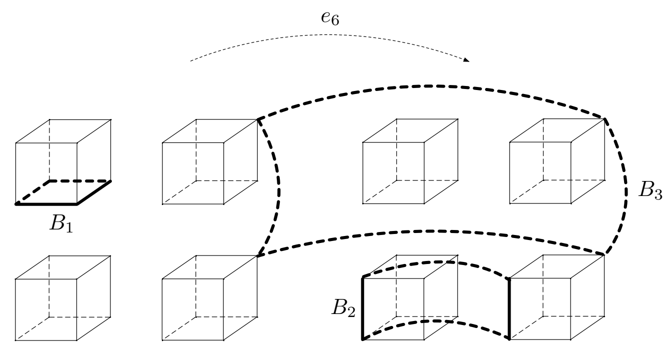
Observe now that the restriction of S obtained after 0-restricting coordinates 1 and 2 is not 1-resistant, a contradiction.

Assume in the remaining case that $\mathbf{0}, e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. A repeated application of Lemma 5.2 (ii) to B_1, followed by an application of it to B_2, B_3 gives us the left figure below:

Applying Lemma 5.2 (ii) to B_2, B_3 gives us the right figure above, thereby yielding a contradiction as 0-restricting coordinates $4, 5$ of S yields a set that is not 1-resistant. This finishes the proof of the claim. $\Diamond$

Thus, $n = 6$. After twisting coordinates $3, 4, 5, 6$, if necessary, we may assume that B_1 initiates at $\mathbf{0}$. Applying Claim 2 (i), we see that after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates at $e_5 + e_6$. Using Claim 2 (i), we see that B_3 must initiate at $e_1 + e_2 + e_3 + e_4$:

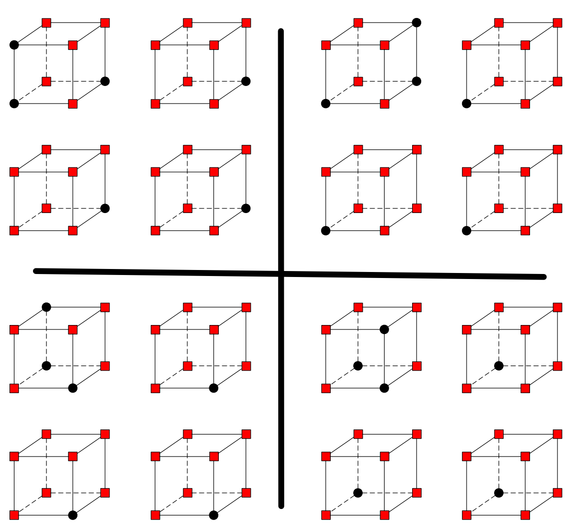


Recall from Claim 4 that B_1, B_2, B_3 are the only bridges of S. Let $S' \subseteq \{0, 1\}^5$ be the restriction of S obtained after 0-restricting coordinate 6. By assumption, every minor of S' has only parallel bridges. As a bridge in S' is not necessarily a bridge in S, we see that S' may have bridges other than B_1 (that will necessarily be parallel to it).

Claim 6. *B_1 does not have a neighboring bridge in S'.*

Proof of Claim. Suppose for a contradiction that B_1 has a neighboring bridge B in S'. Since B is not a bridge of S by Claim 4, it follows that the points in $B \cap S'$ are in the same feasible component of S. After applying Lemma 5.2 (ii) to B_1, we see that the points in $B_1 \cap S'$ also lie in this feasible component of S, a contradiction. $\Diamond$

We may now apply Lemma 5.3 to the bridge B_1 of S'. Depending on which points of B_1 are in S', and how coordinates $1, 2$ are twisted, we get that S' takes on one of the four possibilities shown below:



Consider the 3-dimensional restriction F of S containing B_2 and $B_2 \triangle e_6$. If S' takes one of the top-left, bottom-left or bottom-right possibilities, then F is not 1-resistant, which is not possible. Otherwise, S' takes the top-right possibility, in which case $F \cong F_1$, a contradiction. This finally finishes the proof of Proposition 5.4. $\square$

6 Proof of Theorem 4.1

Take an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$. We say that S is *separable* if there exist a partition of S into nonempty parts S_1, S_2 and distinct coordinates $i, j \in [n]$ such that either $S_1 \subseteq \{x : x_i = 0, x_j = 1\}$ and $S_2 \subseteq \{x : x_i = 1, x_j = 0\}$, or $S_1 \subseteq \{x : x_i = x_j = 0\}$ and $S_2 \subseteq \{x : x_i = x_j = 1\}$. Notice that if S is separable, then it is not connected.

Remark 6.1. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If a projection of S is separable, then so is S.*

We will need the following:

Proposition 6.2. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Suppose there is a partition of S into nonempty parts S_1, S_2 such that $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. Then S_1 and S_2 are hypercubes. In particular, S is polar.*

Proof. The hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is infeasible. As S is 1-resistant, Lemma 1.12 implies that in each of the parallel hypercubes $\{x : x_{n-1} = x_n = 0\}$ and $\{x : x_{n-1} = x_n = 1\}$, the feasible points form a hypercube. That is, the two sets

$$S \cap \{x : x_{n-1} = x_n = 0\} = S_1,$$
$$S \cap \{x : x_{n-1} = x_n = 1\} = S_2$$

are hypercubes. We leave it as an easy exercise for the reader to check that S is polar. $\square$

We are now ready to prove Theorem 4.1, stating the following:

> Take an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and is not connected. Then either
>
> - $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \dots, n\}$, where $A_k = \{\mathbf{0}^k, \mathbf{1}^k\}$,
> - $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$, where $B_k = \{\mathbf{0}^k, e_1, \mathbf{1}^k\}$, or
> - S has a D_3 minor, where $D_3 = \{000, 010, 100, 101\}$.

Proof. Let us start with the following claim:

Claim 1. *S is separable.*

Proof of Claim. Let $k \geq 2$ be the number of feasible components of S. Let $S' \subseteq \{0,1\}^m$ be a projection of S of smallest dimension with exactly k feasible components. It then follows from Lemma 5.1 that every direction of $\{0,1\}^m$ is active in a bridge of S'. However, as S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, Proposition 5.4 implies that every pair of bridges of S' are parallel. As a result, $m = k = 2$ and S' is either $\{00, 11\}$ or $\{10, 01\}$. In particular, S' is separable, so S is separable by Remark 6.1. $\Diamond$

Thus, there is a partition of S into nonempty parts S_1, S_2 such that, after a possible twisting and relabeling, $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. As S is 1-resistant, Proposition 6.2 implies that S_1 and S_2 are hypercubes, and that S is polar. In particular, since S is not a hypercube, Lemma 1.12 implies that the points in S do not agree on a coordinate; notice that this property is preserved in every projection of dimension at least one.

Claim 2. *Either S has a D_3 minor, or one of S_1, S_2 is contained in the antipode of the other.*

Proof of Claim. Suppose that neither of S_1, S_2 is contained in the antipode of the other. We will prove that S has a D_3 projection. Clearly, $n > 2$. We may assume that for each $i \in [n-2]$,

> if S', S'_1, S'_2 are obtained from S, S_1, S_2 after projecting away coordinate i, then one of S'_1, S'_2 is contained in the antipode of the other.

As the points in the polar set S do not agree on a coordinate, there exists a point $x \in S_1$ such that $\mathbf{1} - x \in S_2$. As neither of S_1, S_2 is contained in the antipode of the other, there exist distinct coordinates $i, j \in [n-2]$ such that $x \triangle e_i \in S_1, x \triangle e_j \notin S_1, \mathbf{1} \triangle x \triangle e_i \notin S_2$ and $\mathbf{1} \triangle x \triangle e_j \in S_2$. Our minimality assumption implies that the only feasible neighbors of $x, \mathbf{1} \triangle x$ are $x \triangle e_i, \mathbf{1} \triangle x \triangle e_j$, respectively. As a result, $S_1 = \{x, x \triangle e_i\}$ and $S_2 = \{\mathbf{1} \triangle x, \mathbf{1} \triangle x \triangle e_j\}$, so $S = \{x, x \triangle e_i, \mathbf{1} \triangle x, \mathbf{1} \triangle x \triangle e_j\}$. Clearly, S has a D_3 projection. $\Diamond$

If S has a D_3 minor, then we are done. Otherwise, one of S_1, S_2 is contained in the antipode of the other. After possibly relabeling S_1, S_2, we may assume that S_2 is contained in the antipode of S_1.

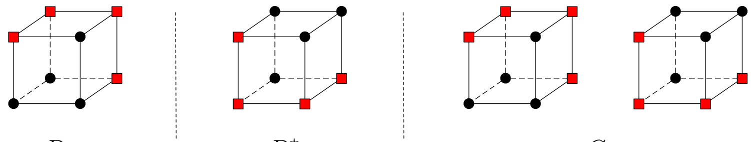
Claim 3. $2|S_2| \geq |S_1| \geq |S_2|$.

Proof of Claim. Clearly, $|S_1| \geq |S_2|$. Suppose for a contradiction that $|S_1| \geq 4|S_2|$. Since S_2 is contained in the antipode of S_1, it can be readily checked that S has an F_3 minor, a contradiction. $\Diamond$

As a result, either $|S_1| = |S_2|$ or $|S_1| = 2|S_2|$. It can now be readily checked that either $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, or $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, thereby finishing the proof of Theorem 4.1. $\square$

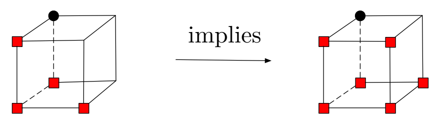
7 D_3 minors and proof of Theorem 4.2

To prove Theorem 4.2 we will need three lemmas. Let $D_3^\star := \{010, 011, 111, 101\} \subseteq \{0,1\}^3$. Observe that $D_3^\star$ is a twisting of $D_3 = \{000, 100, 010, 101\}$, and $C_8 = (D_3 \times \{0\}) \cup (D_3^\star \times \{1\})$.



D_3 $D_3^\star$ C_8

In the following lemma, we will use the following implication of Lemma 5.2 (i):

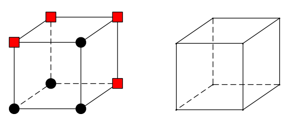


Lemma 7.1. *Let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, where the 0-restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$. Then,*

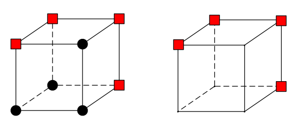
(i) every restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$, and

(ii) either $S \cong D_3 \times \{0,1\}^{n-3}$ or $S \cong C_8 \times \{0,1\}^{n-4}$.

Proof. **(i)** By a recursive argument, it suffices to show that each 3-dimensional restriction of S neighboring a $D_3, D_3^\star$ restriction is also a D_3 or a $D_3^\star$. Thus, we may assume that $n = 4$. After twisting coordinates $1, 2, 3$, if necessary, we may assume that the 0-restriction of S over coordinate 4 is D_3. So $S \cap \{x : x_4 = 0\} = \{0000, 1000, 0100, 1010\}$:

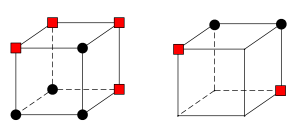


Assume in the first case that $\{0111, 1111\} \cap \overline{S} \neq \emptyset$. After applying Lemma 5.2 (i) twice, we see that $\{0111, 1111, 0011, 1101\} \subseteq \overline{S}$:



Since the two restrictions over coordinate 1 are 1-resistant, $|\{0101, 0001\} \cap S| \neq 1$ and $|\{1001, 1011\} \cap S| \neq 1$. In fact, as S has no F_3 minor, $\{0101, 0001\} \subseteq S$ if and only if $\{1001, 1011\} \subseteq S$. Moreover, as the 0-restriction of S over coordinate 3 is 1-resistant, it follows that $\{0101, 0001, 1001, 1011\} \cap S \neq \emptyset$. As a result, $\{0101, 0001, 1001, 1011\} \subseteq S$, implying in turn that 1-restricting S over coordinate 4 yields D_3.

Assume in the remaining case that $\{0111, 1111\} \cap \overline{S} = \emptyset$. As the 1-restriction of S over coordinate 3 (resp. coordinate 2) is not isomorphic to either of F_1, F_3, we get that $0011 \in \overline{S}$ (resp. $1101 \in \overline{S}$).



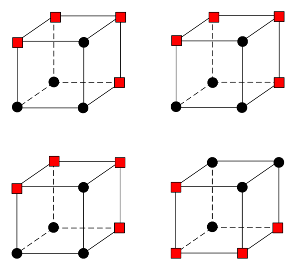
Since S has no F_1, F_3, S_3 restrictions, it follows that $0001, 1001 \in \overline{S}$. Since the 0-restriction of S over coordinate 2 (resp. coordinate 3) is 1-resistant, $1011 \in S$ (resp. $0101 \in S$), implying in turn that 1-restricting S over coordinate 4 yeilds $D_3^\star$.

(ii) It follows from (i) that $S = \bigcup_{y \in \{0,1\}^{n-3}} \left(F \times \{y\} : F \in \{D_3, D_3^\star\} \right)$. Let $R \subseteq \{0,1\}^{n-3}$ be the set of points y such that $S \cap \{x : x_i = y_{i-3} \quad 4 \leq i \leq n\} = D_3 \times \{y\}$.

Claim 1. *Every feasible component of R is a hypercube. Similarly, every infeasible component of R is a hypercube.*

Proof of Claim. By Lemma 1.13, it suffices to prove that for each $y \in R$ and distinct coordinates $i, j \in [n-3]$, if $y, y\triangle e_i, y\triangle e_j \in R$ then $y\triangle e_i\triangle e_j \in R$.

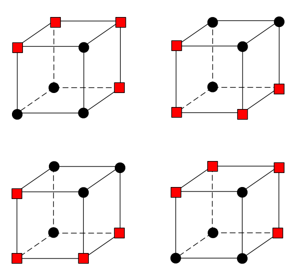
Suppose otherwise. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe that the 0-restriction of S' over coordinates $1, 2$ is not 1-resistant, a contradiction. $\Diamond$

Claim 2. *R is connected. Similarly, $\overline{R}$ is connected.*

Proof of Claim. Suppose for a contradiction that $R \subseteq \{0,1\}^{n-3}$ is not connected. By Claim 1, every feasible component of R is a hypercube, and as there are at least two feasible components, each feasible component is a hypercube of dimension at most $(n-3) - 2 = n - 5$. Thus, there exist $y \in \{0,1\}^{n-3}$ and distinct coordinates $i, j \in [n-3]$ such that $y \in R$ and $y\triangle e_i, y\triangle e_j \in \overline{R}$. Since every infeasible component of R is also a hypercube by Claim 1, it follows that $y\triangle e_i\triangle e_j \in R$. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe however that the 0-restriction of S' over coordinates $1, 2$ is not 1-resistant, a contradiction. $\Diamond$

As a result, both $R, \overline{R}$ are hypercubes, implying in turn that $R \cong \emptyset, \{0,1\}^{n-4} \times \{0\}, \{0,1\}^{n-3}$. If $R \cong \emptyset, \{0,1\}^{n-3}$ then $S \cong D_3 \times \{0,1\}^{n-3}$, and if $R \cong \{0,1\}^{n-4} \times \{0\}$ then $S \cong C_8 \times \{0,1\}^{n-4}$, thereby finishing the proof. $\square$

For each $k \geq 4$, recall that $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$, and let $D_k^\star := D_k \triangle e_k$.

Lemma 7.2. *Take integers $n \geq 3$ and $k \in \{3, \dots, n\}$. Let $S \subseteq \{0,1\}^{n+1}$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then the following statements hold:*

(i) if the projection of S over coordinate $n+1$ is D_n, then S is either $D_{n+1}, D_{n+1}^\star$ or $D_n \times \{0,1\}$,

(ii) if the projection of S over coordinate $k+1$ is $D_k \times \{0,1\}^{n-k}$, then S is either $D_{k+1} \times \{0,1\}^{n-k}, D_{k+1}^\star \times \{0,1\}^{n-k}$ or $D_k \times \{0,1\}^{n-k+1}$.

Proof. **(i)** Assume that the projection of S over coordinate $n+1$ is D_n. Let

$$S_0 := S \cap \{x : x_i = 0, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1},$$
$$S_1 := S \cap \{x : x_i = 1, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1}.$$

Let $\mathbf{1} := \mathbf{1}^{n+1}$ and $\mathbf{1}' := \mathbf{1}^n$. Then

- $S = S_0 \cup S_1$,
- $S_0 \subseteq \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, and the projection of S_0 over coordinate $n+1$ is $\{\mathbf{0}, e_2\}$, and
- $S_1 \subseteq \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, and the projection of S_1 over coordinate $n+1$ is $\{\mathbf{1}' - e_2, \mathbf{1}' - e_2 - e_3\}$.

After twisting coordinate $n+1$, if necessary, we may assume that $\mathbf{0} \in S_0$. Then, since S_0 and S_1 are 1-resistant, we get that

$$\begin{aligned} S_0 &= \{\mathbf{0}, e_2\} \quad \text{or} \quad \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}, \quad \text{and} \\ S_1 &= \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\} \quad \text{or} \quad \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \quad \text{or} \\ &\quad \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}. \end{aligned}$$

Claim 1. *If $S_0 = \{\mathbf{0}, e_2\}$, then $S = D_{n+1}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2\}$.

Assume in the first case that $n = 3$. If $S_1 = \{\mathbf{1} - e_2 - e_4, \mathbf{1} - e_2 - e_3 - e_4\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 3 is not 1-resistant, which is not the case. If $S_1 = \{\mathbf{1} - e_2 - e_4, \mathbf{1} - e_2 - e_3 - e_4, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 2 is isomorphic to F_3, which is again not the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_4$, as claimed.

Assume in the remaining case that $n \geq 4$. If $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\}$, then the points in $S = S_0 \cup S_1$ all agree on coordinate $n+1$, so by Lemma 1.12, S is a hypercube, which is not the case. If

$S_1 = \{\mathbf{1}-e_2-e_{n+1}, \mathbf{1}-e_2-e_3-e_{n+1}, \mathbf{1}-e_2, \mathbf{1}-e_2-e_3\}$, then the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is isomorphic to F_3, which cannot be the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_{n+1}$, as claimed. $\Diamond$

Claim 2. *If $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, then $S = D_n \times \{0, 1\}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2+e_{n+1}\}$. As the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is not isomorphic to F_3, it follows that $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = D_n \times \{0, 1\}$, as required. $\Diamond$

Thus, after twisting coordinate $n + 1$, if necessary, S is either D_{n+1} or $D_n \times \{0, 1\}$, so (i) holds.

(ii) Assume that the projection of S over coordinate $k+1$ is $D_k \times \{0, 1\}^{n-k}$. For each point $y \in \{0, 1\}^{n-k}$, let $S_y := S \cap \{x : x_{i+k+1} = y_i, i \in [n-k]\} \subseteq \{0, 1\}^{n+1}$. Notice that $S = \bigcup_{y \in \{0,1\}^{n-k}} S_y$. For each $y \in \{0, 1\}^{n-k}$, pick an appropriate $S'_y \subseteq \{0, 1\}^{k+1}$ such that $S_y = S'_y \times \{y\}$. Notice that the projection of each S'_y over coordinate $k + 1$ is D_k. We therefore get from (i) that each S'_y is either D_{k+1}, $D^\star_{k+1}$ or $D_k \times \{0, 1\}$.

Claim 3. *All of $(S'_y : y \in \{0, 1\}^{n-k})$ are equal to one another.*

Proof of Claim. Suppose otherwise. Then there exists $y_1, y_2 \in \{0, 1\}^{n-k}$ such that $\mathrm{dist}(y_1, y_2) = 1$ and $S'_{y_1} \neq S'_{y_2}$. In particular, S has either $S' := (D_{k+1} \times \{0\}) \cup (D_k \times \{01, 11\})$ or $S'' := (D_{k+1} \times \{0\}) \cup (D^\star_{k+1} \times \{1\})$ as a restriction. However, the restriction of S' (resp. S'') obtained after 0-restricting coordinates $[n+1] - \{3, k+1, k+2\}$ is not 1-resistant, so S cannot have either of S', S'' as a restriction, a contradiction. $\Diamond$

As a consequence, $S = D_{k+1} \times \{0, 1\}^{n-k}$, $D^\star_{k+1} \times \{0, 1\}^{n-k}$ or $D_k \times \{0, 1\}^{n-k+1}$, so (ii) holds. $\square$

Lemma 7.3. *Take an integer $n \geq 5$ and a set $S \subseteq \{0, 1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. If the projection of S over coordinate n is $C_8 \times \{0, 1\}^{n-5}$, then $S = C_8 \times \{0, 1\}^{n-4}$.*

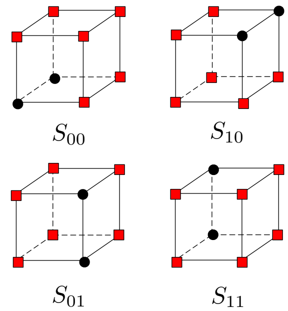
Proof. It suffices to prove this for $n = 5$. Assume that the projection of S over coordinate 5 is $C_8 = (D_3 \times \{0\}) \cup (D^\star_3 \times \{1\})$. For $i, j \in \{0, 1\}$, let $S_{ij} \subseteq \{0, 1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. After twisting coordinate 5, if necessary, we may assume that $\mathbf{0} \in S$.

Claim. *S has a D_3 restriction.*

Proof of Claim. Suppose for a contradiction that S does not have a D_3 restriction. In particular, $S_{00}, S_{01} \neq D_3$ and $S_{10}, S_{11} \neq D^\star_3$. Thus by Lemma 7.2 (i),

$$(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4 \quad \text{or} \quad D^\star_4,$$
$$(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D'_4 \quad \text{or} \quad D'_4 \triangle e_4,$$

where $D'_4 = \{0100, 0110, 1011, 1111\} \subseteq \{0, 1\}^4$. Since $\mathbf{0} \in S$, we must have that $(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4$. Thus, $S_{00} = \{000, 010\}$ and $S_{01} = \{100, 101\}$. Since the restriction of S obtained after 0-restricting coordinates 1 and 5 is not isomorphic to D_3, it follows that $(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D'_4 \triangle e_4$. So, $S_{10} = \{101, 111\}$ and $S_{11} = \{010, 011\}$:



Observe however that the 1-restriction of S over coordinates $2, 3$ is not 1-resistant, a contradiction. $\Diamond$

Thus, $S \cong D_3 \times \{0,1\}^2$ or $C_8 \times \{0,1\}$ by Lemma 7.1 (ii). It can be readily checked that S must be in fact equal to $C_8 \times \{0,1\}$, as required. $\square$

We are now ready to prove Theorem 4.2, stating the following:

> Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S has a D_3 minor, then either
>
> - $S \cong C_8 \times \{0,1\}^{n-4}$, or
> - $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$.

Proof. Among all projections of S with a D_3 restriction, pick the one $S' \subseteq \{0,1\}^\ell$ of largest dimension $\ell \in \{3, \ldots, n\}$. We may assume, after a possible relabeling, that S' is obtained from S after projecting away coordinates $[n] - [\ell]$. It follows from Lemma 7.1 (ii) that, after a possible twisting and relabeling, $S' = C_8 \times \{0,1\}^{\ell-4}$ or $S' = D_3 \times \{0,1\}^{\ell-3}$.

Claim. *If $S' = C_8 \times \{0,1\}^{\ell-4}$, then $\ell = n$.*

Proof of Claim. This follows immediately from Lemma 7.3 and the maximal choice of S'. $\Diamond$

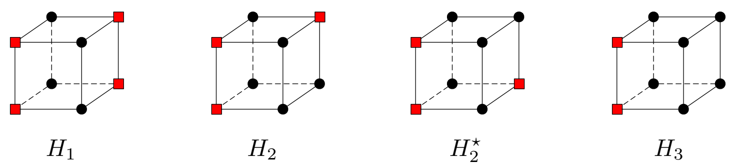
Thus, if $S' = C_8 \times \{0,1\}^{\ell-4}$, then $S \cong C_8 \times \{0,1\}^{n-4}$. Otherwise, $S' = D_3 \times \{0,1\}^{\ell-3}$. In this case, a repeated application of Lemma 7.2 (ii) implies that $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{\ell, \ldots, n\}$, thereby finishing the proof of Theorem 4.2. $\square$

8 Infeasible hypercubes and Theorem 4.3

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. In this section, we will prove the following statement:

> Assume that is S is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a hypercube.

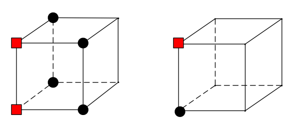
Proving this statement requires three technical lemmas. Let $H_1 := \{100, 010, 101, 011\} \subseteq \{0,1\}^3$, $H_2 := \{100, 010, 101, 011, 110\} \subseteq \{0,1\}^3$, $H_2^\star := \{100, 010, 101, 011, 111\} \subseteq \{0,1\}^3$ and $H_3 := \{100, 010, 101, 011, 110, 111\} \subseteq \{0,1\}^3$, as displayed below:



Given $i \in \{0,1\}$, denote by $S_i \subseteq \{0,1\}^{n-1}$ the i-restriction of S over coordinate n.

Lemma 8.1. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. If $S_0 \in \{H_1, H_2, H_2^\star, H_3\}$, then $|\{000, 001\} \cap S_1| \neq 1$.*

Proof. Suppose, for a contradiction, that $H_1 \subseteq S_0 \subseteq H_3$ and $|\{000, 001\} \cap S_1| = 1$. After twisting coordinate 3, if necessary, we may assume that $000 \in S_1$ and $001 \in \overline{S_1}$. So S may be displayed as below:



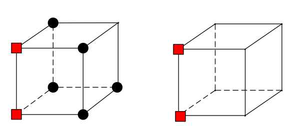
Since the 0-restriction of S over coordinate 1 is not isomorphic to either F_1 or F_3, we get that $011 \in \overline{S_1}$, and since this restriction is not isomorphic to D_3, we get that $010 \in \overline{S_1}$. By the symmetry between coordinates $1, 2$, we get that $\{100, 101\} \subseteq \overline{S_1}$. But then the 0-restriction of S over coordinate 3 is isomorphic to either $P_3, R_{1,1}, F_1$ or F_2, a contradiction. $\square$

Lemma 8.2. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_0 \in \{H_2, H_2^\star, H_3\}$ and $\{000, 001\} \cap S_1 = \emptyset$. Then the following statements hold:*

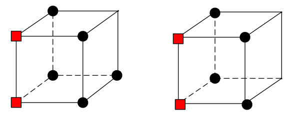
(i) $S_1 \in \{H_1, H_2, H_2^\star, H_3\}$, *and*

(ii) *if $S_1 = H_1$, then $S_0 = H_3$.*

Proof. **(i)** After twisting coordinate 3, if necessary, we may assume that $S_0 \in \{H_2, H_3\}$. We may therefore display S as:



Since the 0-restriction of S over coordinate 1 is 1-resistant, it follows that $|\{010, 011\} \cap S_1| \neq 1$, and since the 0-restriction of S over coordinate 2 is 1-resistant, it follows that $|\{100, 101\} \cap S_1| \neq 1$. Thus, as the 0-restriction of S over coordinate 3 is 1-resistant, either $\{010, 011\} \subseteq S_1$ or $\{100, 101\} \subseteq S_1$. After relabeling coordinates $1, 2$, if necessary, $\{010, 011\} \subseteq S_1$. Since the 0-restriction of S over coordinate 3 is not isomorphic to D_3 or F_3, it follows that $\{100, 101\} \subseteq S_1$ also:



Hence, $S_1 \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_1 = H_1$, then as the 1-restriction of S over coordinate 1 is not isomorphic to F_3, it follows that $111 \in S_0$, so $S_0 = H_3$, as required. $\square$

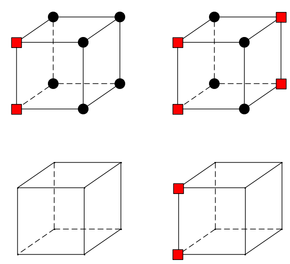
Given that $n \geq 2$ and $i, j \in \{0,1\}$, denote by $S_{ij} \subseteq \{0,1\}^{n-2}$ the restriction of S obtained after i-restricting coordinate $n-1$ and j-restricting coordinate n.

Lemma 8.3. *Let $S \subseteq \{0,1\}^5$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_{00} = H_3, S_{10} = H_1$ and $\{000, 001\} \cap S_{11} = \emptyset$. Then the following statements hold:*

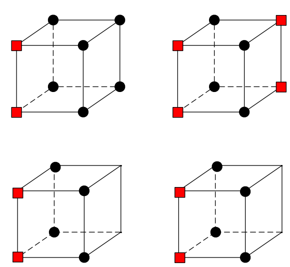
(i) $S_{01}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$*, and*

(ii) if $S_{11} = H_1$ then $S_{01} = H_3$, and therefore $S_1 = S_0$.

Proof. **(i)** For $i, j \in \{0,1\}$, denote by $R_{ij} \subseteq \{0,1\}^5$ the restriction of S obtained after i-restricting coordinate 3 and j-restricting coordinate 5.



Notice that $R_{00} = R_{10} = H_2$ and $001 \notin R_{01} \cup R_{11}$. It therefore follows from Lemma 8.1 that $000 \notin R_{01} \cup R_{11}$. We get from Lemma 8.2 (i)-(ii) that $R_{01}, R_{11} \in \{H_2, H_2^\star, H_3\}$:



As a result, $S_{00}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_{11} = H_1$, then R_{01} and R_{11} must be equal to H_2, implying in turn that $S_{01} = H_3$, as required. $\square$

We are now ready to prove the first main result of this section:

Proposition 8.4. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that has no $R_{1,1}, F_1, F_2, F_3$ and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a hypercube.*

Proof. We prove this by induction on $n \geq 2$. The base case $n = 2$ holds trivially. For the induction step, assume that $n \geq 3$. Let K be the infeasible component of S containing x. If every neighbor of x belongs to S, then $K = \{x\}$ and we are done. Otherwise, we may assume that $x \in \{\mathbf{0}, e_3\} \subseteq K$ and $i = 1, j = 2$. For each $y \in \{0,1\}^{n-3}$, let $S_y := S \cap \{x : x_{3+i} = y_i, i \in [n-3]\}$ and choose an appropriate $R_y \subseteq \{0,1\}^3$ such that $S_y = R_y \times \{y\}$. Notice that $\{000, 001\} \subseteq \overline{R_{\mathbf{0}}}$, and either $\{100, 010, 110\} \subseteq R_{\mathbf{0}}$ or $\{101, 011, 111\} \subseteq R_{\mathbf{0}}$. Since $R_{\mathbf{0}}$ is 1-resistant and not isomorphic to D_3, F_3, it follows that $R_{\mathbf{0}} \in \{H_2, H_2^\star, H_3\}$. In particular, if $n = 3$, then $K = \{\mathbf{0}, e_3\}$ and the induction step is complete. We may therefore assume that $n \geq 4$.

Let S' be the projection of S over coordinate 3. Then S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. Hence, since $\mathbf{0} \in \overline{S'}$ and $\{e_1, e_2, e_1 + e_2\} \subseteq S'$, the induction hypothesis implies that the infeasible component of S' containing $\mathbf{0}$, call it K', is a hypercube. Notice that the set of points in $\{0,1\}^n$ projecting onto a point in K' belong to K and form a hypercube whose dimension is larger by one.

Therefore, it suffices to show that K consists precisely of the points in $\{0,1\}^n$ projecting onto K'. Suppose otherwise. Then there must exist points $z, z + e_3 \in \{0,1\}^n$ projecting onto a point $z' \in \{0,1\}^{n-1}$ such that

- z' belongs to S' and is adjacent to a point in K', and
- $|\{z, z + e_3\} \cap S| = 1$.

Notice that $|\{z, z + e_3\} \cap K| = 1$.

Call a point $y \in \{0,1\}^{n-3}$ *involved* if

- $R_y \in \{H_2, H_2^\star, H_3\}$, and
- $00y \in K'$.

Notice that $\mathbf{0} \in \{0,1\}^{n-3}$ is involved. Now, pick a point $t' \in \{0,1\}^{n-1}$ minimizing $\operatorname{dist}(t', z')$ subject to

- $t' \in K'$, and
- there exists an involved $y \in \{0,1\}^{n-3}$ such that $t' = 00y$,

in this order of priority. We may assume that $t' = \mathbf{0} \in \{0,1\}^{n-1}$. Since $z' \notin K'$, we get that $\operatorname{dist}(\mathbf{0}, z') \geq 1$. It follows from Lemma 8.1 that $\operatorname{dist}(\mathbf{0}, z') \geq 2$. Since K' is a hypercube, there exist an integer $d \geq 2$ and distinct coordinates $j_1, j_2, \ldots, j_d \in [n] - \{3\}$ such that $z' = \sum_{i=1}^d e_{j_i}$ and

$$\sum_{i=1}^k e_{j_i} \in K' \qquad k = 1, \ldots, d-1.$$

Notice that

$$\sum_{i=1}^k e_{j_i} \in K \quad \text{and} \quad e_3 + \sum_{i=1}^k e_{j_i} \in K \qquad k = 1, \ldots, d-1.$$

Thus, since $R_{\mathbf{0}} \in \{H_2, H_3\}$, we have $j_1 \in [n] - \{1, 2, 3\}$. We may therefore assume that $j_1 = 4$. Since $R_{\mathbf{0}} \in \{H_2, H_3\}$ and $\{000, 001\} \cap R_{e_1} = \emptyset$, it follows from Lemma 8.2 (i) that $R_{e_1} \in \{H_1, H_2, H_2^\star, H_3\}$. Our minimal choice of $t' = \mathbf{0}$ implies that $R_{e_1} = H_1$ (otherwise, $t' = e_4$ contradicts the minimality of $t' = \mathbf{0}$). We now get from Lemma 8.1 that $d \geq 3$, and from Lemma 8.2 (ii) that $R_{\mathbf{0}} = H_3$. Since $j_2 \in [n] - \{1, 2, 3, 4\}$, we may assume that $j_2 = 5$. So $e_4 + e_5 \in K'$. As $\mathbf{0}, e_4, e_4 + e_5 \in K'$ and K' is a hypercube, it follows that $e_5 \in K'$. Since $\{000, 001\} \cap R_{e_1+e_2} = \emptyset$, we get from Lemma 8.3 that either

- $R_{e_1+e_2} \in \{H_2, H_2^\star, H_3\}$, or
- $R_{e_2} = H_3$ and $R_{e_1+e_2} = H_1$.

The first case is not possible as it contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_4 + e_5$ would be a better choice. However, the second case is not possible either as it also contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_5$ would be a better choice. This finishes the proof of Proposition 8.4. $\square$

8.1 Proof of Theorem 4.3

We are now ready to prove Theorem 4.3, stating that

> Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0, 1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is connected and has no D_3 minor, then either
>
> - S is a hypercube, or
> - every infeasible component of S is a hypercube.

Proof. Assume that there is an infeasible component K that is not a hypercube.

Claim 1. *Take a point x and distinct coordinates $i, j \in [n]$ such that $x \in K$ and $x \triangle e_i \in S$. If $x \triangle e_i \triangle e_j \in S$, then $x \triangle e_j \in K$.*

Proof of Claim. For if not, $x \triangle e_j \in S$, so by Proposition 8.4, the infeasible component of S containing x, which is K, is a hypercube, a contradiction. $\Diamond$

This claim has the following subtle implication:

Claim 2. *The points in S agree on a coordinate.*

Proof of Claim. Take a point $y \in K$ and a direction $i \in [n]$ such that $y \triangle e_i \in S$. We may assume that $y = \mathbf{0}$ and $i = 1$. As S is connected, it follows from Claim 1 that $S \subseteq \{x : x_1 = 1\}$, as required. $\Diamond$

As S is 1-resistant, it follows from Lemma 1.12 that S is a hypercube, thereby proving Theorem 4.3. $\square$

9 Every ± 1-resistant set is strictly polar.

We will need the following immediate remark:

Remark 9.1. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If S is strictly polar, then so is $S \times \{0,1\}$.*

We will also need the following variant of Lemma 2.5:

Lemma 9.2. *Take an integer $n \geq 1$ and a nonempty set $S \subseteq \{0,1\}^n$ without an $R_{1,1}$ restriction, where every infeasible component is a hypercube. Then*

- *$|S| \geq 2^{n-1}$, and*
- *if $|S| = 2^{n-1}$, then S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.*

In particular, S is strictly polar.

Proof. We prove this by induction on $n \geq 1$. The base cases $n \in \{1,2\}$ are clear. For the induction step, assume that $n \geq 3$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a hypercube of dimension $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty. Since every infeasible component of both S_0, S_1 is a hypercube, we may apply the induction hypothesis. Thus, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$, so by the induction hypothesis, each S_i is either a hypercube of dimension $n-2$ or the union of antipodal hypercubes of dimension $n-3$. If one of S_0, S_1 is a hypercube, then as every infeasible component of S is a hypercube, S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$. Otherwise, each one of S_0, S_1 is the union of two antipodal hypercubes of dimension $n-3$. As S has no $R_{1,1}$ restriction, it must be that $S_0 = S_1$, implying in turn that S is the union of antipodal hypercubes of dimension $n-2$, thereby completing the induction step. $\square$

We are now able to prove Theorem 1.8, stating that every ± 1-resistant set is strictly polar:

Proof of Theorem 1.8. Take an integer $n \geq 1$ and a ± 1-resistant set $S \subseteq \{0,1\}^n$. Then by Theorem 1.5, either

(i) $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \dots, n\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \dots, n\}$,

(v) S is a hypercube, or

(vi) every infeasible component of S is a hypercube, and every feasible point has at most two infeasible neighbors.

Observe that $\{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\}$ and C_8 are strictly polar sets. As a result, in cases (i)-(iv), the set S is strictly polar by Remark 9.1. A hypercube is strictly polar, so in case (v), S is also strictly polar. For the last case (vi), as S has no $R_{1,1}$ restriction by Remark 3.1, Lemma 9.2 implies that S is strictly polar. $\square$

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