Structure theorems for two classes of resistant sets

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Abstract

A subset of the unit hypercube \( \{0, 1\}^n \) is cube-ideal if its convex hull is described by hypercube and generalized set covering inequalities. In this paper, we provide structure theorems for two classes of cube-ideal sets that remain cube-ideal even after making local changes. We will also discuss applications.

1 Introduction

Take an integer \( n \geq 1 \). Denote by \( \{0, 1\}^n \) the extreme points of the \( n \)-dimensional unit hypercube \([0, 1]^n\). For a coordinate \( i \in [n] := \{1, \ldots, n\} \), we refer to \( x_i \geq 0 \) and \( x_i \leq 1 \) as hypercube inequalities. Generalized set covering inequalities are ones of the form

\[
\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [n], I \cap J = \emptyset,
\]

which are precisely the inequalities that cut off (sub-)hypercubes of \( \{0, 1\}^n \). Interpreted as clause satisfaction inequalities for the Boolean satisfiability problem, generalized set covering inequalities are prevalent in the literature. Also referred to as cropping inequalities [8, 13], these inequalities have surfaced as cocircuit inequalities valid for cycle polytopes of binary matroids [6], as set covering inequalities \((J = \emptyset)\) for various set covering problems [7, 11, 9], and as cover inequalities \((I = \emptyset)\) for the knapsack problem [5, 12, 15].

Take a set \( S \subseteq \{0, 1\}^n \). We say that \( S \) is cube-ideal if its convex hull, denoted \( \text{conv}(S) \), can be described by hypercube and generalized set covering inequalities. This notion was introduced and studied in [2]. What is the structure of cube-ideal sets? This question lays the underpinning theme of our paper.

Among basic classes of cube-ideal sets are: the cycle space of a graph [14]; the up-monotone set associated with an ideal clutter (see [2]); and a set where each infeasible component is a hypercube or has maximum degree at most two [3]. Given such basic classes, there are three binary operations that preserve cube-idealness and can be used to generate more cube-ideal sets: the product of two cube-ideal sets is cube-ideal; the coproduct of two cube-ideal sets is cube-ideal; given two cube-ideal sets whose complements are also cube-ideal, their reflective product is cube-ideal [2]. Subsequently, cube-ideal sets form a rich class and have a complex structure, to say the least. Nonetheless, we conjecture the following:
Conjecture 1.1. There exists an algorithm that given an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \) determines in time polynomial in \( n \) and \( |S| \) whether or not \( S \) is cube-ideal.

(It is worth mentioning that testing the idealness of an explicitly given clutter is co-NP-complete, as was shown by Ding, Feng and Zang [10].)

In this paper, we provide structure theorems for cube-ideal sets that remain cube-ideal even after making local changes.

1.1 Definitions and notation

Given points \( a, b \in \{0, 1\}^n \), the (Hadamard) distance between \( a, b \), denoted \( \text{dist}(a, b) \), is the number of coordinates \( a \) and \( b \) differ on. Denote by \( G_n \) the skeleton graph of \( \{0, 1\}^n \), whose vertices are the points in \( \{0, 1\}^n \), where two vertices \( a, b \in \{0, 1\}^n \) are adjacent if \( \text{dist}(a, b) = 1 \). We refer to the points in \( S \) as feasible and to the points in \( \overline{S} := \{0, 1\}^n - S \) as infeasible. The (connected) components of \( G_n[S] \) are feasible components, while the components of \( G_n[\overline{S}] \) are infeasible components.

For \( i \in [n] \), denote by \( e_i \) the \( i \)th unit vector. To twist coordinate \( i \in [n] \) is to replace \( S \) by

\[
S \triangle e_i := \{ x \triangle e_i : x \in S \}.
\]

We say that \( S' \subseteq \{0, 1\}^n \) is isomorphic to \( S \), and write \( S' \cong S \), if \( S' \) is obtained from \( S \) after relabeling and twisting some coordinates.

The set obtained from \( S \cap \{x : x_i = 0\} \) after dropping coordinate \( i \) is called the 0-restriction of \( S \) over coordinate \( i \), and the set obtained from \( S \cap \{x : x_i = 1\} \) after dropping coordinate \( i \) is called the 1-restriction of \( S \) over coordinate \( i \). A restriction of \( S \) is a set obtained after a series of 0- and 1-restrictions. The projection of \( S \) over coordinate \( i \) is the set obtained from \( S \) after dropping coordinate \( i \). A minor of \( S \) is what is obtained after a series of restrictions and projections. A minor is proper if at least one operation is applied.

Remark 1.2 ([2]). If a set is cube-ideal, then so is every isomorphic minor of it.\(^1\)

1.2 Resistance and structure theorems

Let \( P_3 := \{110, 101, 011\} \subseteq \{0, 1\}^3 \) and \( S_3 := \{110, 101, 011, 111\} \subseteq \{0, 1\}^3 \), as displayed in Figure 1. Then

\[
\text{conv}(P_3) = \{ x \in [0, 1]^3 : x_1 + x_2 + x_3 = 2 \} \quad \text{and} \quad \text{conv}(S_3) = \{ x \in [0, 1]^3 : x_1 + x_2 + x_3 \geq 2 \},
\]

implying in turn that \( P_3, S_3 \) are not cube-ideal. In particular, a cube-ideal set has no \( P_3, S_3 \) minor by Remark 1.2.

We say that \( S \) is 1-resistant if, for every subset \( X \subseteq \{0, 1\}^n \) of cardinality at most one, \( S \cup X \) has no \( P_3, S_3 \) minor. The notion of 1-resistance was introduced and studied by Abdi, Cornuéjols and Lee [3], though the prefix 1- was omitted there. There the authors showed that 1-resistance is a multifaceted property, and they demonstrate that the class of 1-resistant sets is quite rich. In particular, they show that,

\(^1\)Going forward, the prefix “isomorphic” will be omitted from “isomorphic restriction” and “isomorphic minor”.

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Figure 1: An illustration of $P_3$ and $S_3$. Round points are feasible while square points are infeasible.

**Theorem 1.3** ([3]). A 1-resistant set is cube-ideal.

Although a structure theorem remains elusive even for this special class of cube-ideal sets, we are able to provide structure theorems for two natural classes.

We say that $S$ is 2-resistant if, for every subset $X \subseteq \{0, 1\}^n$ of cardinality at most two, $S \cup X$ has no $P_3, S_3$ minor. We will prove the following theorem, part (iii) of which explains the structure of 2-resistant sets:

**Theorem 1.4.** Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:

(i) $S$ is 2-resistant,

(ii) $S$ has no restriction $F \subseteq \{0, 1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\},$

(iii) every infeasible component is a hypercube or has maximum degree at most two,

(iv) $S$ has no minor $F \subseteq \{0, 1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}.$

This theorem is proved in §2. There we will also show that 2-resistance of a set $S \subseteq \{0, 1\}^n$ can be tested in time $O(n^3|S|)$.

We say that $S$ is $\pm 1$-resistant if, for every subset $X \subseteq \{0, 1\}^n$ of cardinality at most one, $S \triangle X$ has no $P_3, S_3$ minor. In §3, we provide excluded minor and excluded restriction characterizations for $\pm 1$-resistant sets.

There we will also show that $\pm 1$-resistance of a set $S \subseteq \{0, 1\}^n$ can be tested in time $O(n^2|S|^2)$.

Given integers $n_1, n_2 \geq 0$ and $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$, the product of $S_1, S_2$ is

$$S_1 \times S_2 := \{(x, y): x \in S_1, y \in S_2\} \subseteq \{0, 1\}^{n_1+n_2}.$$

We will prove the following structure theorem:

**Theorem 1.5.** Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then $S$ is $\pm 1$-resistant if, and only if, one of the following statements holds:

(i) $S \cong A_k \times \{0, 1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, where $A_k = \{0^k, 1^k\},$

(ii) $S \cong B_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, where $B_k = \{0^k, e_1, 1^k\},$

(iii) $S \cong C_8 \times \{0, 1\}^{n-4}$, where $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\},$

(iv) $S \cong D_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, where $D_k = \{0^k, e_2, 1^k-e_2, 1^k-e_2-e_3\},$
(v) $S$ is a hypercube, or

(vi) every infeasible component of $S$ is a hypercube, and every feasible point has at most two infeasible neighbors.

Here, $0^k, 1^k$ denote the $k$-dimensional vectors of all-zeros and all-ones, respectively. (The superscripts will be dropped when there is no ambiguity.) A proof outline of this theorem is given in §4; the proof spans §§5, 6, 7 and §8.²

1.3 Strict polarity and applications

We say that $S$ is polar if either there are antipodal feasible points, or the feasible points agree on a coordinate:

$$\{x, 1 - x\} \subseteq S \quad \text{for some } x \in \{0, 1\}^n \quad \text{or} \quad S \subseteq \{x : x_i = a\} \quad \text{for some } i \in [n] \text{ and } a \in \{0, 1\}.$$

We say that $S$ is strictly non-polar if it is not polar, but every proper restriction is polar. It is shown in [3] that if a set is strictly non-polar, and 1-resistant, then it gives rise to an ideal minimally non-packing clutter. Motivated by this fact,

**Question 1.6.** What are the 1-resistant strictly non-polar sets?

![Figure 2: The 2-resistant strictly non-polar sets.](image)

Even though the 1-resistant strictly non-polar sets $S \subseteq \{0, 1\}^n$ satisfying $|S| = 2^{n-1}$ are completely found [3], Question 1.6 is still open. Using the structure theorems obtained, we answer this question for 2- and ±1-resistant sets. To this end, let

$$R_{1,1} := \{000, 110, 101, 011\} \subseteq \{0, 1\}^3$$

$$R_{2,1} := \{0000, 1110, 1001, 0101, 0011, 1101, 1011, 0111\} \subseteq \{0, 1\}^4$$

$$R_5 := \{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\}$$

$$\cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0, 1\}^5,$$

as displayed in Figure 2. We prove the following theorem in §2:

²For an explanation of the origin of resistance, see [1], Chapter 6.
Theorem 1.7. Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only 2-resistant strictly non-polar sets.

We say that $S$ is strictly polar if every restriction of it, including $S$ itself, is polar. Notice that a set is strictly polar if, and only if, it has no strictly non-polar restriction.

Theorem 1.8. A ±1-resistant set is strictly polar.

This theorem is proved in §9.

1.4 Preliminaries

Throughout the paper, we will make use of several results from [3]. Let us state them all here.

Remark 1.9 ([3]). If a set is 1-resistant, then so is every minor of it.

Take a set $F \subseteq \{0, 1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}.$$  

We refer to $F$, and any set isomorphic to it, as fragile. (See Figure 3.)

Theorem 1.10 ([3]). Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:

(i) $S$ is 1-resistant,

(ii) $S$ has no fragile restriction and no $\{0^k, 1^k - e_1\}, k \geq 4$ restriction,

(iii) $S$ has no fragile minor.

Testing 1-resistance can be done efficiently:

Theorem 1.11 ([3]). Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then in time $O(n^4|S|^3)$, one can test whether or not $S$ is 1-resistant.

We will also need the following lemma for 1-resistant sets:

Lemma 1.12 ([3]). Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0, 1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then $S$ is a hypercube.

Lastly, we will need the following lemma for general sets:
Figure 4: The excluded minor, and restriction, defining 2-resistance.

Lemma 1.13 ([3]). Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \), where for all \( x \in \{0, 1\}^n \) and distinct \( i, j \in [n] \), the following statement holds:

\[
\text{if } x, x \triangle e_i, x \triangle e_j \in S \text{ then } x \triangle e_i \triangle e_j \in S.
\]

Then every feasible component of \( S \) is a hypercube.

2 The structure of 2-resistant sets and consequences

In this section, we prove Theorem 1.4 on the structure of 2-resistant sets. We will then prove three applications, including Theorem 1.7 characterizing the 2-resistant strictly non-polar sets. Let us start with the following remark:

Remark 2.1. If a set is 2-resistant, then so is every minor of it.

Proof. Being 2-resistant is clearly closed under restrictions; it remains to show that it is also closed under projections. To this end, take an integer \( n \geq 1 \) and a 2-resistant set \( S \subseteq \{0, 1\}^n \). Let \( S' \subseteq \{0, 1\}^{n-1} \) be the projection of \( S \) over coordinate \( n \). Suppose for a contradiction that \( S' \) is not 2-resistant. Then for some \( X' \subseteq \{0, 1\}^{n-1} \) of cardinality at most two, \( S' \cup X' \) has a \( P_3, S_3 \) minor. Let \( X := \{(x', 0) : x' \in X'\} \). Then \( S \cup X \) has \( S' \cup X' \) as a projection, and has a \( P_3, S_3 \) minor as a consequence, implying in turn that \( S \) is not 2-resistant, a contradiction. \( \square \)

We are now ready to prove Theorem 1.4, stating the following:

Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \). Then the following statements are equivalent:

(i) \( S \) is 2-resistant,

(ii) \( S \) has no restriction \( F \subseteq \{0, 1\}^3 \) such that \( F \cap \{000, 100, 010, 001, 110\} = \{110\} \),

(iii) every infeasible component is a hypercube or has maximum degree at most two,

(iv) \( S \) has no minor \( F \subseteq \{0, 1\}^3 \) such that \( F \cap \{000, 100, 010, 001, 110\} = \{110\} \).

Proof. (i) \( \Rightarrow \) (ii) Observe that \( F \) is not 2-resistant, because \( F \cup \{101, 011\} \) is either \( P_3 \) or \( S_3 \). Thus, a 2-resistant set has no \( F \) restriction by Remark 2.1.

(ii) \( \Rightarrow \) (iii): Assume that \( S \) has no \( F \) restriction.
Claim 1. Let \( x \) be an infeasible point with at least three infeasible neighbors. If \( x \triangle e_i, x \triangle e_j \) are infeasible for some distinct \( i, j \in [n] \), then \( x \triangle e_i \triangle e_j \) is also infeasible.

Proof of Claim. Suppose for a contradiction that \( x \triangle e_i \triangle e_j \) is feasible. Since \( x \) has at least three infeasible neighbors, there is a coordinate \( k \in [n] - \{i, j\} \) such that \( x \triangle e_k \) is infeasible. Then the 3-dimensional restriction of \( S \) containing \( x \triangle e_i, x \triangle e_j, x \triangle e_k \) is a set \( F \subseteq \{0, 1\}^3 \) such that \( F \cap \{000, 100, 010, 001, 110\} = \{110\} \), a contradiction.  

Claim 2. Let \( x \) be an infeasible point with at least three infeasible neighbors. Let \( k \geq 3 \) be the number of infeasible neighbors of \( x \). Then the \( k \)-dimensional hypercube containing \( x \) and its infeasible neighbors is infeasible.

Proof of Claim. After a possible twisting and relabeling, if necessary, we may assume that \( x = 0 \) and its infeasible neighbors are \( e_1, \ldots, e_k \). We need to show that for all subsets \( I \subseteq [k] \), \( \sum_{i \in I} e_i \in \mathbb{S} \). We will proceed by induction on \( |I| \geq 0 \). The base cases \( |I| \in \{0, 1\} \) hold by assumption, and the case \( |I| = 2 \) follows from Claim 1. For the induction step, assume that \( |I| \geq 3 \). After a possible relabeling, if necessary, we may assume that \( I = [\ell] \).

Let \( y := \sum_{i=1}^{\ell} e_i \). By the induction hypothesis, \( y \) and its three neighbors \( y \triangle e_{\ell-2}, y \triangle e_{\ell-1}, y \triangle e_{\ell} \) are all infeasible. It therefore follows from Claim 1 that \( y \triangle e_{\ell-1} \triangle e_{\ell} = \sum_{i=1}^{\ell} e_i \) is infeasible, thereby completing the induction step.  

Let \( K \subseteq \mathbb{S} \) be an infeasible component, and let \( k \) be the maximum number of infeasible neighbors of a point in \( K \). If \( k \leq 2 \), then \( K \) has maximum degree at most two. Otherwise, \( k \geq 3 \). It then follows from Claim 2 that \( K \) contains a \( k \)-dimensional hypercube. Our maximal choice of \( k \) in turn implies that \( K \) is in fact the \( k \)-dimensional hypercube. Thus, every infeasible component is a hypercube or has maximum degree at most two.

(iii) \( \Rightarrow \) (iv): Assume that every infeasible component of \( S \) is a hypercube or has maximum degree at most two.

Claim 3. If \( S' \) is a minor of \( S \), then every infeasible component of \( S' \) is a hypercube or has maximum degree at most two.

Proof of Claim. It suffices to prove this for restrictions and projections. The claim clearly holds for restrictions. As for projections, assume that \( S' \) is obtained from \( S \) after projecting away coordinate \( n \). Let \( K' \subseteq \{0, 1\}^{n-1} \) be an infeasible component of \( S' \). Clearly, \( \{(x, 0), (x, 1) : x \in K'\} \subseteq \{0, 1\}^n \) is connected and infeasible, so it is contained in an infeasible component \( K \) of \( S \). If \( K \) has maximum degree at most two, then so does \( \{(x, 0), (x, 1) : x \in K'\} \), implying in turn that \( K' \) has maximum degree at most two. Otherwise, \( K \) is a hypercube. In this case, as \( K' \) is an infeasible component of \( S' \), it must be that \( K = \{(x, 0), (x, 1) : x \in K'\} \), implying in turn that \( K' \) is a hypercube. Thus, \( K' \) is a hypercube or has maximum degree at most two, as claimed.  

Thus, since the infeasible component of \( F \) containing 000 is neither a hypercube or of maximum degree at most two, \( S \) does not have an \( F \) minor.
(iv) $\Rightarrow$ (i): Assume that $S$ is not 2-resistant. Then there is a subset $X \subseteq \{0, 1\}^n$ of cardinality at most two such that $S \cup X$ has a $P_3, S_3$ minor. Thus there is a subset $Y \subseteq \{0, 1\}^3$ of cardinality at most two such that $S$ has a $P_3 - Y, S_3 - Y$ minor. After relabeling the coordinates, if necessary, we see that both $P_3 - Y, S_3 - Y$ are the desired minor.

2.1 Applications of Theorem 1.4

The first application is that testing 2-resistance can be done efficiently:

**Corollary 2.2.** Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then in time $O(n^4|S|^2)$, one can test whether or not $S$ is 2-resistant.

**Proof.** By Theorem 1.4 (ii), testing whether $S$ is 2-resistant is equivalent to testing whether $S$ has no restriction $F \subseteq \{0, 1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$. Such a restriction can be found according to the following simple algorithm:

Pick a feasible point $x$ and distinct coordinates $i, j, k \in [n]$. Test whether or not the 3-dimensional restriction containing $x e_i, x e_j, x e_k$ is the desired restriction. If so, then $S$ is not 2-resistant.

If not, change $x, i, j, k$.

As a result, testing 2-resistance takes time $n^3|S| \times n|S| = n^4|S|^2$.

The second application is yet another characterization of 2-resistance:

**Corollary 2.3.** Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then $S$ is 2-resistant if, and only if, $S \cup X$ has no $P_3, S_3$ restriction for all subsets $X \subseteq \{0, 1\}^n$ of cardinality at most two.

**Proof.** $(\Rightarrow)$ follows from the definition of 2-resistance. $(\Leftarrow)$ Assume that $S \cup X$ has no $P_3, S_3$ restriction for all subsets $X \subseteq \{0, 1\}^n$ of cardinality at most two. Then $S$ has no restriction $F \subseteq \{0, 1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$. It therefore follows from Theorem 1.4 (ii) that $S$ is 2-resistant, as required.

The third application of Theorem 1.4 is the characterization of the 2-resistant strictly non-polar sets. We need the following lemma:

**Lemma 2.4.** Take an integer $n \geq 5$ and a set $S \subseteq \{0, 1\}^n$, where every infeasible point has at most two infeasible neighbors. Then $|S| \geq 2^{n-1}$.

**Proof.** It suffices to prove this for $n = 5$, as the general case follows from a simple inductive argument. For $i, j \in \{0, 1\}$, let $S_{ij} \subseteq \{0, 1\}^3$ be the restriction of $S$ obtained after $i$-restricting coordinate 4 and $j$-restricting coordinate 5. We may assume that $|S_{00}| + |S_{10}| \leq 7$ and $|S_{00}| \leq 3$. After a possible twisting of coordinates 1, 2, 3, we may assume that $\{000, 111\} \subseteq S_{00} \subseteq \{000, 111, 110\}$. This implies that $\{001, 101, 011\} \subseteq S_{10}$. Since $|S_{00}| + |S_{10}| \leq 7$, we get that $S_{00} = \{000, 111, 110\}$ and therefore $S_{10} = \{001, 101, 011, 110\}$. Since
every infeasible point of $S$ has at most two infeasible neighbors, it follows that $\{100, 010, 001, 101, 011\} \subseteq S_{01}$ and $\{000, 100, 010\} \subseteq S_{11}$, implying in turn that $|S_{01}| + |S_{11}| \geq 8$. In fact, as every infeasible point of $S$ has at most two infeasible neighbors, $|S_{01}| + |S_{11}| > 8$, so $|S| \geq 7 + 9 = 16$, as required.

Recall that $R_{1,1} = \{000, 110, 101, 011\}$. Using Lemma 2.4, we prove the following:

**Lemma 2.5.** Take an integer $n \geq 5$ and a nonempty set $S \subseteq \{0,1\}^n$, where every infeasible component is a hypercube or has maximum degree at most two. If $S$ has no $R_{1,1}$ restriction and one of its infeasible components is a hypercube of dimension at least 3, then

- $|S| \geq 2^{n-1}$, and
- if $|S| = 2^{n-1}$, then $S$ is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.

**Proof.** We will prove this by induction on $n \geq 5$. The base case $n = 5$ is clear. For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the $i$-restriction of $S$ over coordinate $n$. If one of $S_0, S_1$ is empty, then the other one must be $\{0,1\}^{n-1}$, so $S$ is a hypercube of dimension $n-1$ and the induction step is complete. We may therefore assume that $S_0, S_1$ are nonempty.

Assume in the first case that $S$ has an infeasible hypercube of dimension $\geq 4$ active in, say, direction $e_n$. Then both $S_0, S_1$ have infeasible hypercubes of dimension $\geq 3$. Thus by the induction hypothesis, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. By the induction hypothesis, one of the following cases holds:

- $S_0$ is a hypercube of dimension $n-2 \geq 4$: In this case, we may assume that $S_0 = \{x : x_n = 0\} = \{x : x_{n-1} = x_n = 0\}$. Since every infeasible component of $S$ is a hypercube or has maximum degree at most two, the hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is either (all) feasible or infeasible. Since $|S_1| = 2^{n-2}$, it follows that $S \cap \{x : x_{n-1} = 0, x_n = 1\}$ is either
  $$\{x : x_{n-1} = 0, x_n = 1\} \text{ or } \{x : x_{n-1} = x_n = 1\}.$$  
  Thus, $S$ is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.

- $S_1$ is the union of two antipodal hypercubes of dimension $n-3 \geq 3$: In this case, we may assume that $S_1 = \{x : x_n = 1\} = \{x : x_{n-2} = x_{n-1}, x_n = 0\}$. Since every infeasible component of $S$ is a hypercube or has maximum degree at most two, and $|S_1| = 2^{n-2}$, it follows that $S_1 \cap \{x : x_n = 1\}$ is either
  $$\{x : x_{n-2} = x_{n-1}, x_n = 1\} \text{ or } \{x : x_{n-2} + x_{n-1} = 1, x_n = 1\}.$$  
  However, since $S$ has no $R_{1,1}$ restriction, the latter is not possible. Thus, $S = \{x : x_{n-2} = x_{n-1}\}$, so $S$ is the union of antipodal hypercubes of dimension $n-2$, thereby completing the induction step.
Assume in the remaining case that every infeasible component of \( S \) has maximum degree at most two or is a (3-dimensional) cube. By assumption, one of the infeasible components is a cube, which we may assume is contained in \( S_0 \). By the induction hypothesis, \( |S_0| \geq 2^{n-2} \) and if equality holds, then \( S_0 \) is either a hypercube of dimension \( n - 2 \) or the union of antipodal hypercubes of dimension \( n - 3 \). If \( S_1 \) has an infeasible component that is a cube, then the induction hypothesis implies that \( |S_1| \geq 2^{n-2} \), and if not, \( S_1 \) has maximum degree at most two, so by Lemma 2.4, \( |S_1| \geq 2^{n-2} \). Either way, \( |S_1| \geq 2^{n-2} \), so \( |S| = |S_0| + |S_1| \geq 2^{n-1} \). We claim that equality does not hold. Suppose for a contradiction that \( |S| = 2^{n-1} \). Then \( |S_0| = |S_1| = 2^{n-2} \). Then \( S_0 \) is either a hypercube of dimension \( n - 2 \geq 4 \) or the union of antipodal hypercubes of dimension \( n - 3 \geq 3 \). As \( S \) has no infeasible hypercube of dimension \( \geq 4 \), it follows that \( n = 6 \) and \( S_0 \) is the union of antipodal cubes, say
\[
S \cap \{x : x_6 = 0\} = \{x : x_4 = x_5, x_6 = 0\},
\]
and so
\[
S \cap \{x : x_6 = 1\} = \{x : x_4 + x_5 = 1, x_6 = 1\}
\]
as \( |S_1| = 2^{n-2} = 16 \). But then \( S \) has an \( R_{1,1} \) restriction, a contradiction to our assumption. This completes the induction step.

Recall the sets \( R_{2,1}, R_5 \) displayed in Figure 2. We need the following last ingredient:

**Theorem 2.6** ([4]). Up to isomorphism, \( R_{1,1}, R_{2,1}, R_5 \) are the only strictly non-polar sets where every infeasible point has at most two infeasible neighbors.

We are now ready to prove Theorem 1.7, stating that up to isomorphism, \( R_{1,1}, R_{2,1}, R_5 \) are the only 2-resistant strictly non-polar sets:

**Proof of Theorem 1.7.** We know that \( R_{1,1}, R_{2,1}, R_5 \) are strictly non-polar sets, and since their infeasible components have maximum degree at most two, they are 2-resistant by Theorem 1.4 (iii). To prove that they are, up to isomorphism, the only 2-resistant strictly non-polar sets, pick an integer \( n \geq 1 \) and a 2-resistant set \( S \subseteq \{0, 1\}^n \) without an \( R_{1,1}, R_{2,1}, R_5 \) restriction. It suffices to show that \( S \) is polar. By Theorem 1.4 (iii), every infeasible component is a hypercube or has maximum degree at most two. If \( S \) has maximum degree at most two, then by Theorem 2.6, \( S \) is polar. Otherwise, \( S \) has an infeasible hypercube of dimension at least 3. If \( n = 4 \) or \( S = \emptyset \), then \( S \) is clearly polar. Otherwise, \( n \geq 5 \) and \( S \neq \emptyset \). By Lemma 2.5, \( |S| \geq 2^{n-1} \); if equality holds, then \( S \) is either a hypercube or the union of antipodal hypercubes, so \( S \) is clearly polar. Otherwise, \( |S| > 2^{n-1} \), implying in particular that there are antipodal feasible points, so \( S \) is polar, as required.

### 3 A co-NP characterization of ±1-resistant sets

Let us start with the following obvious remark:

**Remark 3.1.** If a set is ±1-resistant, then so is every restriction of it.
The class of ±1-resistant sets turns out to be closed under projections as well, but the reason is not as straightforward as it was for 2-resistant sets. In fact, there is a subtle difference between ±1- and 2-resistance, which becomes manifest by the following example, showing that there is no ±1-resistant analogue of Corollary 2.3:

**Example.** Let $S := \{111011, 000001, 001110, 110100\} \subseteq \{0, 1\}^6$. The feasible points are at pairwise distance 4, so for every subset $X \subseteq \{0, 1\}^6$, $S \triangle X$ has no $P_3, S_3$ restriction. However, as $S - \{110100\}$ has a $P_3$ projection, $S$ is not ±1-resistant.

Nevertheless, in this section, we find the excluded minors and restrictions defining ±1-resistance. We will then see two applications, that ±1-resistance is a minor-closed property, and that it can be tested efficiently.

To start, consider the 3-dimensional sets displayed below,

![Sets](image)

written as

$$R_{1,1} = \{000, 110, 011\}$$
$$F_1 = \{000, 100, 010, 111\}$$
$$F_2 = \{000, 100, 010, 001, 111\}$$
$$F_3 = \{000, 100, 010, 001, 110\}.$$

Moreover, for each $k \geq 4$, let $F_k := \{0, e_1, e_2, e_1 + e_2, 1 - e_1 - e_2\} \subseteq \{0, 1\}^k$, which has an $F_3$ projection obtained after projecting away coordinates $4, \ldots, n$.

**Remark 3.2.** The sets $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ±1-resistant.

**Proof.** Notice that $R_{1,1} - \{000\} = P_3$, $F_1 - \{000\} \cong P_3$, $F_2 - \{111\} \cong S_3$, and that for each $k \geq 3$, $F_k - \{e_1 + e_2\}$ has an $S_3$ projection obtained after projecting away coordinates $[k] - \{1, 2, 3\}$. As a result, $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ±1-resistant. □

We are now ready to prove the following:

**Theorem 3.3.** Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:

(i) $S$ is ±1-resistant,

(ii) $S$ has no $\{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}$ restriction,

(iii) $S$ has no $\{R_{1,1}, F_1, F_2, F_3\}$ minor.
Proof. It follows from Remark 1.9 that every minor of $S$ is 1-resistant. We will use this throughout the proof. (i) $\Rightarrow$ (ii): It follows from Remarks 3.1 and 3.2 that $S$ has no $\{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}$ restriction. (ii) $\Rightarrow$ (iii): We will need the following three claims:

Claim 1. Let $R \subseteq \{0,1\}^4$ be 1-resistant. Let $N \subseteq \{0,1\}^3$ be the projection of $R$ over coordinate 4. Then the following statements hold:

(1) if $N = R_{1,1}$, then $R$ has an $R_{1,1}$ restriction,

(2) if $N = F_1$, then $R$ has an $F_1$ restriction,

(3) if $N = F_2$, then $R$ has one of $F_1, F_2$ as a restriction.

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the $i$-restriction of $R$ over coordinate 4. Notice that $R_0 \cup R_1 = N$. (1) Since $R_0$ and $R_1$ are 1-resistant, it follows that $|R_0| \in \{0,1,4\}$ and $|R_1| \in \{0,1,4\}$. Since $|R_0| + |R_1| \geq 4$, it follows that one of $R_0, R_1$ is $R_{1,1}$, so $R$ has an $R_{1,1}$ restriction. (2) We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since the 0-restriction of $R$ over coordinate 1 is 1-resistant, it follows that $010 \in R_0$. Similarly, as the 0-restriction of $R$ over coordinate 2 is 1-resistant, $100 \in R_0$. Because $R_0$ is 1-resistant, we have that $R_0 = F_1$, so $R$ has an $F_1$ restriction. (3) We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since $R$ has no $P_3, S_3$ restriction, at least two of $100, 010, 001$ must belong to $R_0$. Without loss of generality, $100, 010 \in R_0$. As $R_0$ is 1-resistant, $111 \in R_0$, so $R_0$ is either $F_1$ or $F_2$, implying in turn that $R$ has one $F_1, F_2$ as a restriction. ◊

Claim 2. Let $R \subseteq \{0,1\}^4$ be 1-resistant and have no $F_1, F_3$ restriction. If the projection of $R$ over coordinate 4 is $F_3$, then $R \cong F_4$.

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the $i$-restriction of $R$ over coordinate 4. Notice that $R_0 \cup R_1 = F_3$. Assume in the first case that $110 \in R_0 \cap R_1$. Since $100 \in R_0 \cup R_1$ and the 1-restriction of $R$ over coordinate 1 is 1-resistant, it follows that $100 \in R_0 \cap R_1$. Similarly, $001 \in R_0 \cap R_1$. After possibly twisting coordinate 4 of $R$, we may assume that $001 \in R_0$. This implies that $R_0$ is isomorphic to either $F_1$ or $F_3$, which is not the case as $R$ has no $F_1, F_3$ restriction. Assume in the remaining case that $110 \notin R_0 \cap R_1$. After possibly twisting coordinate 4 of $R$, we may assume that $110 \in R_0$ and $110 \notin R_1$. As $100 \in R_0 \cup R_1$ and the 1-restriction of $R$ over coordinate 1 is 1-resistant, it follows that $100 \in R_0$ and $100 \notin R_1$. Similarly, $010 \in R_0$ and $010 \notin R_1$. Since $R_0 \not= F_1, F_3$, it follows that $001 \notin R_0$ and so $001 \in R_1$. As $R_0$ is 1-resistant, $000 \in R_0$. Since $R$ has no $F_3$ restriction, it follows that $000 \notin R_1$, implying in turn that $R \cong F_4$, as required. ◊

Claim 3. Take an integer $k \geq 4$ and a 1-resistant set $R \subseteq \{0,1\}^{k+1}$ that has no $F_3, F_k$ restriction. If the projection of $R$ over coordinate $k + 1$ is $F_k$, then $R \cong F_{k+1}$.

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^k$ be the $i$-restriction of $R$ over coordinate $k + 1$. Then $R_0 \cup R_1 = F_k$. For $i \in \{0,1\}$, since $R_i$ is 1-resistant, it follows that $|R_i \cap \{0, e_1, e_2, e_1 + e_2\}| \neq 3$, and if
\(|R_i \cap \{0, e_1, e_2, e_1 + e_2\}| = 2\) then the two points in \(R_i \cap \{0, e_1, e_2, e_1 + e_2\}\) are adjacent. Since the restriction of \(R\) obtained after 0-restricting coordinates \(3, \ldots, k\) is not isomorphic to \(F_3\), one of the following holds:

- \(|R_0 \cap \{0, e_1, e_2, e_1 + e_2\}| = |R_1 \cap \{0, e_1, e_2, e_1 + e_2\}| = 2\): in this case, the 0-restriction of \(R\) over coordinates \([k + 1] - \{1, 2, 3, k + 1\}\) is not 1-resistant,
- \(|R_0 \cap \{0, e_1, e_2, e_1 + e_2\}| = |R_1 \cap \{0, e_1, e_2, e_1 + e_2\}| = 4\): in this case, one of \(R_0, R_1\) is \(F_k\),
- one of \(|R_0 \cap \{0, e_1, e_2, e_1 + e_2\}|, |R_1 \cap \{0, e_1, e_2, e_1 + e_2\}|\) is 2 and the other one is 4: in this case, the 0-restriction of \(R\) over coordinates \([k + 1] - \{1, 2, 3, k + 1\}\) is not 1-resistant,
- one of \(|R_0 \cap \{0, e_1, e_2, e_1 + e_2\}|, |R_1 \cap \{0, e_1, e_2, e_1 + e_2\}|\) is 0 and the other one is 4.

Thus, the last case is the only possibility. In this case, since \(R\) has no \(F_k\) restriction, it follows that \(R \cong F_{k+1}\), as required.

\[\square\]

Assume that \(S\) has an \(N \in \{R_{1,1}, F_1, F_2, F_3\}\) minor, obtained after applying \(\ell\) single projections and \(n-3-\ell\) single restrictions, for some \(\ell \in \{0, \ldots, n-3\}\). We need to show that \(S\) has one of \(R_{1,1}, \{F_k : 1 \leq k \leq n\}\) as a restriction. A repeated application of Claim 1 implies that if \(N \in \{R_{1,1}, F_1, F_2\}\), then \(S\) has one of \(\{R_{1,1}, F_1, F_2\}\) as a restriction. We may therefore assume that \(N = F_3\), and that \(S\) has no \(\{R_{1,1}, F_1, F_2\}\) restriction. If \(\ell = 0\), then \(S\) has an \(F_3\) restriction, so we are done. We may therefore assume that \(\ell \geq 1\) and \(S\) has no \(F_3\) restriction. If \(\ell = 1\), then by Claim 2, \(S\) has an \(F_3\) restriction and we are done. We may therefore assume that \(\ell \geq 2\) and \(S\) has no \(F_3, F_4\) restriction. By repeatedly applying Claim 3, we see that \(S\) has one of \(F_5, \ldots, F_n\) as a restriction, as required.

(iii) \(\Rightarrow\) (i): Assume that \(S\) is not \(\pm 1\)-resistant. Since \(S\) is 1-resistant, there exists an \(x \in S\) such that \(S \setminus \{x\}\) has an \(N \in \{P_3, S_3\}\) minor. Thus, for some point \(y \in \{0, 1\}^3\), \(S\) has an \(N \cup \{y\}\) minor. Since \(N \cup \{y\}\) is 1-resistant, it must be isomorphic to one of \(R_{1,1}, F_1, F_2, F_3\). Thus, \(S\) has one of \(\{R_{1,1}, F_1, F_2, F_3\}\) as a minor.

Thus,

**Corollary 3.4.** Take an integer \(n \geq 1\) and a set \(S \subseteq \{0, 1\}^n\). Then the following statements are equivalent:

(i) \(S\) is \(\pm 1\)-resistant,

(ii) \(S\) has none of the following restrictions:

\[
\{F : F \text{ is fragile}\} \cup \{0^k, 1^k - e_1 : k \geq 4\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\},
\]

(iii) \(S\) has none of the following minors:

\[
\{F : F \text{ is fragile}\} \cup \{R_{1,1}, F_1, F_2, F_3\}.
\]

In particular, \(\pm 1\)-resistance is a minor-closed property.
Proof. This is an immediate consequence of Theorems 3.3 and 1.10.

Corollary 3.5. Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \). Then in time \( O(n^4|S|^3) \), one can test whether or not \( S \) is \( \pm 1 \)-resistant.

Proof. We will appeal to Theorem 3.3 (ii). By Theorem 1.11, we can test in time \( O(n^4|S|^3) \) whether or not \( S \) is \( 1 \)-resistant. Finding an \( R_{1,1} \) restriction can be found as follows:

1. For every pair of points \( x, y \) of \( S \) at distance 2:
   - (a) let \( I := \{i \in [n] : x_i = y_i\} \),
   - (b) for every coordinate \( i \in I \),
     i. let \( S' \subseteq \{0, 1\}^3 \) be the restriction of \( S \) over coordinates \( I - \{i\} \) containing (the images of) \( x \) and \( y \),
     ii. if \( S' \cong R_{1,1} \), then output “\( S \) has an \( R_{1,1} \) restriction”,
   - (c) if (ii) fails for every \( i \in I \), then change the pair \( x, y \).

2. If (ii) fails for every pair \( x, y \), then output “\( S \) has no \( R_{1,1} \) restriction”.

The correctness of this algorithm is clear, and its running time is \( n\binom{|S|}{2} \times (n-2) \times n|S| \). Finding an \( \{F_k : 1 \leq k \leq n\} \) can be found as follows:

1. For \( k \in \{3, 4, \ldots, n\} \):
   - (a) for every pair of points \( x, y \) of \( S \) at distance \( k \),
     i. let \( S' \subseteq \{0, 1\}^k \) be the smallest restriction of \( S \) containing \( x \) and \( y \),
     ii. if \( k = 3 \) and \( S' \cong F_1, F_2, F_3 \), then output “\( S \) has an \( F_1, F_2, F_3 \) restriction”,
     iii. if \( k \geq 4 \) and \( S' \cong F_k \), then output “\( S \) has an \( F_k \) restriction”,
   - (b) if (ii)-(iii) fail for every pair \( x, y \) and \( k < n \), then increment \( k \).

2. If (ii)-(iii) fail for every pair \( x, y \) and \( k = n \), then output “\( S \) has no \( \{F_k : 1 \leq k \leq n\} \) restriction”.

The correctness of this algorithm follows from the fact that each one of \( \{F_k : 1 \leq k \leq n\} \) has antipodal feasible points; its running time is \( \sum_{k=3}^{n} n\binom{|S|}{2} \times n|S| = n\binom{|S|}{2} \times n|S| \times (n-2) \). Thus, by Theorem 3.3 (ii), testing whether or not \( S \) is \( \pm 1 \)-resistant can be done in time \( O(n^4|S|^3) + n\binom{|S|}{2} \times (n-2) \times n|S| + n\binom{|S|}{2} \times n|S| \times (n-2) = O(n^4|S|^3) \), as required.

4 An outline of the proof of Theorem 1.5

Theorem 1.5 on the structure of \( \pm 1 \)-resistant sets is a consequence of three results, which we summarize here. Assuming the correctness of these results, we then prove Theorem 1.5.
For an integer $k \geq 2$ recall that $A_k = \{0, 1\} \subseteq \{0, 1\}^k$, and for an integer $k \geq 3$ recall that $B_k = \{0, e_1, 1\} \subseteq \{0, 1\}^k$.

**Theorem 4.1.** Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0, 1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If $S$ is not connected, then either

- $S \cong A_k \times \{0, 1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$,
- $S \cong B_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, or
- $S$ has a $D_3$ minor.

Here, $D_3 = \{000, 100, 010, 101\} \subseteq \{0, 1\}^3$. Recall that $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\} \subseteq \{0, 1\}^4$.

![Diagram of $D_3$ and $C_8$]

**Theorem 4.2.** Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0, 1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If $S$ has a $D_3$ minor, then either

- $S \cong C_8 \times \{0, 1\}^{n-4}$, or
- $S \cong D_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$.

Here, for each integer $k \geq 4$, $D_k = \{0, e_2, 1-e_2, 1-e_2-e_3\} \subseteq \{0, 1\}^k$. See the figure below for an illustration of $D_4$ and $D_5$:

![Diagram of $D_4$ and $D_5$]

**Theorem 4.3.** Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0, 1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If $S$ is connected and has no $D_3$ minor, then either

- $S$ is a hypercube, or
- every infeasible component of $S$ is a hypercube.

As a consequence of these three results, let us prove Theorem 1.5, stating the following:
Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0,1\}^n \). Then \( S \) is \( \pm 1 \)-resistant if, and only if, one of the following statements holds:

(i) \( S \cong A_k \times \{0,1\}^{n-k} \) for some \( k \in \{2, \ldots, n\} \),

(ii) \( S \cong B_k \times \{0,1\}^{n-k} \) for some \( k \in \{3, \ldots, n\} \),

(iii) \( S \cong C_8 \times \{0,1\}^{n-4} \),

(iv) \( S \cong D_k \times \{0,1\}^{n-k} \) for some \( k \in \{3, \ldots, n\} \),

(v) \( S \) is a hypercube, or

(vi) every infeasible component of \( S \) is a hypercube, and every feasible point has at most two infeasible neighbors.

**Proof of Theorem 1.5, assuming Theorems 4.1, 4.2 and 4.3.** (\( \Rightarrow \)) Clearly, \( S \) is \( 1 \)-resistant, so by Theorem 3.3 (iii), \( S \) has no \( R_1,1, F_1, F_2, F_3 \) minor. If \( S \) is not connected and has no \( D_3 \) minor, then (i) or (ii) holds by Theorem 4.1. If \( S \) has a \( D_3 \) minor, then (iii) or (iv) holds by Theorem 4.2. Otherwise, \( S \) is connected and has no \( D_3 \) minor. If (v) holds, then we are done. Otherwise, by Theorem 4.3, then every infeasible component of \( S \) is a hypercube. We claim that (vi) holds. Suppose otherwise. Then there is a feasible point \( x \) with three infeasible neighbors \( x \triangle e_i, x \triangle e_j, x \triangle e_k \), for distinct \( i, j, k \in [n] \). Since every infeasible component is a hypercube, it follows that \( x \triangle e_i \triangle e_j, x \triangle e_j \triangle e_k, x \triangle e_k \triangle e_i \) are feasible. But then the 3-dimensional restriction of \( S \) containing \( x \triangle e_i, x \triangle e_j, x \triangle e_k \) is isomorphic to either \( R_1,1 \) or \( F_2 \), a contradiction. Hence, (vi) holds, as required.

(\( \Leftarrow \)) We will need the following claim:

**Claim.** If \( S \) is \( \pm 1 \)-resistant, then so is \( S \times \{0,1\} \).

**Proof of Claim.** By Corollary 3.4 (ii), the excluded restrictions defining \( \pm 1 \)-resistance are

\[
\{F : F \text{ is fragile}\} \cup \left\{ \{0^k, 1^k - e_1\} : k \geq 4 \right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.
\]

In particular, every excluded restriction of \( \pm 1 \)-resistance is not isomorphic to \( F \times \{0,1\} \) for any set \( F \). This proves the claim. \( \diamond \)

It can be readily checked that the sets \( \{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\} \) and \( C_8 \) are \( \pm 1 \)-resistant. Thus, after repeatedly applying the claim above, we see that the four classes (i)-(iv) are \( \pm 1 \)-resistant. It can also be readily checked that (v) is a \( \pm 1 \)-resistant class.

It remains to show that the restriction-closed class (vi) is \( \pm 1 \)-resistant. To this end, pick a set \( S \) from (vi). Suppose for a contradiction that \( S \) is not \( \pm 1 \)-resistant. By Corollary 3.4 (ii), \( S \) has one of the following restrictions:

\[
\{F : F \text{ is fragile}\} \cup \left\{ \{0^k, 1^k - e_1\} : k \geq 4 \right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.
\]

Out of these sets, \( R_{1,1} \) is the only set whose infeasible components are hypercubes. Thus, \( S \) has an \( R_{1,1} \) restriction. However, \( R_{1,1} \) has a feasible point with three infeasible neighbors, implying in turn that \( S \) has a feasible point with three infeasible neighbors, a contradiction. \( \square \)
It remains to prove Theorems 4.1, 4.2 and 4.3; they are proved in §6, §7 and §8.1, respectively.

5 Bridges

Take an integer \( n \geq 2 \). For a point \( x \in \{0, 1\}^n \) and distinct coordinates \( i, j \in [n] \) such that \( x_i = x_j = 0 \), we refer to \( \{x, x + e_i, x + e_j, x + e_i + e_j\} \) as a square that initiates at \( x \) and is active in directions \( e_i, e_j \). Two squares are parallel if they are active in the same pair of directions. Two parallel squares are neighbors if the points they initiate from are neighbors.

Take a set \( S \subseteq \{0, 1\}^n \). A bridge is a square that contains feasible points from different feasible components. Notice that a bridge contains exactly two feasible points, which are non-adjacent and belong to different feasible components. In this section, we will prove the following statement:

Take an integer \( n \geq 3 \) and let \( S \subseteq \{0, 1\}^n \) be a set that is 1-resistant and has no \( R_{1, 1}, F_1, F_2, F_3 \) minor. Then every pair of bridges are parallel.

We will need three lemmas to prove this statement.

**Lemma 5.1.** Take an integer \( n \geq 3 \) and a set \( S \subseteq \{0, 1\}^n \), where direction \( e_n \) is not active in any bridge. If \( S' \) is obtained from \( S \) after projecting away coordinate \( n \), then the feasible components of \( S \) project onto different feasible components of \( S' \).

**Proof.** For a point \( x \in \{0, 1\}^n \), denote by \( x' \subseteq \{0, 1\}^{n-1} \) the point obtained from \( x \) after dropping the \( n^{\text{th}} \) coordinate. To prove the lemma, it suffices to show that if \( K \) is a feasible component of \( S \) and \( x \in S - K \), then \( \text{dist}(x', y') \geq 2 \) for all \( y \in K \). Well, since \( x \) does not belong to the component \( K \), \( \text{dist}(x, y) \geq 2 \) for all \( y \in K \), implying in turn that

\[
\text{dist}(x', y') \geq \text{dist}(x, y) - 1 \geq 1 \quad \forall y \in K.
\]

In particular, \( x' \notin \{y' : y \in K\} \). Suppose for a contradiction that \( \text{dist}(x', y') = 1 \) for some \( y \in K \). As the inequalities above are held at equality, there must be a coordinate \( i \in [n - 1] \) such that \( y = x \triangle e_i e_n \). But then \( \{x, x \triangle e_i, x \triangle e_n, x \triangle e_i \triangle e_n\} \) would be a bridge that is active in direction \( e_n \), contrary to our assumption. Hence,

\[
\text{dist}(x', y') \geq 2 \quad \forall y \in K,
\]

as required. \( \square \)

**Lemma 5.2.** Take an integer \( n \geq 3 \) and a set \( S \subseteq \{0, 1\}^n \) that is 1-resistant and has no \( R_{1, 1}, F_1, F_2 \) restriction. Take a point \( x \in \{0, 1\}^n \) and distinct coordinates \( i, j, k \in [n] \). Then the following statements hold:

(i) If \( x \triangle e_i, x \triangle e_j, x \triangle e_k \in \overline{S} \), then \( |\{x \triangle e_i \triangle e_j, x \triangle e_j \triangle e_k, x \triangle e_k \triangle e_i\} \cap S| \leq 1 \).

(ii) If \( x \in S \) and \( \{x, x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j\} \) is a bridge, then \( \{x \triangle e_i \triangle e_k, x \triangle e_j \triangle e_k\} \cap S = \emptyset \).

(iii) If \( x \in S \) and \( \{x, x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j\} \) is a bridge, then \( |\{x \triangle e_k, x \triangle e_i \triangle e_j \triangle e_k\} \cap S| \geq 1 \).
Proof. After a possible twisting and relabeling, if necessary, we may assume that \( x = 0 \) and \( i = 1, j = 2, k = 3 \). Let \( S' \subseteq \{0,1\}^3 \) be the restriction of \( S \) obtained after 0-restricting coordinates 4, \ldots, \( n \). (i): Suppose that \( e_1, e_2, e_3 \in \overline{S} \). Assume for a contradiction that two of \( e_1 + e_2, e_2 + e_4, e_3 + e_1 \), say \( e_1 + e_2, e_2 + e_3 \) belong to \( S \). If \( e_1 + e_3 \in S \), then \( S' \) is isomorphic to one of \( P_3, S_3, R_{1,1}, F_2 \), which cannot occur as \( S \) is 1-resistant and has no \( R_{1,1}, F_2 \) restriction. Otherwise, \( e_1 + e_3 \in \overline{S} \). Since \( S' \not\equiv P_3 \) and \( S \) is 1-resistant, it follows that \( 0, e_1 + e_2 + e_3 \in S \), implying in turn that \( S' \equiv F_1 \), a contradiction as \( S \) has no \( F_1 \) restriction. (ii), (iii): Suppose that \( 0 \in S \) and \( \{0, e_1, e_2, e_1 + e_2\} \) is a bridge. Then \( e_1 + e_2 \in S \) and \( e_1, e_2 \in \overline{S} \). Let us first prove (ii), that \( \{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset \). Suppose otherwise. After possibly relabeling coordinates 1, 2, we may assume that \( e_1 + e_3 \in S \). Since \( 0, e_1 + e_2 \) are in different feasible components, it follows that \( |\{e_3, e_1 + e_2 + e_3\} \cap S| \leq 1 \). After possibly twisting coordinates 1, 2, we may assume that \( e_3 \in \overline{S} \). Since \( e_1, e_2, e_3 \in \overline{S} \), we get from (i) that \( |\{e_1 + e_2, e_2 + e_3, e_3 + e_1\} \cap S| \leq 1 \), a contradiction. Thus, \( \{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset \), so (ii) holds. Since \( S \) is 1-resistant, it follows immediately that \( \{e_3, e_1 + e_2 + e_3\} \cap S \neq \emptyset \), so (iii) holds.

Lemma 5.3. Take a set \( S \subseteq \{0,1\}^3 \) that is 1-resistant, has no \( R_{1,1}, F_1, F_2, F_3 \) minor, and in every minor, including \( S \) itself, every pair of bridges are parallel. If \( 0 \in S \) and \( \{0, e_1, e_2, e_1 + e_2\} \) is a bridge without neighboring bridges, then after possibly twisting coordinates 1 and 2, we have that \( S = \{0, e_3, e_1 + e_2, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_2 + e_4 + e_3\} \):

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1}
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array}
\]

Proof. Let \( B := \{0, e_1, e_2, e_1 + e_2\} \). As \( B \) is a bridge and \( 0 \in S, e_1 + e_2 \in S \) and \( e_1, e_2 \in \overline{S} \). It follows from Lemma 5.2 (ii) that \( e_1 + e_3, e_2 + e_3 \in \overline{S} \). By Lemma 5.2 (iii) and the fact that \( B \) has no neighboring bridge, we get that exactly one of \( e_3, e_1 + e_2 + e_3 \) belongs to \( S \). After twisting coordinates 1 and 2, if necessary, we may assume that \( e_3 \in S \) and \( e_1 + e_2 + e_3 \in \overline{S} \). Moreover, by Lemma 5.2 (ii), we have that \( \{e_1 + e_4, e_2 + e_4\} \subseteq \overline{S} \). Let \( S' \) be the 0-restriction of \( S \) over coordinate 5, which looks as follows:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3}
\includegraphics[width=0.2\textwidth]{diagram4}
\end{array}
\]

Claim 1. \( e_4 \in \overline{S} \) and \( e_1 + e_2 + e_4 \in S \).

Proof of Claim. Suppose otherwise. Since \( B \) has no neighboring bridge in \( S \), it follows from Lemma 5.2 (iii) that \( e_4 \in S \) and \( e_1 + e_2 + e_4 \in S \). If \( e_2 + e_3 + e_4 \in S \), then the 0-restriction of \( S' \) over coordinate 1 is either \( F_1 \) or \( F_3 \), which is not the case. Thus, \( e_2 + e_3 + e_4 \in \overline{S} \). Since the 0-restriction of \( S' \) over coordinate 1 is 1-resistant, it follows that \( e_3 + e_4 \in S \). As the 0-restriction of \( S' \) over coordinate 2 is not \( F_3 \), we have \( e_1 + e_3 + e_4 \in \overline{S} \).
Since the 1-restriction of $S'$ over coordinate 1 is 1-resistant, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so $S'$ looks as follows:

```
[diagram]
```

Observe however now that $F_3$ is obtained from $S'$ after projecting away coordinate 1, a contradiction. ♦

**Claim 2.** $\{e_1 + e_3, e_2 + e_3, e_4\} \subseteq \overline{S}$.

**Proof of Claim.** Suppose otherwise. After interchanging the roles of 1, 2, if necessary, we may assume that $e_1 + e_3 + e_4 \in S$. If $e_3 + e_4 \in \overline{S}$, then $\{0, e_3\}$ is a feasible component of $S'$, so the square initiating from $e_3$ and active in directions $e_1, e_4$ is a bridge of $S'$ that is not parallel to $B$, which is contrary to our assumption. Thus, $e_3 + e_4 \in S$. Since 0, $e_1 + e_2$ belong to different feasible components of $S$, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so $S'$ looks as follows:

```
[diagram]
```

Observe however that $S'$ has two non-parallel bridges, namely $B$ and the square that initiates from $e_1 + e_4$ and is active in directions $e_2, e_3$, a contradiction. ♦

**Claim 3.** $\{e_3 + e_4, e_1 + e_2 + e_3 + e_4\} \subseteq \overline{S}$.

**Proof of Claim.** Since the 0-restriction of $S'$ over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in \overline{S}$. Since the 1-restriction of $S'$ over coordinate 1 is also 1-resistant, we see that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, as required. ♦

We just determined the status of all the points in $\{x : x_5 = 0\}$. A similar argument applied to $\{x : x_4 = 0\}$ gives us the left figure below:

```
[diagram]
```

Consider the set obtained from $S$ after 1-restricting over coordinate 1 and 0-restricting over coordinate 3; since this set is 1-resistant and not isomorphic to $F_1, F_3$, we get that $e_1 + e_4 + e_5 \in \overline{S}$ and $e_1 + e_2 + e_4 + e_5 \in \overline{S}$. As the 1-restriction of $S$ over coordinates 1, 2 is not $F_3$, we get that $1 \in \overline{S}$. Now consider the set obtained from $S$ after 1-restricting coordinate 2 and 0-restricting over coordinate 3; since this set is not $F_3$, we get that $e_2 + e_4 + e_5 \in \overline{S}$. Note that $\{e_1 + e_2, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_2 + e_4 + e_5\}$ forms a feasible component
of $S$. Hence, as $S$ does not have non-parallel bridges, it follows that $e_2 + e_3 + e_4 + e_5, e_1 + e_3 + e_4 + e_5 \in \overline{S}$, and also that $e_3 + e_4 + e_5 \in \overline{S}$. (See the right figure above.) Once again, as $S$ does not have non-parallel bridges, it follows that $e_4 + e_5 \in \overline{S}$, thereby finishing the proof.

We are now ready to prove the main result of this section:

**Proposition 5.4.** Take an integer $n \geq 3$ and let $S \subseteq \{0, 1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.

**Proof.** Suppose for a contradiction that $S$ has a pair of non-parallel bridges. (In particular, $S$ is not connected.) We may assume that in every proper minor of $S$, every pair of bridges, if any, are parallel.

**Claim 1.** Every direction is active in a bridge.

**Proof of Claim.** Suppose for a contradiction that direction $e_n$ is not active in any bridge. For a point $x \in \{0, 1\}^n$, denote by $x' \subseteq \{0, 1\}^{n-1}$ the point obtained from $x$ after dropping the $n$th coordinate. Notice first that by Lemma 5.1, the feasible components of $S$ project onto different feasible components of $S'$, the subset of $\{0, 1\}^{n-1}$ obtained from $S$ after projecting away coordinate $n$. We will derive a contradiction to the minimality of $S$ by showing that $S'$ has non-parallel bridges.

We will show that if $B$ is a bridge of $S$, then $B' := \{x' : x \in B\}$ is still a bridge of $S'$ that is active in the same directions as before. Since $e_n$ is not active in any bridge of $S$, we may assume that $n \geq 3$ and $B = \{0, e_1, e_2, e_1 + e_2\}$ where $0, e_1 + e_2$ belong to different feasible components of $S$, and $e_1, e_2 \in \overline{S}$. It follows from Lemma 5.2 (ii) that $0, e_1 + e_2 \in S'$ and $e_1, e_2 \in \overline{S'}$. Moreover, since the feasible components of $S$ project onto different feasible components of $S'$, we see that $0, e_1 + e_2$ belong to different feasible components of $S'$. Thus, $B'$ is still a bridge of $S'$ that is active in the same directions as before.

As a corollary, $S'$ still has non-parallel bridges, thereby contradiciting the minimality of $S$.  

**Claim 2.** The following statements hold:

(i) if $B, B'$ are non-parallel bridges that are not active in direction $e_i$, then $\{x : x_i = 0\}$ contains one of the bridges and $\{x : x_i = 1\}$ contains the other one,

(ii) if $B, B', B''$ are non-parallel bridges, then every direction is active in one of the bridges, and

(iii) $n \in \{4, 5, 6\}$. 

**Proof of Claim.** (i) For if not, then one of the restrictions of $S$ over coordinate $i$ contains $B$ and $B'$, thereby contradicting the minimality of $S$. (ii) Suppose for a contradiction that $e_i$ is not active in any of $B, B', B''$. Then one of the hyperplanes $\{x : x_i = 0\}, \{x : x_i = 1\}$ contains at least two of $B, B', B''$, thereby contradicting (i). (iii) Let $B, B'$ be non-parallel bridges. It follows from Lemma 5.2 (ii) that $n \geq 4$. If every direction is active in one of $B, B'$, we get that $n = 4$. Otherwise, there is a direction $e_i$ inactive in both $B, B'$. By Claim 1, there is a bridge $B''$ active in $e_i$. Clearly, $B, B', B''$ are pairwise non-parallel bridges. It now follows from (ii) that $n \leq 6$, as required.
Claim 3. \( n \neq 4 \).

Proof of Claim. Suppose for a contradiction that \( n = 4 \). Let \( B, B' \) be non-parallel bridges of \( S \). We may assume that \( B = \{0, e_1, e_2, e_1+e_2\}, 0, e_1+e_2 \in S \) and \( e_1, e_2 \in \overline{S} \). By Lemma 5.2 (ii), \( e_1+e_3, e_2+e_3, e_1+e_4, e_2+e_4 \in \overline{S} \):

Assume in the first case that \( B' \) shares an active direction with \( B \). After possibly relabeling coordinates 1, 2, we may assume that \( B' \) is active in directions \( e_1, e_3 \). It follows from Claim 2 (i) that \( B' \) is contained in \( \{x : x_4 = 1\} \). After possibly twisting coordinates 1, 2, we may assume that \( B' = \{e_4, e_1+e_4, e_3+e_4, e_1+e_3+e_4\} \). Since \( e_1+e_4 \in \overline{S} \), it follows that \( e_4, e_1+e_4, e_3+e_4 \in S \) and \( e_3+e_4 \in \overline{S} \). Applying Lemma 5.2 (ii), we get that \( e_3, e_2+e_3+e_4, e_1+e_2+e_4 \in \overline{S} \). Since the two restrictions of \( S \) over coordinate 4 are 1-resistant, it follows that \( e_1+e_2+e_3, 1 \in S \):

Observe, however, that 1-restricting \( S \) over coordinate 3 yields a set that is not 1-resistant, a contradiction.

Assume in the remaining case that \( B' \) is active in directions \( e_3, e_4 \). Observe that \( B' \) is not contained in \( \{x : x_1 + x_2 = 1\} \). After possibly twisting coordinates 1, 2, we may assume that \( B' \) initiates from 0. This means that \( e_3, e_4 \in \overline{S} \) and \( e_3+e_4 \in S \). Applying Lemma 5.2 (iii), we get that \( e_1+e_2+e_4 \in S \) and \( e_1+e_3+e_4, e_2+e_3+e_4 \in \overline{S} \):

The 1-restriction of \( S \) over coordinate 4, however, is isomorphic to either \( F_1 \) or \( F_3 \), a contradiction.

Thus, we have that \( n \in \{5, 6\} \). It follows from Claim 1 that there are \( \left\lceil \frac{n}{2} \right\rceil = 3 \) pairwise non-parallel bridges \( B_1, B_2, B_3 \). We get from Claim 2 (ii) that, after a possible relabeling, \( B_1 \) is active in \( e_1, e_2 \), \( B_2 \) is active in \( e_3, e_4 \), and

- if \( n = 5 \), then \( B_3 \) is active in \( e_3, e_5 \),
- if \( n = 6 \), then \( B_3 \) is active in \( e_5, e_6 \).

We can further say that,

Claim 4. If \( B \) is a bridge different from \( B_1, B_2, B_3 \), then \( n = 5 \).
Proof of Claim. Suppose for a contradiction that $n = 6$. It follows from Claim 2 (ii) that $B$ is parallel to one of $B_1, B_2, B_3$. Consider the bridge $B_2$. Since $B_2, B_3$ are inactive in $e_1, e_2$, it follows from Claim 2 (i) that the hyperplanes $\{x : x_1 = 0\}, \{x : x_2 = 0\}$ split $B_2, B_3$. Moreover, since $B_2, B_1$ are inactive in $e_5$, the hyperplane $\{x : x_5 = 0\}$ splits $B_2, B_1$. Hence, the residing square of $B_2$ – and any bridge parallel to it – is uniquely determined once $B_1$ and $B_3$ are given, implying that $B$ is not parallel to $B_2$. By the symmetry between $B_1, B_2, B_3$, we get that $B$ is not parallel to $B_1, B_3$ either, a contradiction.

Claim 5. $n \neq 5$.

Proof of Claim. Suppose for a contradiction that $n = 5$. After twisting coordinates 3, 4, 5, if necessary, we may assume that $B_1$ initiates at 0. By Claim 2 (i), and after possibly twisting coordinates 1, 2, we may assume that $B_2$ initiates at $e_5$. Another application of Claim 2 (i) tells us that $B_3$ initiates at $e_1 + e_2 + e_4$:

![Diagram of bridge](image)

Assume in the first case that $0, e_1 + e_2 \in \overline{S}$ and $e_1, e_2 \in S$. Then a repeated application of Lemma 5.2 (ii) tells us that $e_3, e_1 + e_2 + e_3, e_5, e_1 + e_2 + e_5, e_4, e_1 + e_2 + e_4 \in \overline{S}$. As a result, in the bridge $B_2$, we have that $e_3 + e_5, e_4 + e_5 \in S$:

![Diagram of bridge](image)

Observe now that the restriction of $S$ obtained after 0-restricting coordinates 1 and 2 is not 1-resistant, a contradiction.

Assume in the remaining case that $0, e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. A repeated application of Lemma 5.2 (ii) to $B_1$, followed by an application of it to $B_2, B_3$ gives us the left figure below:
Applying Lemma 5.2 (ii) to $B_2, B_3$ gives us the right figure above, thereby yielding a contradiction as 0-restricting coordinates 4, 5 of $S$ yields a set that is not 1-resistant. This finishes the proof of the claim. ♦

Thus, $n = 6$. After twisting coordinates 3, 4, 5, 6, if necessary, we may assume that $B_1$ initiates at 0. Applying Claim 2 (i), we see that after possibly twisting coordinates 1, 2, we may assume that $B_2$ initiates at $e_5 + e_6$. Using Claim 2 (i), we see that $B_3$ must initiate at $e_1 + e_2 + e_3 + e_4$:

Recall from Claim 4 that $B_1, B_2, B_3$ are the only bridges of $S$. Let $S' \subseteq \{0, 1\}^5$ be the restriction of $S$ obtained after 0-restricting coordinate 6. By assumption, every minor of $S'$ has only parallel bridges. As a bridge in $S'$ is not necessarily a bridge in $S$, we see that $S'$ may have bridges other than $B_1$ (that will necessarily be parallel to it).

Claim 6. $B_1$ does not have a neighboring bridge in $S'$.

Proof of Claim. Suppose for a contradiction that $B_1$ has a neighboring bridge $B$ in $S'$. Since $B$ is not a bridge of $S$ by Claim 4, it follows that the points in $B \cap S'$ are in the same feasible component of $S$. After applying Lemma 5.2 (ii) to $B_1$, we see that the points in $B_1 \cap S'$ also lie in this feasible component of $S$, a contradiction. ♦

We may now apply Lemma 5.3 to the bridge $B_1$ of $S'$. Depending on which points of $B_1$ are in $S'$, and how coordinates 1, 2 are twisted, we get that $S'$ takes on one of the four possibilities shown below:
Consider the 3-dimensional restriction $F$ of $S$ containing $B_2$ and $B_2 \triangle e_6$. If $S'$ takes one of the top-left, bottom-left or bottom-right possibilities, then $F$ is not 1-resistant, which is not possible. Otherwise, $S'$ takes the top-right possibility, in which case $F \cong F_1$, a contradiction. This finally finishes the proof of Proposition 5.4.

\section{Proof of Theorem 4.1}

Take an integer $n \geq 2$ and a set $S \subseteq \{0, 1\}^n$. We say that $S$ is \textit{separable} if there exist a partition of $S$ into nonempty parts $S_1, S_2$ and distinct coordinates $i, j \in [n]$ such that either $S_1 \subseteq \{x : x_i = 0, x_j = 1\}$ and $S_2 \subseteq \{x : x_i = 1, x_j = 0\}$, or $S_1 \subseteq \{x : x_i = x_j = 0\}$ and $S_2 \subseteq \{x : x_i = x_j = 1\}$. Notice that if $S$ is separable, then it is not connected.

\textbf{Remark 6.1.} Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. If a projection of $S$ is separable, then so is $S$.

We will need the following:

\textbf{Proposition 6.2.} Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0, 1\}^n$. Suppose there is a partition of $S$ into nonempty parts $S_1, S_2$ such that $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. Then $S_1$ and $S_2$ are hypercubes. In particular, $S$ is polar.

\textbf{Proof.} The hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is infeasible. As $S$ is 1-resistant, Lemma 1.12 implies that in each of the parallel hypercubes $\{x : x_{n-1} = x_n = 0\}$ and $\{x : x_{n-1} = x_n = 1\}$, the feasible points form a hypercube. That is, the two sets

\[ S \cap \{x : x_{n-1} = x_n = 0\} = S_1, \]
\[ S \cap \{x : x_{n-1} = x_n = 1\} = S_2 \]

are hypercubes. We leave it as an easy exercise for the reader to check that $S$ is polar. \hfill $\square$

We are now ready to prove Theorem 4.1, stating the following:

Take an integer $n \geq 2$ and a set $S \subseteq \{0, 1\}^n$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and is not connected. Then either

- $S \cong A_k \times \{0, 1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, where $A_k = \{0^k, 1^k\}$,
- $S \cong B_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, where $B_k = \{0^k, e_1, 1^k\}$, or
- $S$ has a $D_3$ minor, where $D_3 = \{000, 010, 100, 101\}$.

\textbf{Proof.} Let us start with the following claim:

\textbf{Claim 1.} $S$ is separable.
Proof of Claim. Let \( k \geq 2 \) be the number of feasible components of \( S \). Let \( S' \subseteq \{0, 1\}^m \) be a projection of \( S \) of smallest dimension with exactly \( k \) feasible components. It then follows from Lemma 5.1 that every direction of \( \{0, 1\}^m \) is active in a bridge of \( S' \). However, as \( S' \) is 1-resistant and has no \( R_{1,1}, F_1, F_2, F_3 \) minor, Proposition 5.4 implies that every pair of bridges of \( S' \) are parallel. As a result, \( m = k = 2 \) and \( S' \) is either \( \{00, 11\} \) or \( \{10, 01\} \). In particular, \( S' \) is separable, so \( S \) is separable by Remark 6.1.

Thus, there is a partition of \( S \) into nonempty parts \( S_1, S_2 \) such that, after a possible twisting and relabeling, \( S_1 \subseteq \{x : x_{n-1} = x_n = 0\} \) and \( S_2 \subseteq \{x : x_{n-1} = x_n = 1\} \). As \( S \) is 1-resistant, Proposition 6.2 implies that \( S_1 \) and \( S_2 \) are hypercubes, and that \( S \) is polar. In particular, since \( S \) is not a hypercube, Lemma 1.12 implies that the points in \( S \) do not agree on a coordinate; notice that this property is preserved in every projection of dimension at least one.

Claim 2. Either \( S \) has a \( D_3 \) minor, or one of \( S_1, S_2 \) is contained in the antipode of the other.

Proof of Claim. Suppose that neither of \( S_1, S_2 \) is contained in the antipode of the other. We will prove that \( S \) has a \( D_3 \) projection. Clearly, \( n > 2 \). We may assume that for each \( i \in [n-2] \),

\[
\text{if } S', S'_1, S'_2 \text{ are obtained from } S, S_1, S_2 \text{ after projecting away coordinate } i, \text{ then one of } S'_1, S'_2 \text{ is contained in the antipode of the other.}
\]

As the points in the polar set \( S \) do not agree on a coordinate, there exists a point \( x \in S_1 \) such that \( 1 - x \in S_2 \). As neither of \( S_1, S_2 \) is contained in the antipode of the other, there exist distinct coordinates \( i, j \in [n-2] \) such that \( x \Delta e_i \in S_1, x \Delta e_j \notin S_1, 1 \Delta x \Delta e_i \notin S_2 \) and \( 1 \Delta x \Delta e_j \in S_2 \). Our minimality assumption implies that the only feasible neighbors of \( x, 1 \Delta x \) are \( x \Delta e_i, 1 \Delta x \Delta e_j \), respectively. As a result, \( S_1 = \{x, x \Delta e_i\} \) and \( S_2 = \{1 \Delta x, 1 \Delta x \Delta e_j\} \), so \( S = \{x, x \Delta e_i, 1 \Delta x, 1 \Delta x \Delta e_j\} \). Clearly, \( S \) has a \( D_3 \) projection.

If \( S \) has a \( D_3 \) minor, then we are done. Otherwise, one of \( S_1, S_2 \) is contained in the antipode of the other. After possibly relabeling \( S_1, S_2 \), we may assume that \( S_2 \) is contained in the antipode of \( S_1 \).

Claim 3. \( 2|S_2| \geq |S_1| \geq |S_2| \).

Proof of Claim. Clearly, \( |S_1| \geq |S_2| \). Suppose for a contradiction that \( |S_1| \geq 4|S_2| \). Since \( S_2 \) is contained in the antipode of \( S_1 \), it can be readily checked that \( S \) has an \( F_3 \) minor, a contradiction.

As a result, either \( |S_1| = |S_2| \) or \( |S_1| = 2|S_2| \). It can now be readily checked that either \( S \cong A_k \times \{0, 1\}^{n-k} \) for some \( k \in \{2, \ldots, n\} \), or \( S \cong B_k \times \{0, 1\}^{n-k} \) for some \( k \in \{3, \ldots, n\} \), thereby finishing the proof of Theorem 4.1. \( \square \)

7 \( D_3 \) minors and proof of Theorem 4.2

To prove Theorem 4.2 we will need three lemmas. Let \( D^*_3 := \{010, 011, 111, 101\} \subseteq \{0, 1\}^3 \). Observe that \( D^*_3 \) is a twisting of \( D_3 = \{000, 100, 010, 101\} \), and \( C_8 = (D_3 \times \{0\}) \cup (D^*_3 \times \{1\}) \).
In the following lemma, we will use the following implication of Lemma 5.2 (i):

![Diagram]

**Lemma 7.1.** Let $S \subseteq \{0, 1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, where the 0-restriction of $S$ over coordinates $4, \ldots, n$ is either $D_3$ or $D_3^*$. Then,

(i) every restriction of $S$ over coordinates $4, \ldots, n$ is either $D_3$ or $D_3^*$, and

(ii) either $S \cong D_3 \times \{0, 1\}^{n-3}$ or $S \cong C_8 \times \{0, 1\}^{n-4}$.

**Proof.** (i) By a recursive argument, it suffices to show that each 3-dimensional restriction of $S$ neighboring a $D_3, D_3^*$ restriction is also a $D_3$ or a $D_3^*$. Thus, we may assume that $n = 4$. After twisting coordinates 1, 2, 3, if necessary, we may assume that the 0-restriction of $S$ over coordinate 4 is $D_3$. So $S \cap \{x : x_4 = 0\} = \{0000, 1000, 0100, 1010\}$:

![Diagram]

Assume in the first case that $\{0111, 1111\} \cap \overline{S} \neq \emptyset$. After applying Lemma 5.2 (i) twice, we see that $\{0111, 1111, 0011, 1101\} \subseteq \overline{S}$:

![Diagram]

Since the two restrictions over coordinate 1 are 1-resistant, $|\{0101, 0001\} \cap S| \neq 1$ and $|\{1001, 1011\} \cap S| \neq 1$. In fact, as $S$ has no $F_3$ minor, $\{0101, 0001\} \subseteq S$ if and only if $\{1001, 1011\} \subseteq S$. Moreover, as the 0-restriction of $S$ over coordinate 3 is 1-resistant, it follows that $\{0101, 0001, 1001, 1011\} \cap S \neq \emptyset$. As a result, $\{0101, 0001, 1001, 1011\} \subseteq S$, implying in turn that 1-restricting $S$ over coordinate 4 yields $D_3$.

Assume in the remaining case that $\{0111, 1111\} \cap \overline{S} = \emptyset$. As the 1-restriction of $S$ over coordinate 3 (resp. coordinate 2) is not isomorphic to either of $F_1, F_3$, we get that $0011 \in \overline{S}$ (resp. $1101 \in \overline{S}$).
Since $S$ has no $F_1, F_3, S_3$ restrictions, it follows that 0001, 1001 $\in S$. Since the 0-restriction of $S$ over coordinate 2 (resp. coordinate 3) is 1-resistant, 1011 $\in S$ (resp. 0101 $\in S$), implying in turn that 1-restricting $S$ over coordinate 4 yeilds $D_3^\star$.

(ii) It follows from (i) that $S = \bigcup_{y \in \{0, 1\}^{n-3}} (F \times \{y\} : F \in \{D_3, D_3^\star\})$. Let $R \subseteq \{0, 1\}^{n-3}$ be the set of points $y$ such that $S \cap \{x : x_i = y_{i-3} \quad 4 \leq i \leq n\} = D_3 \times \{y\}$.

**Claim 1.** Every feasible component of $R$ is a hypercube. Similarly, every infeasible component of $R$ is a hypercube.

**Proof of Claim.** By Lemma 1.13, it suffices to prove that for each $y \in R$ and distinct coordinates $i, j \in [n - 3]$, if $y, y \triangle e_i, y \triangle e_j \in R$ then $y \triangle e_i \triangle e_j \in R$.

Suppose otherwise. After a possible twisting and relabeling, we may assume that $y = 0, i = 1, j = 2$. Let $S'$ be the 0-restriction of $S$ over coordinates 6, $\ldots$, $n$:

$$
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure1.png}
\end{array}
$$

Observe that the 0-restriction of $S'$ over coordinates 1, 2 is not 1-resistant, a contradiction. $\diamond$

**Claim 2.** $R$ is connected. Similarly, $\overline{R}$ is connected.

**Proof of Claim.** Suppose for a contradiction that $R \subseteq \{0, 1\}^{n-3}$ is not connected. By Claim 1, every feasible component of $R$ is a hypercube, and as there are at least two feasible components, each feasible component is a hypercube of dimension at most $(n - 3) - 2 = n - 5$. Thus, there exist $y \in \{0, 1\}^{n-3}$ and distinct coordinates $i, j \in [n - 3]$ such that $y \in R$ and $y \triangle e_i, y \triangle e_j \in \overline{R}$. Since every infeasible component of $R$ is also a hypercube by Claim 1, it follows that $y \triangle e_i \triangle e_j \in R$. After a possible twisting and relabeling, we may assume that $y = 0, i = 1, j = 2$. Let $S'$ be the 0-restriction of $S$ over coordinates 6, $\ldots$, $n$:

$$
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure2.png}
\end{array}
$$

Observe however that the 0-restriction of $S'$ over coordinates 1, 2 is not 1-resistant, a contradiction. $\diamond$
As a result, both $R, \overline{R}$ are hypercubes, implying in turn that $R \cong \emptyset, \{0, 1\}^{n-4} \times \{0\}, \{0, 1\}^{n-3}$. If $R \cong \emptyset, \{0, 1\}^{n-3}$ then $S \cong D_3 \times \{0, 1\}^{n-3}$, and if $R \cong \{0, 1\}^{n-4} \times \{0\}$ then $S \cong C_8 \times \{0, 1\}^{n-4}$, thereby finishing the proof. 

For each $k \geq 4$, recall that $D_k = \{0, e_2, 1 - e_2, 1 - e_2 - e_3\} \subseteq \{0, 1\}^k$, and let $D^*_k := D_k \triangle e_k$.

**Lemma 7.2.** Take integers $n \geq 3$ and $k \in \{3, \ldots, n\}$. Let $S \subseteq \{0, 1\}^{n+1}$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then the following statements hold:

(i) if the projection of $S$ over coordinate $n + 1$ is $D_n$, then $S$ is either $D_{n+1}, D^*_n \cup D_n \times \{0, 1\}$,

(ii) if the projection of $S$ over coordinate $k + 1$ is $D_k \times \{0, 1\}^{n-k}$, then $S$ is either $D_{k+1} \times \{0, 1\}^{n-k}, D^*_k \times \{0, 1\}^{n-k+1}$.

**Proof.** (i) Assume that the projection of $S$ over coordinate $n + 1$ is $D_n$. Let

$$ S_0 := S \cap \{x : x_i = 0, i \neq 2, 3, n + 1\} \subseteq \{0, 1\}^{n+1}, $$

$$ S_1 := S \cap \{x : x_i = 1, i \neq 2, 3, n + 1\} \subseteq \{0, 1\}^{n+1}. $$

Let $1 := 1^{n+1}$ and $1' := 1^n$. Then

- $S = S_0 \cup S_1$,
- $S_0 \subseteq \{0, e_2, e_{n+1}, e_2 + e_{n+1}\}$, and the projection of $S_0$ over coordinate $n + 1$ is $\{0, e_2\}$, and
- $S_1 \subseteq \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}, 1 - e_2, 1 - e_2 - e_3\}$, and the projection of $S_1$ over coordinate $n + 1$ is $\{1' - e_2, 1' - e_2 - e_3\}$.

After twisting coordinate $n + 1$, if necessary, we may assume that $0 \in S_0$. Then, since $S_0$ and $S_1$ are 1-resistant, we get that

$$ S_0 = \{0, e_2\} \quad \text{or} \quad \{0, e_2, e_{n+1}, e_2 + e_{n+1}\}, \quad \text{and} $$

$$ S_1 = \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}\} \quad \text{or} \quad \{1 - e_2, 1 - e_2 - e_3\} \quad \text{or} $$

$$ \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}, 1 - e_2, 1 - e_2 - e_3\}. $$

**Claim 1.** If $S_0 = \{0, e_2\}$, then $S = D_{n+1}$.

**Proof of Claim.** Suppose that $S_0 = \{0, e_2\}$.

Assume in the first case that $n = 3$. If $S_1 = \{1 - e_2 - e_4, 1 - e_2 - e_3 - e_4\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 3 is not 1-resistant, which is not the case. If $S_1 = \{1 - e_2 - e_4, 1 - e_2 - e_3 - e_4, 1 - e_2, 1 - e_2 - e_3\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 2 is isomorphic to $F_3$, which is again not the case. Therefore, $S_1 = \{1 - e_2, 1 - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_4$, as claimed.

Assume in the remaining case that $n \geq 4$. If $S_1 = \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}\}$, then the points in $S = S_0 \cup S_1$ all agree on coordinate $n + 1$, so by Lemma 1.12, $S$ is a hypercube, which is not the case. If
$S_1 = \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}, 1 - e_2, 1 - e_2 - e_3\}$, then the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is isomorphic to $F_3$, which cannot be the case. Therefore, $S_1 = \{1 - e_2, 1 - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_{n+1}$, as claimed. $
abla$

Claim 2. If $S_0 = \{0, e_2, e_{n+1}, e_2 + e_{n+1}\}$, then $S = D_n \times \{0, 1\}$.

Proof of Claim. Suppose that $S_0 = \{0, e_2, e_{n+1}, e_2 + e_{n+1}\}$. As the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is not isomorphic to $F_3$, it follows that $S_1 = \{1 - e_2 - e_{n+1}, 1 - e_2 - e_3 - e_{n+1}, 1 - e_2, 1 - e_2 - e_3\}$, implying in turn that $S = D_n \times \{0, 1\}$, as required. $
abla$

Thus, after twisting coordinate $n + 1$, if necessary, $S$ is either $D_{n+1}$ or $D_n \times \{0, 1\}$, so (i) holds.

(ii) Assume that the projection of $S$ over coordinate $k + 1$ is $D_k \times \{0, 1\}^{n-k}$. For each point $y \in \{0, 1\}^n$, let $S_y := S \cap \{x : x_{i+k+1} = y_i, i \in [n-k] \} \subseteq \{0, 1\}^{n+1}$. Notice that $S = \bigcup_{y \in \{0, 1\}^n} S_y$. For each $y \in \{0, 1\}^{n-k}$, pick an appropriate $S'_y \subseteq \{0, 1\}^{k+1}$ such that $S_y = S'_y \times \{y\}$. Notice that the projection of each $S'_y$ over coordinate $k + 1$ is $D_k$. We therefore get from (i) that each $S'_y$ is either $D_{k+1}$, $D^*_k$ or $D_k \times \{0, 1\}$. 

Claim 3. All of $(S'_y : y \in \{0, 1\}^{n-k})$ are equal to one another.

Proof of Claim. Suppose otherwise. Then there exists $y_1, y_2 \in \{0, 1\}^{n-k}$ such that $\text{dist}(y_1, y_2) = 1$ and $S'_{y_1} \neq S'_{y_2}$. In particular, $S$ has either $S' := (D_{k+1} \times \{0\}) \cup (D_k \times \{01, 11\})$ or $S'' := (D_{k+1} \times \{0\}) \cup (D^*_k \times \{1\})$ as a restriction. However, the restriction of $S'$ (resp. $S''$) obtained after 0-restricting coordinates $[n+1] - \{3, k+1, k+2\}$ is not 1-resistant, so $S$ cannot have either of $S'$, $S''$ as a restriction, a contradiction. $
abla$

As a consequence, $S = D_{k+1} \times \{0, 1\}^{n-k}, D^*_{k+1} \times \{0, 1\}^{n-k}$ or $D_k \times \{0, 1\}^{n-k+1}$, so (ii) holds. $
abla$

Lemma 7.3. Take an integer $n \geq 5$ and a set $S \subseteq \{0, 1\}^n$ that is 1-resistant and has no $R_{1, 1}, F_1, F_2, F_3$ minor. If the projection of $S$ over coordinate $n$ is $C_8 \times \{0, 1\}^{n-5}$, then $S = C_8 \times \{0, 1\}^{n-4}$.

Proof. It suffices to prove this for $n = 5$. Assume that the projection of $S$ over coordinate 5 is $C_8 = (D_3 \times \{0\}) \cup (D^*_3 \times \{1\})$. For $i, j \in \{0, 1\}$, let $S_{ij} \subseteq \{0, 1\}^3$ be the restriction of $S$ obtained after $i$-restricting coordinate 4 and $j$-restricting coordinate 5. After twisting coordinate 5, if necessary, we may assume that $0 \in S$.

Claim. $S$ has a $D_3$ restriction.

Proof of Claim. Suppose for a contradiction that $S$ does not have a $D_3$ restriction. In particular, $S_{00}, S_{01} \neq D_3$ and $S_{10}, S_{11} \neq D^*_3$. Thus by Lemma 7.2 (i),

$$(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4 \text{ or } D^*_4,$$

$$(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D'_4 \text{ or } D'_4 \triangle e_4,$$

where $D'_4 = \{0100, 0110, 1011, 1111\} \subseteq \{0, 1\}^4$. Since $0 \in S$, we must have that $(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4$. Thus, $S_{00} = \{000, 010\}$ and $S_{01} = \{100, 101\}$. Since the restriction of $S$ obtained after 0-restricting coordinates 1 and 5 is not isomorphic to $D_3$, it follows that $(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D'_4 \triangle e_4$. So, $S_{10} = \{101, 111\}$ and $S_{11} = \{010, 011\}$.
Observe however that the 1-restriction of $S$ over coordinates 2, 3 is not 1-resistant, a contradiction.

Thus, $S \cong D_3 \times \{0,1\}^2$ or $C_8 \times \{0,1\}$ by Lemma 7.1 (ii). It can be readily checked that $S$ must be in fact equal to $C_8 \times \{0,1\}$, as required.

We are now ready to prove Theorem 4.2, stating the following:

Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_1, R_1, F_1, F_2, F_3$ minor. If $S$ has a $D_3$ minor, then either

- $S \cong C_8 \times \{0,1\}^{n-4}$, or
- $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3,\ldots,n\}$.

**Proof.** Among all projections of $S$ with a $D_3$ restriction, pick the one $S' \subseteq \{0,1\}^{\ell}$ of largest dimension $\ell \in \{3,\ldots,n\}$. We may assume, after a possible relabeling, that $S'$ is obtained from $S$ after projecting away coordinates $[n] - [\ell]$. It follows from Lemma 7.1 (ii) that, after a possible twisting and relabeling, $S' = C_8 \times \{0,1\}^{\ell-4}$ or $S' = D_3 \times \{0,1\}^{\ell-3}$.

**Claim.** If $S' = C_8 \times \{0,1\}^{\ell-4}$, then $\ell = n$.

**Proof of Claim.** This follows immediately from Lemma 7.3 and the maximal choice of $S'$.

Thus, if $S' = C_8 \times \{0,1\}^{\ell-4}$, then $S \cong C_8 \times \{0,1\}^{n-4}$. Otherwise, $S' = D_3 \times \{0,1\}^{\ell-3}$. In this case, a repeated application of Lemma 7.2 (ii) implies that $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{\ell,\ldots,n\}$, thereby finishing the proof of Theorem 4.2.

8 Infeasible hypercubes and Theorem 4.3

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. In this section, we will prove the following statement:

Assume that is $S$ is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and no $D_3$ minor. Take a point $x$ and distinct coordinates $i, j \in [n]$ such that $x$ is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing $x$ is a hypercube.
Proof. Suppose, for a contradiction, that
\[ S \subseteq \{0,1\}^3, H_1 := \{100,010,101,011\} \subseteq \{0,1\}^3, H_2 := \{100,010,101,011,110\} \subseteq \{0,1\}^3, H'_2 := \{100,010,101,011,111\} \subseteq \{0,1\}^3 \text{ and } \]
\[ H_3 := \{100,010,101,011,110,111\} \subseteq \{0,1\}^3, \]
as displayed below:

Given \( i \in \{0,1\} \), denote by \( S_i \subseteq \{0,1\}^{n-1} \) the \( i \)-restriction of \( S \) over coordinate \( n \).

**Lemma 8.1.** Let \( S \subseteq \{0,1\}^4 \) be a set that is 1-resistant and has no \( R_{1,1}, F_1, F_2, F_3, D_3 \) minor. If \( S_0 \in \{H_1, H_2, H'_2, H_3\} \), then \( |\{000,001\} \cap S_1| \neq 1 \).

**Proof.** Suppose, for a contradiction, that \( H_1 \subseteq S_0 \subseteq H_3 \) and \( |\{000,001\} \cap S_1| = 1 \). After twisting coordinate 3, if necessary, we may assume that 000 \( \in S_1 \) and 001 \( \notin S_1 \). So \( S \) may be displayed as below:

Since the 0-restriction of \( S \) over coordinate 1 is not isomorphic to either \( F_1 \) or \( F_3 \), we get that 011 \( \in \overline{S_1} \), and since this restriction is not isomorphic to \( D_3 \), we get that 010 \( \in \overline{S_1} \). By the symmetry between coordinates 1, 2, we get that \( \{100,101\} \subseteq \overline{S_1} \). But then the 0-restriction of \( S \) over coordinate 3 is isomorphic to either \( P_3, R_{1,1}, F_1 \) or \( F_2 \), a contradiction. \( \square \)

**Lemma 8.2.** Let \( S \subseteq \{0,1\}^4 \) be a set that is 1-resistant and has no \( R_{1,1}, F_1, F_2, F_3, D_3 \) minor, where \( S_0 \in \{H_2, H'_2, H_3\} \) and \( \{000,001\} \cap S_1 = \emptyset \). Then the following statements hold:

(i) \( S_1 \in \{H_1, H_2, H'_2, H_3\} \), and

(ii) if \( S_1 = H_1 \), then \( S_0 = H_3 \).

**Proof.** (i) After twisting coordinate 3, if necessary, we may assume that \( S_0 \in \{H_2, H_3\} \). We may therefore display \( S \) as:

Since the 0-restriction of \( S \) over coordinate 1 is 1-resistant, it follows that \( |\{010,011\} \cap S_1| \neq 1 \), and since the 0-restriction of \( S \) over coordinate 2 is 1-resistant, it follows that \( |\{100,101\} \cap S_1| \neq 1 \). Thus, as the 0-restriction of \( S \) over coordinate 3 is 1-resistant, either \( \{010,011\} \subseteq S_1 \) or \( \{100,101\} \subseteq S_1 \). After relabeling coordinates 1, 2, if necessary, \( \{010,011\} \subseteq S_1 \). Since the 0-restriction of \( S \) over coordinate 3 is not isomorphic to \( D_3 \) or \( F_3 \), it follows that \( \{100,101\} \subseteq S_1 \) also.
Hence, $S_1 \in \{H_1, H_2, H_2^*, H_3\}$. (ii) If $S_1 = H_1$, then as the 1-restriction of $S$ over coordinate 1 is not isomorphic to $F_3$, it follows that 111 $\in S_0$, so $S_0 = H_3$, as required.

Given that $n \geq 2$ and $i, j \in \{0, 1\}$, denote by $S_{ij} \subseteq \{0, 1\}^{n-2}$ the restriction of $S$ obtained after $i$-restricting coordinate $n-1$ and $j$-restricting coordinate $n$.

**Lemma 8.3.** Let $S \subseteq \{0, 1\}^5$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_{00} = H_3, S_{10} = H_1$ and $\{000, 001\} \cap S_{11} = \emptyset$. Then the following statements hold:

(i) $S_{01}, S_{11} \in \{H_1, H_2, H_2^*, H_3\}$, and

(ii) if $S_{11} = H_1$ then $S_{01} = H_3$, and therefore $S_1 = S_0$.

**Proof.** (i) For $i, j \in \{0, 1\}$, denote by $R_{ij} \subseteq \{0, 1\}^5$ the restriction of $S$ obtained after $i$-restricting coordinate 3 and $j$-restricting coordinate 5.

Notice that $R_{00} = R_{10} = H_2$ and $001 \notin R_{01} \cup R_{11}$. It therefore follows from Lemma 8.1 that $000 \notin R_{01} \cup R_{11}$. We get from Lemma 8.2 (i)-(ii) that $R_{01}, R_{11} \in \{H_2, H_2^*, H_3\}$:

As a result, $S_{00}, S_{11} \in \{H_1, H_2, H_2^*, H_3\}$. (ii) If $S_{11} = H_1$, then $R_{01}$ and $R_{11}$ must be equal to $H_2$, implying in turn that $S_{01} = H_3$, as required.

We are now ready to prove the first main result of this section:

**Proposition 8.4.** Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0, 1\}^n$ that has no $R_{1,1}, F_1, F_2, F_3$ and no $D_3$ minor. Take a point $x$ and distinct coordinates $i, j \in [n]$ such that $x$ is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing $x$ is a hypercube.
Proof. We prove this by induction on \( n \geq 2 \). The base case \( n = 2 \) holds trivially. For the induction step, assume that \( n \geq 3 \). Let \( K \) be the infeasible component of \( S \) containing \( x \). If every neighbor of \( x \) belongs to \( S \), then \( K = \{ x \} \) and we are done. Otherwise, we may assume that \( x \in \{ 0, e_3 \} \subseteq K \) and \( i = 1, j = 2 \). For each \( y \in \{ 0, 1 \}^{n-3} \), let \( S_y := S \cap \{ x : x_{3+i} = y_i, i \in [n-3] \} \) and choose an appropriate \( R_y \subseteq \{ 0, 1 \}^3 \) such that \( S_y = R_y \times \{ y \} \). Notice that \( \{ 000, 001 \} \subseteq R_0 \), and either \( \{ 100, 010, 110 \} \subseteq R_0 \) or \( \{ 101, 011, 111 \} \subseteq R_0 \). Since \( R_0 \) is 1-resistant and not isomorphic to \( D_3, F_3 \), it follows that \( R_0 \in \{ H_2, H_2^*, H_3 \} \). In particular, if \( n = 3 \), then \( K = \{ 0, e_3 \} \) and the induction step is complete. We may therefore assume that \( n \geq 4 \).

Let \( S' \) be the projection of \( S \) over coordinate 3. Then \( S' \) is 1-resistant and has no \( R_{1,1}, F_1, F_2, F_3, D_3 \) minor. Hence, since \( 0 \in S' \), \( e_1, e_2, e_1 + e_2 \subseteq S' \), the induction hypothesis implies that the infeasible component of \( S' \) containing \( 0 \), call it \( K' \), is a hypercube. Notice that the set of points in \( \{ 0, 1 \}^n \) projecting onto a point in \( K' \) belong to \( K \) and form a hypercube whose dimension is larger by one.

Therefore, it suffices to show that \( K \) consists precisely of the points in \( \{ 0, 1 \}^n \) projecting onto \( K' \). Suppose otherwise. Then there must exist points \( z, z + e_3 \in \{ 0, 1 \}^n \) projecting onto a point \( z' \in \{ 0, 1 \}^{n-1} \) such that

- \( z' \) belongs to \( S' \) and is adjacent to a point in \( K' \), and
- \( \{ z, z + e_3 \} \cap S \neq 1 \).

Notice that \( \{ z, z + e_3 \} \cap K \neq 1 \).

Call a point \( y \in \{ 0, 1 \}^{n-3} \) involved if

- \( R_y \in \{ H_2, H_2^*, H_3 \} \), and
- \( 00y \in K' \).

Notice that \( 0 \in \{ 0, 1 \}^{n-3} \) is involved. Now, pick a point \( t' \in \{ 0, 1 \}^{n-1} \) minimizing \( \text{dist}(t', z') \) subject to

- \( t' \in K' \), and
- there exists an involved \( y \in \{ 0, 1 \}^{n-3} \) such that \( t' = 00y \),

in this order of priority. We may assume that \( t' = 0 \in \{ 0, 1 \}^{n-1} \). Since \( z' \notin K' \), we get that \( \text{dist}(0, z') \geq 1 \). It follows from Lemma 8.1 that \( \text{dist}(0, z') \geq 2 \). Since \( K' \) is a hypercube, there exist an integer \( d \geq 2 \) and distinct coordinates \( j_1, j_2, \ldots, j_d \in [n] \setminus \{ 3 \} \) such that \( z' = \sum_{i=1}^{d} e_{j_i} \), and

\[
\sum_{i=1}^{k} e_{j_i} \in K' \quad k = 1, \ldots, d - 1.
\]

Notice that

\[
\sum_{i=1}^{k} e_{j_i} \in K \quad \text{and} \quad e_3 + \sum_{i=1}^{k} e_{j_i} \in K \quad k = 1, \ldots, d - 1.
\]

Thus, since \( R_0 \in \{ H_2, H_3 \} \), we have \( j_1 \in [n] \setminus \{ 1, 2, 3 \} \). We may therefore assume that \( j_1 = 4 \). Since \( R_0 \in \{ H_2, H_3 \} \) and \( \{ 000, 001 \} \cap R_{e_3} = \emptyset \), it follows from Lemma 8.2 (i) that \( R_{e_1} \in \{ H_1, H_2, H_2^*, H_3 \} \). Our
minimal choice of \( t' = 0 \) implies that \( R_{e_1} = H_1 \) (otherwise, \( t' = e_4 \) contradicts the minimality of \( t' = 0 \)). We now get from Lemma 8.1 that \( d \geq 3 \), and from Lemma 8.2 (ii) that \( R_0 = H_3 \). Since \( j_2 \in [n] - \{1, 2, 3, 4\} \), we may assume that \( j_2 = 5 \). So \( e_4 + e_5 \in K' \). As \( 0, e_4, e_4 + e_5 \in K' \) and \( K' \) is a hypercube, it follows that \( e_5 \in K' \). Since \( \{000, 001\} \cap R_{e_1+e_2} = \emptyset \), we get from Lemma 8.3 that either

- \( R_{e_1+e_2} \in \{H_2, H_2^*, H_3\} \), or
- \( R_{e_2} = H_3 \) and \( R_{e_1+e_2} = H_1 \).

The first case is not possible as it contradicts the minimal choice of \( t' = 0 \), for \( t' = e_4 + e_5 \) would be a better choice. However, the second case is not possible either as it also contradicts the minimal choice of \( t' = 0 \), for \( t' = e_5 \) would be a better choice. This finishes the proof of Proposition 8.4.

\[ \square \]

### 8.1 Proof of Theorem 4.3

We are now ready to prove Theorem 4.3, stating that

Take an integer \( n \geq 1 \) and a 1-resistant set \( S \subseteq \{0, 1\}^n \) without an \( R_{1,1}, F_1, F_2, F_3 \) minor. If \( S \) is connected and has no \( D_3 \) minor, then either

- \( S \) is a hypercube, or
- every infeasible component of \( S \) is a hypercube.

**Proof.** Assume that there is an infeasible component \( K \) that is not a hypercube.

**Claim 1.** Take a point \( x \) and distinct coordinates \( i, j \in [n] \) such that \( x \in K \) and \( x\triangle e_i \in S \). If \( x\triangle e_i \triangle e_j \in S \), then \( x\triangle e_j \in K \).

**Proof of Claim.** For if not, \( x\triangle e_j \in S \), so by Proposition 8.4, the infeasible component of \( S \) containing \( x \), which is \( K \), is a hypercube, a contradiction.

\[ \diamond \]

This claim has the following subtle implication:

**Claim 2.** The points in \( S \) agree on a coordinate.

**Proof of Claim.** Take a point \( y \in K \) and a direction \( i \in [n] \) such that \( y\triangle e_i \in S \). We may assume that \( y = 0 \) and \( i = 1 \). As \( S \) is connected, it follows from Claim 1 that \( S \subseteq \{x : x_1 = 1\} \), as required.

\[ \diamond \]

As \( S \) is 1-resistant, it follows from Lemma 1.12 that \( S \) is a hypercube, thereby proving Theorem 4.3.

\[ \square \]
9 Every $\pm 1$-resistant set is strictly polar.

We will need the following immediate remark:

**Remark 9.1.** Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. If $S$ is strictly polar, then so is $S \times \{0, 1\}$.

We will also need the following variant of Lemma 2.5:

**Lemma 9.2.** Take an integer $n \geq 1$ and a nonempty set $S \subseteq \{0, 1\}^n$ without an $R_{1,1}$ restriction, where every infeasible component is a hypercube. Then

- $|S| \geq 2^{n-1}$, and

- if $|S| = 2^{n-1}$, then $S$ is either a hypercube of dimension $n - 1$ or the union of antipodal hypercubes of dimension $n - 2$.

In particular, $S$ is strictly polar.

**Proof.** We prove this by induction on $n \geq 1$. The base cases $n \in \{1, 2\}$ are clear. For the induction step, assume that $n \geq 3$. For $i \in \{0, 1\}$, let $S_i \subseteq \{0, 1\}^{n-1}$ be the $i$-restriction of $S$ over coordinate $n$. If one of $S_0, S_1$ is empty, then the other one must be $\{0, 1\}^{n-1}$, so $S$ is a hypercube of dimension $n - 1$ and the induction step is complete. We may therefore assume that $S_0, S_1$ are nonempty. Since every infeasible component of both $S_0, S_1$ is a hypercube, we may apply the induction hypothesis. Thus, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$, so by the induction hypothesis, each $S_i$ is either a hypercube of dimension $n - 2$ or the union of antipodal hypercubes of dimension $n - 3$. If one of $S_0, S_1$ is a hypercube, then as every infeasible component of $S$ is a hypercube, $S$ is either a hypercube of dimension $n - 1$ or the union of antipodal hypercubes of dimension $n - 2$. Otherwise, each one of $S_0, S_1$ is the union of two antipodal hypercubes of dimension $n - 3$. As $S$ has no $R_{1,1}$ restriction, it must be that $S_0 = S_1$, implying in turn that $S$ is the union of antipodal hypercubes of dimension $n - 2$, thereby completing the induction step.

We are now able to prove Theorem 1.8, stating that every $\pm 1$-resistant set is strictly polar:

**Proof of Theorem 1.8.** Take an integer $n \geq 1$ and a $\pm 1$-resistant set $S \subseteq \{0, 1\}^n$. Then by Theorem 1.5, either

(i) $S \cong A_k \times \{0, 1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$,

(ii) $S \cong B_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$,

(iii) $S \cong C_8 \times \{0, 1\}^{n-4}$,

(iv) $S \cong D_k \times \{0, 1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$,

(v) $S$ is a hypercube, or
(vi) every infeasible component of $S$ is a hypercube, and every feasible point has at most two infeasible neighbors.

Observe that $\{A_k : k \geq 2\}$, $\{B_k, D_k : k \geq 3\}$ and $C_8$ are strictly polar sets. As a result, in cases (i)-(iv), the set $S$ is strictly polar by Remark 9.1. A hypercube is strictly polar, so in case (v), $S$ is also strictly polar. For the last case (vi), as $S$ has no $R_{1,1}$ restriction by Remark 3.1, Lemma 9.2 implies that $S$ is strictly polar. □

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