On the Relative Strength of Split, Triangle and Quadrilateral Cuts

Amitabh Basu *
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213
abasu1@andrew.cmu.edu

Pierre Bonami †
LIF, Faculté des Sciences de Luminy, Université de Marseille, France
pierre.bonami@lif.univ-mrs.fr

Gérard Cornuéjols ‡
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213
and LIF, Faculté des Sciences de Luminy, Université de Marseille, France
gc0v@andrew.cmu.edu

François Margot §
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213
fmargot@andrew.cmu.edu

July 2008

Abstract

Integer programs defined by two equations with two free integer variables and nonnegative continuous variables have three types of nontrivial facets: split, triangle or quadrilateral inequalities. In this paper, we compare the strength of these three families of inequalities. In particular we study how well each family approximates the integer hull. We show that, in a well defined sense, triangle inequalities provide a good approximation of the integer hull. The same statement holds for quadrilateral inequalities. On the other hand, the approximation produced by split inequalities may be arbitrarily bad.

1 Introduction

In this paper, we consider mixed integer linear programs with two equality constraints, two free integer variables and any number of nonnegative continuous variables. We assume that

*Supported by a Mellon Fellowship.
†Supported by ANR grant BLAN06-1-138894.
‡Supported by NSF grant CMMI0653419, ONR grant N00014-97-1-0196 and ANR grant BLAN06-1-138894.
§Supported by ONR grant N00014-97-1-0196.

the two integer variables are expressed in terms of the remaining variables as follows.

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^2 \\ s & \in & \mathbb{R}^k_+. \end{array} \tag{1}$$

This model is a natural relaxation of a general mixed integer linear program (MILP) and therefore it can be used to generate cutting planes for MILP. Currently, MILP solvers rely on cuts that can be generated from a single equation (such as Gomory mixed integer cuts [13], MIR cuts [16], lift-and-project cuts [3], lifted cover inequalities [8]). Model (1) has attracted attention recently as a way of generating new families of cuts from two equations instead of just a single one [1, 7, 10, 11, 14].

We assume $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, $k \geq 1$, and $r^j \in \mathbb{Q}^2 \setminus \{0\}$. So $s = 0$ is not a solution of (1). Let $R_f(r^1, \dots, r^k)$ be the convex hull of all vectors $s \in \mathbb{R}^k_+$ such that $f + \sum_{j=1}^{k} r^j s_j$ is integral. A classical theorem of Meyer [17] implies that $R_f(r^1, \dots, r^k)$ is a polyhedron. Andersen, Louveaux, Weismantel and Wolsey [1] showed that the only inequalities needed to describe $R_f(r^1, \dots, r^k)$ are $s \geq 0$ (called *trivial inequalities*), split inequalities [6] and intersection cuts (Balas [2]) arising from triangles or quadrilaterals in $\mathbb{R}^2$. Cornuéjols and Margot [7] characterized the facets of $R_f(r^1, \dots, r^k)$: The nontrivial facets are minimal inequalities related to maximal lattice-free convex sets in $\mathbb{R}^2$ with nonempty interior (Borozan and Cornuéjols [5]). By *lattice-free convex set* we mean a convex set with no integral point in its interior. However integral points are allowed on the boundary. These maximal lattice-free convex sets are splits, triangles, and quadrilaterals as proved in the following theorem of Lovász [15].

Theorem 1.1. (Lovász [15]) *In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:*

(i) A split $c \leq ax_1 + bx_2 \leq c + 1$ where a and b are coprime integers and c is an integer;

(ii) A triangle with an integral point in the interior of each of its edges;

(iii) A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.

$R_f(r^1, \dots, r^k)$ is a polyhedron of blocking type and a nontrivial valid inequality for $R_f(r^1, \dots, r^k)$ is of the form

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1 \tag{2}$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}_+$[19]. A nontrivial valid inequality is *minimal* if there is no other nontrivial valid inequality $\sum_{j=1}^{k} \psi'(r^j) s_j \geq 1$ such that $\psi'(r^j) \leq \psi(r^j)$ for all $j = 1, \dots, k$. The following result provides a link between minimal nontrivial valid inequalities and the maximal lattice-free convex sets of Theorem 1.1.

Theorem 1.2. (Borozan and Cornuéjols [5]) *Minimal nontrivial valid inequalities are associated with functions ψ that are nonnegative positively homogeneous piecewise linear and convex. Furthermore, the closure of the set*

$$B_\psi := \{x \in \mathbb{Q}^2 : \ \psi(x - f) \leq 1\} \tag{3}$$

is a maximal lattice-free convex set with nonempty interior.

Conversely, any maximal lattice-free convex set B with f in its interior defines a minimal function $\psi_B : \mathbb{R}^2 \to \mathbb{R}_+$ that can be used to generate a minimal nontrivial valid inequality. Indeed, define $\psi_B(0) = 0$ and $\psi_B(x - f) = 1$ for all points x on the boundary of B. Then, the positive homogeneity of ψ_B implies the value of $\psi_B(r)$ for any vector $r \in \mathbb{R}^2 \setminus \{0\}$: If there is a positive scalar λ such that the point $f + \lambda r$ is on the boundary of B, we get that $\psi_B(r) = 1/\lambda$. Otherwise, if there is no such λ, r is an unbounded direction of B and $\psi_B(r) = 0$.

Following Dey and Wolsey [10], the maximal lattice-free triangles can be partitioned into three types (see Figure 1):

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- *Type 2 triangles*: triangles with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;
- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 1.1.

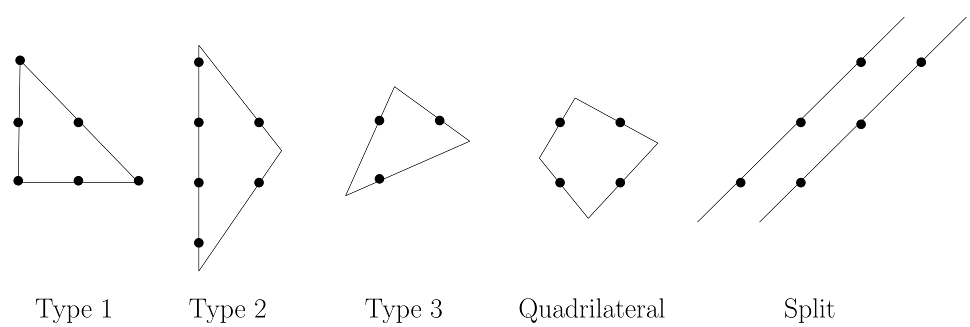

Figure 1: Maximal lattice-free convex sets with nonempty interior in $\mathbb{R}^2$

1.1 Motivation

An unbounded maximal lattice-free set has two edges and is called a *split*. These two edges are parallel and their direction is the *direction* of the split. Split inequalities for (1) are valid inequalities that can be derived by combining the two equations in (1) and by using the integrality of $\pi_1 x_1 + \pi_2 x_2$, where $\pi \in \mathbb{Z}^2$ defines the normal to the unbounded direction of the split. Similarly, for general MILPs, the equations can be combined into a single equality from which a split inequality is derived. Split inequalities are equivalent to Gomory mixed integer cuts [18]. Empirical evidence shows that split inequalities are effective for strengthening the linear programming relaxation of MILPs [4, 9]. Interestingly, triangle and quadrilateral inequalities cannot be derived from a single equation. They can only be derived from (1) without aggregating the two equations. Recent computational experiments by Espinoza [11] indicate that quadrilaterals also induce effective cutting planes in the context of solving general MILPs. In this paper, we consider the relative strength of split, triangle and quadrilateral inequalities from a theoretical point of view. We use an approach for measuring strength initiated by Goemans [12], based on the following definition and results.

Let $Q \subseteq \mathbb{R}^n_+ \setminus \{0\}$ be a polyhedron of the form $Q = \{x : a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$ where $a^i \geq 0$ and $b_i \geq 0$ for $i = 1, \ldots, m$ and let $\alpha > 0$ be a scalar. We define the polyhedron αQ as $\{x : \alpha a^i x \geq b_i \text{ for } i = 1, \ldots, m\}$. Note that αQ contains Q when $\alpha \geq 1$. It will be convenient to define αQ to be $\mathbb{R}^n_+$ when $\alpha = +\infty$.

We need the following generalization of a theorem of Goemans [12].

Theorem 1.3. *Suppose $Q \subseteq \mathbb{R}^n_+ \setminus \{0\}$ is defined as above. If convex set $P \subseteq \mathbb{R}^n_+$ is a relaxation of Q (i.e. $Q \subseteq P$), then the smallest value of $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is*

$$\max_{i=1,\ldots,m} \left\{ \frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \ : \ b_i > 0 \right\}.$$

In other words, the only directions that need to be considered to compute α are those defined by the nontrivial facets of Q. Goemans' paper assumes that both P and Q are polyhedra, but one can easily verify that only the polyhedrality of Q is needed in the proof. We give the proof of Theorem 1.3 in Section 2, for completeness.

1.2 Results

Let the *split closure* $S_f(r^1, \ldots, r^k)$ be the intersection of all split inequalities, let the *triangle closure* $T_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free triangles, and let the *quadrilateral closure* $Q_f(r^1, \ldots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free quadrilaterals. Since all the facets of $R_f(r^1, \ldots, r^k)$ are induced by these three families of maximal lattice-free convex sets, we have

$$R_f(r^1, \ldots, r^k) = S_f(r^1, \ldots, r^k) \cap T_f(r^1, \ldots, r^k) \cap Q_f(r^1, \ldots, r^k).$$

It is known that the split closure is a polyhedron (Cook, Kannan and Schrijver [6]) but such a result is not known for the triangle closure and the quadrilateral closure. In this paper we show the following results.

Theorem 1.4. *$T_f(r^1, \ldots, r^k) \subseteq S_f(r^1, \ldots, r^k)$ and $Q_f(r^1, \ldots, r^k) \subseteq S_f(r^1, \ldots, r^k)$.*

We study the strength of the triangle closure and quadrilateral closure in the sense defined in Section 1.1. We first compute the strength of a single Type 1 triangle facet as f varies in the interior of the triangle, relative to the entire split closure.

Theorem 1.5. *Let T be a Type 1 triangle as depicted in Figure 2. Let f be in its interior and assume that the set of rays $\{r^1, \ldots, r^k\}$ contains rays pointing to the three corners of T. Let $\sum_{i=1}^k \psi(r^i) s_i \geq 1$ be the inequality generated by T. The value*

$$\inf \left\{ \sum_{i=1}^k \psi(r^i) s_i \ : \ s \in S_f(r^1, \ldots, r^k) \right\}$$

is a piecewise linear function of f for which some level curves are depicted in Figure 2. This function varies from a minimum of $\frac{1}{2}$ in the center of T to a maximum of $\frac{2}{3}$ at its corners.

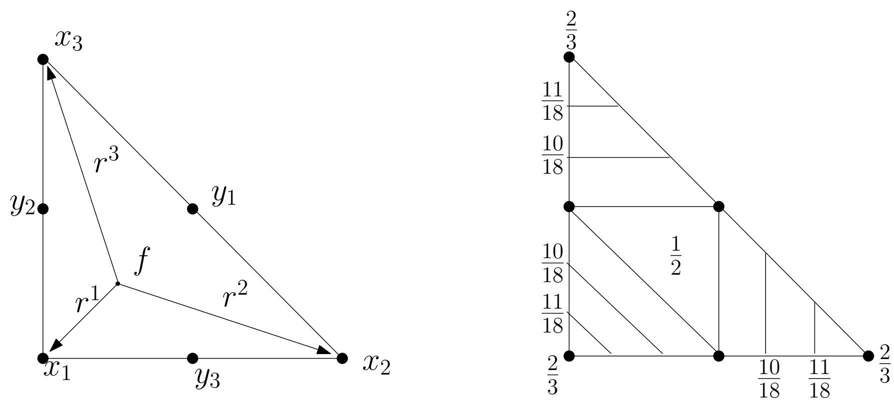

Figure 2: Illustration for Theorem 1.5

Next we show that both the triangle closure and the quadrilateral closure are good approximations of the integer hull $R_f(r^1, \dots, r^k)$ in the sense that

Theorem 1.6.
$R_f(r^1, \dots, r^k) \subseteq T_f(r^1, \dots, r^k) \subseteq 2R_f(r^1, \dots, r^k)$ *and*
$R_f(r^1, \dots, r^k) \subseteq Q_f(r^1, \dots, r^k) \subseteq 2R_f(r^1, \dots, r^k)$.

Finally we show that the split closure may not be a good approximation of the integer hull.

Theorem 1.7. *For any $\alpha > 1$, there is a choice of f, $r^1, \dots, r^k$ such that*
$S_f(r^1, \dots, r^k) \not\subseteq \alpha R_f(r^1, \dots, r^k)$.

These results provide additional support for the recent interest in cuts derived from two or more rows of an integer program [1, 5, 7, 10, 11, 14].

2 Proof of Theorem 1.3

Proof. Let

$$\alpha = \max_{i=1,\dots,m} \left\{ \frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \ : \ b_i > 0 \right\}.$$

We first show that $P \subseteq \alpha Q$. This holds when $\alpha = +\infty$ by definition of αQ. Therefore we may assume $1 \leq \alpha < +\infty$. Consider any point $p \in P$. The inequalities of αQ are of the form $\alpha a^i x \geq b_i$ with $a^i \geq 0$ and $b_i \geq 0$. If $b_i = 0$, then since $p \in P \subseteq \mathbb{R}^n_+$, $a^i p \geq 0$ and hence this inequality is satisfied. If $b_i > 0$, then we know from the definition of α that

$$\frac{b_i}{\inf\{a^i x \ : \ x \in P\}} \leq \alpha.$$

This implies

$$b_i \leq \alpha \inf\{a^i x \ : \ \ x \in P\} \leq \alpha a^i p.$$

Therefore, p satisfies this inequality.

We next show that for any $1 \leq \alpha' < \alpha$, $P \not\subseteq \alpha' Q$. Say $\alpha = \frac{b_j}{\inf\{a^j x \ : \ x \in P\}}$ (i.e. the maximum, possibly $+\infty$, is reached for index j). Let $\delta = \frac{b_j}{\alpha'} - \frac{b_j}{\alpha}$. We have $\delta > 0$. From the definition of α we know that $\inf\{a^j x \ : \ \ x \in P\} = \frac{b_j}{\alpha}$. Therefore, there exists $p \in P$ such that $a^j p < \frac{b_j}{\alpha} + \delta = \frac{b_j}{\alpha'}$. So $\alpha' a^j p < b_j$ and hence $p \notin \alpha' Q$.

□

3 Split closure vs. triangle and quadrilateral closures

In this section, we present the proof of Theorem 1.4.

Proof. (Theorem 1.4). We show that if any point $\bar{s}$ is cut off by a split inequality, then it is also cut off by some triangle inequality.

Consider any split inequality $\sum_{i=1}^{k} \psi_S(r^i) s_i \geq 1$ (see Figure 3) and denote by L_1 and L_2 its two boundary lines. Point f lies in some parallelogram of area 1 whose vertices $y^1, y^2, y^3,$ and y^4 are lattice points on the boundary of the split.

Assume without loss of generality that y^1 and y^2 are on L_1. Consider the family $\mathcal{T}$ of triangles whose edges are supported by L_2 and by two lines passing through y^1 and y^2 and whose interior contains the segment $y^1 y^2$. See Figure 3. Note that all triangles in $\mathcal{T}$ are of Type 2. For $T \in \mathcal{T}$ we will denote by ψ_T the minimal function associated with T.

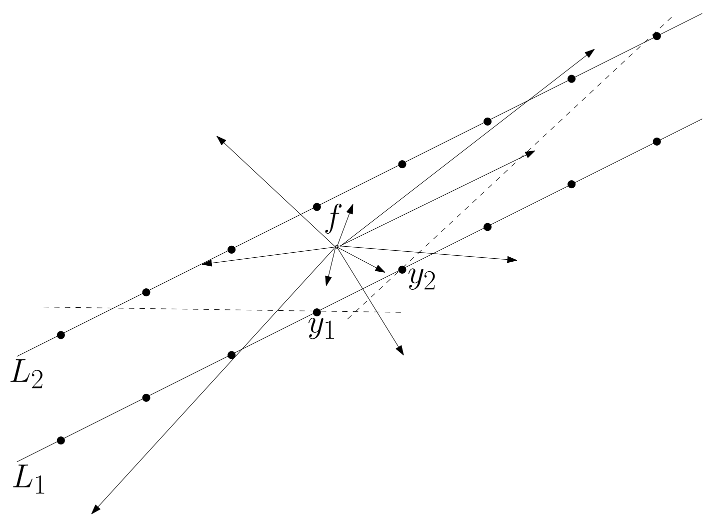

Figure 3: Approximating a split inequality with a triangle inequality. The triangle is formed by L_2 and the two dashed lines

By assumption, $\sum_{i=1}^{k} \psi_S(r^i) \bar{s}_i < 1$. Let $\epsilon = 1 - \sum_{i=1}^{k} \psi_S(r^i) \bar{s}_i$.

We now make the following simple observation.

Observation 3.1. *Given a finite set X of points that lie in the interior of the split S, we can find a triangle $T \in \mathcal{T}$ as defined above, such that all points in X are in the interior of T.*

Proof. Consider the convex hull $\mathcal{C}(X)$ of X. Since all points in X are in the interior of S, so is $\mathcal{C}(X)$. This implies that the tangent lines from y^1 and y^2 to $\mathcal{C}(X)$ are not parallel to L_1. Two of these four tangent lines along with L_2 of S form a triangle in $\mathcal{T}$ with X in its interior. □

Let $s_{max} = \max\{\bar{s}_i \ : \ i = 1\dots,k\}$ and define $\delta = \frac{\epsilon}{2 \cdot k \cdot s_{max}} > 0$. For every ray r^i define $c(r^i) = \psi_S(r^i) + \delta$. Therefore, by definition $p^i = \frac{1}{c(r^i)} \cdot r^i$ is a point strictly in the interior of S. Using Observation 3.1, there exists a triangle $T \in \mathcal{T}$ which contains all the points p^i. It follows that the coefficient $\psi_T(r^i)$ for any ray r^i is less than or equal to $c(r^i)$.

We claim that for this triangle T we have $\sum_{i=1}^k \psi_T(r^i)\bar{s}_i < 1$. Indeed,

$$\begin{aligned}
\sum_{i=1}^k \psi_T(r^i)\bar{s}_i &\leq \sum_{i=1}^k c(r^i)\bar{s}_i \\
&= \sum_{i=1}^k (\psi_S(r^i) + \delta)\bar{s}_i = \sum_{i=1}^k \psi_S(r^i)\bar{s}_i + \sum_{i=1}^k \frac{\epsilon}{2 \cdot k \cdot s_{max}}\bar{s}_i \\
&\leq \sum_{i=1}^k \psi_S(r^i)\bar{s}_i + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} < 1
\end{aligned}$$

The first inequality follows from the definition of $c(r^i)$ and the last equality follows from the fact that $\sum_{i=1}^k \psi_S(r^i)\bar{s}_i = 1 - \epsilon$.

This shows that $T_f(r^1, \dots, r^k) \subseteq S_f(r^1, \dots, r^k)$. For the quadrilateral closure, we also use two lines passing through y^3 and y^4 on L_2 and argue similarly. □

4 Tools

Cornuéjols and Margot [7] characterized the facets of $R_f(r^1, \dots, r^k)$ as follows. Let B_ψ be a maximal lattice-free split, triangle or quadrilateral with f in its interior. For any $j = 1, \dots, k$ such that $\psi(r^j) > 0$, let p^j be the intersection of the half-line $f + \lambda r^j$, $\lambda \geq 0$, with the boundary of B_ψ. The point p^j is called the *boundary point* for r^j. Let P be a set of boundary points. We say that a point $p \in P$ is *active* if it can have a positive coefficient in a convex combination of points in P generating an integral point. Note that $p \in P$ is active if and only if p is integral or there exists $q \in P$ such that the segment pq contains an integral point in its interior. We say that an active point $p \in P$ is *uniquely active* if it has a positive coefficient in *exactly one* convex combination of points in P generating an integral point.

Apply the following *Reduction Algorithm*:

0.) Let $P = \{p^1, \dots, p^k\}$.

1.) While there exists $p \in P$ such that p is active and p is a convex combination of other points in P, remove p from P. At the end of this step, P contains at most two active points on each edge of B_ψ and all points of P are distinct.

2.) While there exists a uniquely active $p \in P$, remove p from P.

3.) If P contains exactly two active points p and q (and possibly inactive points), remove both p and q from P.

The ray condition holds for a triangle or a quadrilateral if $P = \emptyset$ at termination of the Reduction Algorithm.

The ray condition holds for a split if, at termination of the Reduction Algorithm, either $P = \emptyset$, or $P = \{p_1, q_1, p_2, q_2\}$ with p_1, q_1 on one of the boundary lines and p_2, q_2 on the other and both line segments p_1q_1 and p_2q_2 contain at least two integral points.

Theorem 4.1. (Cornuéjols and Margot [7]) *The facets of $R_f(r^1, \ldots, r^k)$ are*

(i) split inequalities where the unbounded direction of B_ψ is r^j for some $j = 1, \ldots, k$ and the line $f + \lambda r^j$ contains no integral point; or where B_ψ satisfies the ray condition,

(ii) triangle inequalities where the triangle B_ψ has its corner points on three half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$; or where the triangle B_ψ satisfies the ray condition,

(iii) quadrilateral inequalities where the corners of B_ψ are on four half-lines $f + \lambda r^j$ for some $j = 1, \ldots, k$ and $\lambda > 0$, and B_ψ satisfies a certain ratio condition (the ratio condition will not be needed in this paper; the interested reader is referred to [7] for details).

Note that the same facet may arise from different convex sets. For example quadrilaterals for which the ray condition holds define facets, but there is always also a triangle that defines the same facet, which is the reason why there is no mention of the ray condition in (iii) of Theorem 4.1.

4.1 Reducing the number of rays in the analysis

The following technical theorem will be used in the proofs of Theorems 1.5, 1.6 and 1.7, where we will be applying Theorem 1.3.

Theorem 4.2. *Let $B_1, \ldots, B_m$ be lattice-free convex sets with f in the interior of B_p, $p = 1, \ldots, m$. Let $R_c \subseteq \{1, \ldots, k\}$ be a subset of the ray indices such that for every ray r^j with $j \notin R_c$, r^j is the convex combination of some two rays in R_c. Define*

$$
\begin{aligned}
z_1 = \quad \min \quad & \sum_{i=1}^{k} s_i \\
& \sum_{i=1}^{k} \psi_{B_p}(r^i) s_i \geq 1 \quad \textit{for} \;\; p = 1, \ldots, m \\
& s \geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
z_c = \quad \min \quad & \sum_{i \in R_c} s_i \\
& \sum_{i \in R_c} \psi_{B_p}(r^i) s_i \geq 1 \quad \textit{for} \;\; p = 1, \ldots, m \\
& s \geq 0\ .
\end{aligned}
$$

Then $z_1 = z_c$.

Proof. Assume there exists $j \notin R_c$ and r^1, r^2 are the rays in R_c such that $r^j = \lambda r^1 + (1-\lambda) r^2$ for some $0 < \lambda < 1$. Let $K = \{1, \dots, k\} - j$ and define

$$\begin{aligned} z_2 = \quad \min \quad & \sum_{i \in K} s_i \\ & \sum_{i \in K} \psi_{B_p}(r^i) s_i \quad \geq 1 \quad \text{for} \quad p = 1, \dots, m \\ & s \geq 0 \ . \end{aligned}$$

We first show that $z_1 = z_2$. Applying the same reasoning repeatedly to all the indices not in R_c yields the proof of the theorem.

Any optimal solution for the LP defining z_2 yields a feasible solution for the one defining z_1 by setting $s_j = 0$, implying $z_1 \leq z_2$. It remains to show that $z_1 \geq z_2$.

Consider any point $\hat{s}$ satisfying $\sum_{i=1}^{k} \psi_{B_p}(r^i)\hat{s}_i \geq 1$ for every $p \in \{1, \dots, m\}$. Consider the following values $\bar{s}$ for the variables corresponding to the indices $t \in K$.

$$\bar{s}_t = \begin{cases} \hat{s}_t & \text{if} \ \ t \notin \{1, 2, j\} \\ \hat{s}_1 + \lambda \hat{s}_j & \text{if} \ \ t = 1 \\ \hat{s}_2 + (1-\lambda)\hat{s}_j & \text{if} \ \ t = 2 \end{cases}$$

One can check that

$$\sum_{i \in K} \bar{s}_i = \hat{s}_j + \sum_{i \in K} \hat{s}_i \ .$$

By Theorem 1.2 ψ_{B_p} is convex, thus $\psi_{B_p}(r^j) \leq \lambda \psi_{B_p}(r^1) + (1-\lambda)\psi_{B_p}(r^2)$ for $p = 1, \dots, m$. It follows that $\sum_{i \in K} \psi_{B_p}(r^i)\bar{s}_i \geq \psi_{B_p}(r^j)\hat{s}_j + \sum_{i \in K} \psi_{B_p}(r^i)\hat{s}_i = \sum_{i=1}^{k} \psi_{B_p}(r^i)\hat{s}_i \geq 1$ for $p = 1, \dots, m$. Hence $\bar{s}$ satisfies all the inequalities restricted to indices in K and has the same objective value as $\hat{s}$. It follows that $z_1 \geq z_2$. □

5 Proof sketch for Theorems 1.5 and 1.6

In this section, we give a brief outline of the proofs of Theorems 1.5 and 1.6. A complete proof will be given in Sections 6 and 7 respectively.

In Theorem 1.5, we need to analyze the optimization problem

$$\inf \left\{ \sum_{i=1}^{k} \psi(r^i) s_i \ : \ s \in S_f(r^1, \dots, r^k) \right\} \ , \tag{4}$$

where ψ is the minimal function derived from the Type 1 triangle.

For Theorem 1.6, recall that all non trivial facet defining inequalities for $R_f(r^1, \dots, r^k)$ are of the form $a^i s \geq 1$ with $a^i \geq 0$. Therefore, Theorem 1.3 shows that to prove Theorem 1.6, we need to consider all nontrivial facet defining inequalities and optimize in the direction a^i over the triangle closure $T_f(r^1, \dots, r^k)$ and the quadrilateral closure $Q_f(r^1, \dots, r^k)$. This task is made easier since all the non trivial facets of $R_f(r^1, \dots, r^k)$ are characterized in Theorem 4.1. Moreover, Theorem 1.4 shows that we can ignore the facets defined by split inequalities.

Formally, consider a maximal lattice-free triangle or quadrilateral B with associated minimal function ψ that gives rise to a facet $\sum_{j=1}^{k} \psi(r^j)s_j \geq 1$ of $R_f(r^1, \ldots, r^k)$. We want to investigate the following optimization problems:

$$\min \left\{ \sum_{j=1}^{k} \psi(r^j)s_j : \ s \in T_f(r^1, \ldots, r^k) \right\} \tag{5}$$

and

$$\min \left\{ \sum_{j=1}^{k} \psi(r^j)s_j : \ s \in Q_f(r^1, \ldots, r^k) \right\} \tag{6}$$

We first observe that, without loss of generality, we can make the following simplifying assumptions for problems (4), (5) and (6).

Assumption 5.1. *Consider the objective function ψ in problems (4), (5) and (6). For every j such that $\psi(r^j) > 0$, the ray r^j is such that the point $f + r^j$ is on the boundary of the lattice-free set B generating ψ.*

Indeed, this amounts to scaling the coefficient for the ray r^j by a constant factor in every inequality derived from all maximal lattice-free sets, including B. Therefore, this corresponds to a simultaneous scaling of variable s_j and corresponding coefficients in problems (4), (5) and (6). This does not change the optimal values of these problems. Moreover, Cornuéjols and Margot [7] show that the equations of all edges of triangles of Type 1, 2, or 3, of quadrilaterals and the direction of all splits generating facets of $R_f(r^1, \ldots, r^k)$ are rational. This implies that the scaling factor for ray r^j is a rational number and that the scaled ray is rational too.

As a consequence, we can assume that the objective function of problems (4),(5) or (6) is $\sum_{j=1}^{k} s_j \geq 1$.

When B_ψ is a triangle or quadrilateral and f is in its interior, define a *corner ray* to be a ray r such that $f + \lambda r$ is a corner of B_ψ for some $\lambda > 0$.

Remark 5.2. *If $\{r^1, \ldots, r^k\}$ contains the corner rays of the convex set defining the objective functions of (4),(5) or (6), then Assumption 5.1 implies that the hypotheses of Theorem 4.2 are satisfied. Therefore, when analyzing (4),(5) or (6), we can assume that $\{r^1, \ldots, r^k\}$ is exactly the set of corner rays.*

6 Type 1 triangle and the split closure

In this section, we present the proof of Theorem 1.5.

Consider any Type 1 triangle T with integral vertices x^j, for $j = 1, 2, 3$, and one integral point y^j for $j = 1, 2, 3$ in the interior of each edge. We want to study the optimization problem (4). Recall that Remark 5.2 says that we only need to consider the case with three corner rays r^1, r^2 and r^3.

We compute the exact value of

$$
\begin{aligned}
z_{SPLIT} = \quad \min \quad & \sum_{j=1}^{3} s_j \\
& \sum_{j=1}^{3} \psi(r^j) s_j \quad \geq 1 \quad \text{for all splits } B_\psi \\
& s \in \mathbb{R}^3_+ .
\end{aligned} \tag{7}
$$

Observe that, using an affine transformation, T can be made to have one horizontal edge x^1x^2 and one vertical edge x^1x^3, as shown in Figure 2. Without loss of generality, we place the origin at point x^1.

We distinguish two cases depending on the position of f in the interior of triangle T: f is in the inner triangle T_I with vertices $y^1 = (1,1)$, $y^2 = (0,1)$ and $y^3 = (1,0)$; and $f \in int(T) \setminus T_I$. We show that $z_{SPLIT} = \frac{1}{2}$ when f is in the inner triangle T_I and that z_{SPLIT} increases linearly from $\frac{1}{2}$ when f is at the boundary of T_I to $\frac{2}{3}$ at the corners of the triangle T when $f \in int(T) \setminus T_I$. See the right part of Figure 2 for some level curves of z_{SPLIT} as a function of the position of f in T. By a symmetry argument, it is sufficient to consider the inner triangle T_I and the corner triangle T_C defined by $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$.

Theorem 6.1. *Let T be a triangle with integral vertices, say $(0,0)$, $(0,2)$ and $(2,0)$. Then*

(i) $z_{SPLIT} = \frac{1}{2}$ when f is interior to the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

(ii) $z_{SPLIT} = 1 - \frac{1}{3-f_1-f_2}$ when $f = (f_1, f_2)$ is interior to the corner triangle $f_1 + f_2 \leq 1$, $f_1, f_2 \geq 0$. The value of z_{SPLIT} when f is in the other corner triangles follows by symmetry.

To prove this theorem, we show that the split closure is completely defined by only three split inequalities. In other words, all other split inequalities are dominated by these three split inequalities.

Define S_1 as the convex set $1 \leq x_1 + x_2 \leq 2$, S_2 as the convex set $0 \leq x_1 \leq 1$ and S_3 as the convex set $0 \leq x_2 \leq 1$. Define *Split 1* (resp. *Split 2*, *Split3*) to be the inequality obtained from S_1 (resp. S_2, S_3).

Let S be a split inequality with f in the interior of S. The *shores* of S are the two half-planes containing the points not in the interior of S.

Lemma 6.2. *Let A, B, and C be three points on a line, with B between A and C and let S be a split. If A and C are not in the interior of S but B is, then A and C are on opposite shores of S.*

Proof. If A and C are on the same shore W of S then, by convexity of W, the segment AC is completely in W, a contradiction. □

Lemma 6.3. *If f is in the interior of the triangle T_I with vertices $(0,1)$, $(1,0)$ and $(1,1)$, then the split closure is defined by Split 1, Split 2 and Split 3.*

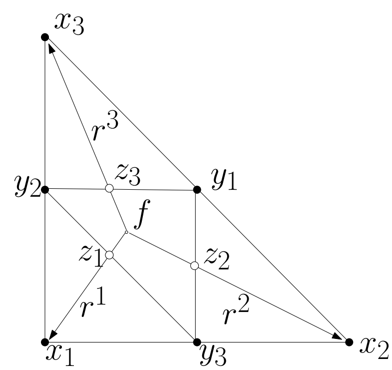

Figure 4: Illustration for the proof of Lemma 6.3

Proof. Let S be a split defining an inequality that is not dominated by either of Split 1, Split 2, or Split 3. Let L_1 and L_2 be the two parallel lines bounding S. For $i = 1, 2, 3$, let z^i (resp. w^i) be the boundary point for r^i on the boundary of T_I (resp. S) (see Figure 4). (Note that since $x^i = f + r^i$ is integer, these points exist.) Observe that if z^1 is not in the interior of S, then the inequality obtained from S is dominated by Split 1, since the three boundary points w_1, w_2, w_3 are closer to f than the corresponding three boundary points z^1, x^2, x^3 for the three rays on the boundary of S_1. A similar observation holds for z^2 and Split 2 and for z^3 and Split 3, yielding that z^2 and z^3 are also in the interior of S.

Since the points y^1, y^2 and y^3 are integer, they are not in the interior of S. Applying Lemma 6.2 to y^1, z^3, y^2, we have that y^1 and y^2 are on opposite shores of S. Now, y^3 is in one of the two shores of S. Assume without loss of generality that it is on the same shore as y^1. Applying Lemma 6.2 to y^1, z^2, y^3, we have that y^1 and y^3 are on opposite shores of S, a contradiction. □

Lemma 6.4. *If f is in the interior of the triangle T_I with vertices* $(0, 1)$, $(1, 0)$ *and* $(1, 1)$, *then* $z_{SPLIT} = \frac{1}{2}$.

Proof. By Lemma 6.3, z_{SPLIT} is given by

$$\begin{aligned} z_{SPLIT} = \quad \min \quad & \sum_{j=1}^{k} s_j \\ & \sum_{j=1}^{k} \psi_i(r^j) s_j \quad \geq 1 \quad \text{for } i = 1, 2, 3 \\ & s \in \mathbb{R}_+^k \end{aligned} \tag{8}$$

where $\psi_i(r^j)$ is the coefficient of s^j in Split i, for $i = 1, 2, 3$. Let $f = (f_1, f_2)$. The coefficient of s^j in the split inequality can be computed from the boundary point for r^j with the corresponding split. For example, the boundary points for r^2 and r^3 with S_1 are the integer points x^2 and x^3. This implies that $\psi_1(r^2) = \psi_1(r^3) = 1$. On the other hand, the boundary point for r^1 is the point $t = \left(\frac{f_1}{f_1+f_2}, \frac{f_2}{f_1+f_2}\right)$. The length of r^1 divided by the length of the segment ft determines the coefficient $\psi_1(r^1)$ of s_1 (This follows from the homogeneity of ψ_1

and the fact that $\psi_1(t-f)=1$ since t is on the boundary of S_1). We get $\psi_1(r^1)=\frac{f_1+f_2}{f_1+f_2-1}$. Repeating this for S_2 and S_3, we get that z_{SPLIT} is the optimal value of the following linear program.

$$
\begin{array}{rlrrrl}
z_{SPLIT} = & \min & s_1 & +s_2 & +s_3 & \\
& & \frac{f_1+f_2}{f_1+f_2-1}s_1 & +s_2 & +s_3 & \geq 1 \\
& & s_1 & +\frac{2-f_1}{1-f_1}s_2 & +s_3 & \geq 1 \\
& & s_1 & +s_2 & +\frac{2-f_2}{1-f_2}s_3 & \geq 1 \\
& & & s \geq 0. & &
\end{array}
\tag{9}
$$

Its optimal solution is $s_1=\frac{f_1+f_2-1}{2}$, $s_2=\frac{1-f_1}{2}$, $s_3=\frac{1-f_2}{2}$ with value $z_{SPLIT}=s_0+s_1+s_2=\frac{1}{2}$. To verify that this solution is optimal, note that the dual of (9) is

$$
\begin{array}{rlrrrl}
z_{SPLIT} = & \max & z_1 & +z_2 & +z_3 & \\
& & \frac{f_1+f_2}{f_1+f_2-1}z_1 & +z_2 & +z_3 & \leq 1 \\
& & z_1 & +\frac{2-f_1}{1-f_1}z_2 & +z_3 & \leq 1 \\
& & z_1 & +z_2 & +\frac{2-f_2}{1-f_2}z_3 & \leq 1 \\
& & & z \geq 0, & &
\end{array}
\tag{10}
$$

with optimal solution $z_1=\frac{f_1+f_2-1}{2}$, $z_2=\frac{1-f_1}{2}$, $z_3=\frac{1-f_2}{2}$, with value $\frac{1}{2}$. □

Now we prove the second part of the theorem, when f is interior to the corner triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ or an inner point on the segment y^2y^3.

Lemma 6.5. *If f is in the interior of the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$, or an inner point on the segment joining $(0,1)$ to $(1,0)$, then the split closure is defined by Split 2 and Split 3.*

Proof. Let S be a split defining a split inequality that is not dominated by either of Split 2, or Split 3. Let L_1 and L_2 be the two parallel lines bounding S. Let z^2 be the intersection of r^2 with y^1y^3 and let z^3 be the intersection of r^3 with y^1y^2. For $i=1,2,3$, let w^i be the intersection of r^i with either L_1 or L_2. (Note that since x^i is integer, r^i has to intersect one of the two lines.) Observe that if z^2 is not in the interior of S, then the inequality obtained from S is dominated by Split 2, since the three intersections w_1, w_2, w_3 are closer to f than the corresponding three intersections x^1, z^2, x^3 for S_2. A similar observation holds for z^3 and S^3, yielding that z^3 is also in the interior of S.

Since the points y^1, y^2 and y^3 are integer, they are not in the interior of S. Applying Lemma 6.2 to y^1, z^3, y^2, we have that y^1 and y^2 are on opposite shores of S. Applying Lemma 6.2 to y^1, z^2, y^3, we have that y^1 and y^3 are on opposite shores of S. It follows that y^2 and y^3 are on the same shore W of S and thus the whole segment y^2y^3 is not in W. This is a contradiction with the fact that both f and z^3 are in the interior of S, as the two segments y^2y^3 and fz^3 intersect. □

Lemma 6.6. *If $f=(f_1,f_2)$ is in the interior of the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$, or an inner point on the segment joining $(0,1)$ to $(1,0)$, then $z_{SPLIT}=1-\frac{1}{3-f_1-f_2}$.*

Proof. The optimal solution of the LP

$$\begin{array}{rllll} z_{SPLIT} = & \min s_1 & +s_2 & +s_3 & \\ & s_1 & +\frac{2-f_1}{1-f_1}s_2 & +s_3 & \geq 1 \\ & s_1 & +s_2 & +\frac{2-f_2}{1-f_2}s_3 & \geq 1 \\ & & s \geq 0. & & \end{array} \tag{11}$$

is $s_1 = 0$, $s_2 = \frac{1-f_1}{3-f_1-f_2}$, $s_3 = \frac{1-f_2}{3-f_1-f_2}$. □

This completes the proof of Theorem 6.1. This theorem in conjunction with Theorem 1.3 implies that including all Type 1 triangle facets can improve upon the split closure only by a factor of 2.

Corollary 6.7. *Let $\mathcal{F}$ be the family of all facet defining inequalities arising from Type 1 triangles. Define*

$$\bar{S}_f = S_f(r^1, \ldots, r^k) \cap \left\{ \sum_{i=1}^{k} \psi(r^i)s_i \geq 1 \ : \ \psi \ in \ \mathcal{F} \right\} .$$

Then $\bar{S}_f \subseteq S_f(r^1 \ldots, r^k) \subseteq 2\bar{S}_f$.

7 Integer hull vs. triangle and quadrilateral closures

In this section we present the proof of Theorem 1.6. We show that the triangle closure $T_f(r^1, \ldots, r^k)$ and the quadrilateral closure $Q_f(r^1, \ldots, r^k)$ both approximate the integer hull $R_f(r^1, \ldots, r^k)$ to within a factor of two. As outlined in Section 5, we can show this by taking a facet of $R_f(r^1, \ldots, r^k)$, and optimizing in that direction over $T_f(r^1, \ldots, r^k)$ or $Q_f(r^1, \ldots, r^k)$. As noted in that section, we need to analyze the optimization problems (5) and (6).

7.1 Approximating the integer hull by the triangle closure

Theorem 1.4 shows that we can ignore the facets defined by split inequalities. We only need to consider facets of $R_f(r^1, \ldots, r^k)$ derived from quadrilaterals to obtain the objective function of problem (5). We prove the following result.

Theorem 7.1. *Let Q be a maximal lattice-free quadrilateral with corresponding minimal function ψ and generating a facet $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ of $R_f(r^1, \ldots, r^k)$. Then*

$$\min \left\{ \sum_{i=1}^{k} \psi(r^i)s_i \ : \ s \in T_f(r^1, \ldots, r^k) \right\} \geq \frac{1}{2} .$$

Proof. The theorem holds if the facet defining inequality can also be obtained as a triangle inequality. Therefore by Theorem 4.1, we may assume that rays $r^1, \ldots r^4$ are the corners rays of Q (See Figure 5). We remind the reader of Remark 5.2, showing that we can assume that $k = 4$ and that the four rays are exactly the corner rays of Q.

By an affine transformation, we may further assume that the four integral points on the boundary of Q are $(0,0),(1,0),(1,1),(0,1)$. Moreover, by symmetry, we may assume that the fractional point f satisfies $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ as rotating this region about $(\frac{1}{2},\frac{1}{2})$ by multiples of $\frac{\pi}{2}$ covers the entire quadrilateral. Note that $f_1 < 0$ and $f_2 < 0$ are possible.

We relax Problem (5) by keeping only two of the triangle inequalities, defined by triangles T_1 and T_2. T_1 has corner $f + r^4$ and edges supported by the two edges of Q incident with that corner and by the line $x = 1$. T_2 has corner $f + r^1$ and edges supported by the two edges of Q incident with that corner and by the line $y = 1$. The two triangles are depicted in dashed lines in Figure 5.

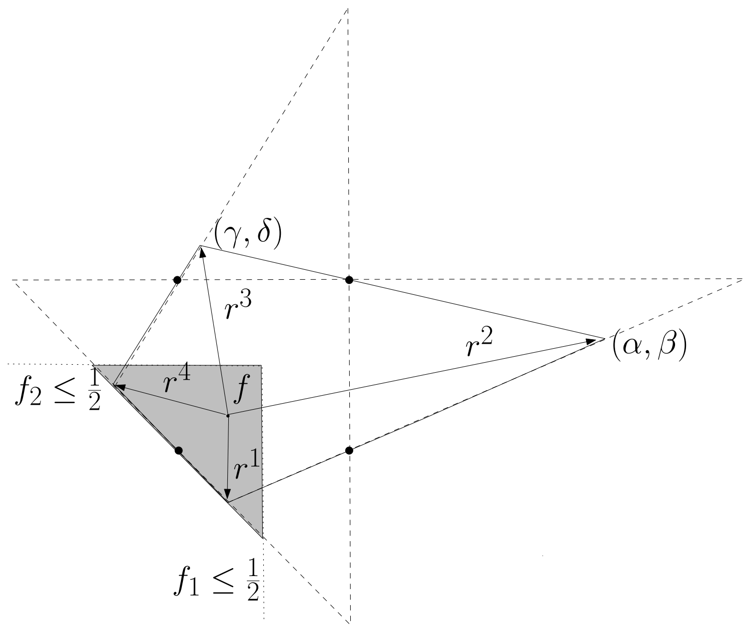

Figure 5: Approximating a quadrilateral inequality with triangle inequalities

Thus, Problem (5) can be relaxed to the LP

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ & \sum_{i=1}^{4} \psi_{T_1}(r^i) s_i \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & \sum_{i=1}^{4} \psi_{T_2}(r^i) s_i \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^4_+. \end{aligned} \tag{12}$$

Let $(\alpha,\beta) = f + r^2$ and $(\gamma,\delta) = f + r^3$. Computing the coefficients $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$, LP (12) becomes

$$
\begin{array}{lll}
\min & s_1+s_2+s_3+s_4 & \\
& s_1+\dfrac{\alpha-f_1}{1-f_1}s_2+s_3+s_4 \quad \geq 1 & (T_1) \\
& s_1+s_2+\dfrac{\delta-f_2}{1-f_2}s_3+s_4 \quad \geq 1 & (T_2) \\
& s\in\mathbb{R}^4_+. &
\end{array} \tag{13}
$$

Using the equation of the edge of Q connecting $f+r^2$ and $f+r^3$, we can find bounds on $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$. The edge has equation $x_1\frac{1}{t}+\frac{t-1}{t}x_2=1$ for some $1<t<\infty$. Therefore $\alpha\leq t$ and $\delta\leq\frac{t}{t-1}$. Using these two inequalities together with $f_1\leq\frac{1}{2}$ and $f_2\leq\frac{1}{2}$ we get

$$
\frac{\alpha-f_1}{1-f_1}=\frac{\alpha-1}{1-f_1}+1\leq 2(t-1)+1=2t-1 \qquad \text{and} \qquad \frac{\delta-f_2}{1-f_2}\leq 2\frac{t}{t-1}-1\ .
$$

Using these bounds, we obtain the relaxation of LP (13)

$$
\begin{array}{lll}
\min & s_1+s_2+s_3+s_4 & \\
& s_1+(2t-1)s_2+s_3+s_4 \qquad \geq 1 & (T_1) \\
& s_1+s_2+(2\dfrac{t}{t-1}-1)s_3+s_4 \quad \geq 1 & (T_2) \\
& s\in\mathbb{R}^4_+. &
\end{array} \tag{14}
$$

Set $\lambda=2t-1$ and $\mu=2\frac{t}{t-1}-1$. Then $t>1$ implies $\lambda>1$ and $\mu>1$. The optimal solution of the above LP is $s_1=s_4=0$, $s_2=\frac{\mu-1}{\lambda\mu-1}$ and $s_3=\frac{\lambda-1}{\lambda\mu-1}$ with value

$$
s_1+s_2+s_3+s_4=\frac{\lambda+\mu-2}{\lambda\mu-1}=\frac{t^2-2t+2}{t^2}\ .
$$

To find the minimum of this expression for $t>1$, we set its derivative to 0, and get the solution $t=2$. Thus the minimum value of $s_1+s_2+s_3+s_4$ is equal to $\frac{1}{2}$.

□

7.2 Approximating the integer hull by the quadrilateral closure

In this section, we study Problem (6). Theorem 1.4 shows that we can ignore the facets defined by split inequalities. Moreover, we can approximate the facets derived from Type 1 and Type 2 triangles using quadrilaterals in a similar manner as the splits were approximated by triangles and quadrilaterals. See Figure 6. We again define the set X of points which lies strictly inside the Type 1 or Type 2 triangle, similarly to Observation 3.1. Then we can find quadrilaterals as shown in Figure 6 that contain the set X. The proof goes through in exactly the same manner.

However triangles of Type 3 pose a problem. They cannot be approximated to any desired precision by a sequence of quadrilaterals.

In this section, we work under Assumption 5.1. By Theorem 4.1, a Type 3 triangle T defines a facet if and only if either the ray condition holds, or all three corner rays are present. First we consider the case where the ray condition holds.

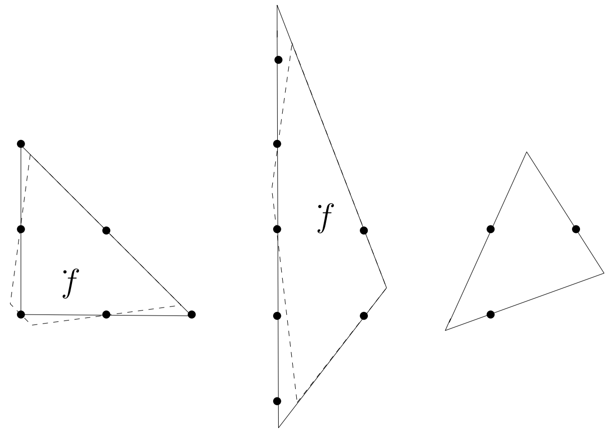

Figure 6: Approximating a triangle inequality with a quadrilateral inequality

Theorem 7.2. *Let T be a triangle of Type 3 with corresponding minimal function ψ_T defining a facet of $R_f(r^1, \ldots, r^k)$. If the ray condition holds for T then Problem (6) has optimal value 1.*

Proof. We first prove that if the ray condition holds the points $p^j = f + r^j$ are integral points on the boundary of T, for $j = 1, \ldots, k$.

For $i = 1, 2, 3$, let P_i be the set of points left at the end of Step i of the Reduction Algorithm given in Section 4. Each $p^j \in P_1$ with p^j integral is uniquely active and is removed during Step 2 of the Reduction Algorithm. Hence, all points in P_2 are non-integral. Observe that Step 3 can only remove boundary points p and q when the segment pq contains at least two integral points in its relative interior. Therefore this step does not remove anything in a Type 3 triangle and $P_3 = P_2$.

Since the ray condition holds, we have $P_3 = P_2 = \emptyset$ and P_1 contains only integral points. But then $P_1 = P$, showing that all boundary points at the beginning of the Reduction Algorithm are integral.

It is then straightforward to construct a maximal lattice-free quadrilateral Q with p^j, $j = 1, \ldots, k$ on its boundary, and containing f in its interior. It follows that the value of Problem (6) is equal to 1. □

We now consider the case where T is a Type 3 triangle with the three corner rays present. In this case, we can approximate the facet obtained from T to within a factor of two by using inequalities derived from triangles of Type 2. Define another relaxation $\bar{T}_f(r^1, \ldots, r^k)$ as the convex set defined by the intersection of the inequalities derived only from Type 1 and Type 2 triangles. By definition, $T_f(r^1, \ldots, r^k) \subseteq \bar{T}_f(r^1, \ldots, r^k)$. From the discussion at the beginning of this section, we also know $Q_f(r^1, \ldots, r^k) \subseteq \bar{T}_f(r^1, \ldots, r^k)$. Hence (6) can be relaxed to

$$\min \left\{ \sum_{i=1}^{k} \psi(r^i) s_i : \ s \in \bar{T}_f(r^1, \ldots, r^k) \right\} . \tag{15}$$

Theorem 7.3. *Let T be a triangle of Type 3 with corresponding minimal function ψ and generating a facet $\sum_{i=1}^{k} \psi(r^i)s_i \geq 1$ of $R_f(r^1, \ldots, r^k)$. Then,*

$$\min \left\{ \psi(r^i)s_i \ : \ s \in \bar{T}_f(r^1, \ldots, r^k) \right\} \geq \frac{1}{2} \ .$$

This theorem implies directly the following corollary.

Corollary 7.4. $Q_f(r^1, \ldots, r^k) \subseteq 2R_f(r^1, \ldots, r^k)$.

Proof of Theorem 7.3. We first make an affine transformation to simplify computations. Let y^1, y^2, y^3 be the three lattice points on the sides of T. We choose the transformation such that the two following properties are satisfied.

(i) The standard lattice is mapped to the lattice generated by the vectors $v^1 = (1, 0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, i.e. all points of the form $z_1 v^1 + z_2 v^2$, where z_1, z_2 are integers.

(ii) y^1, y^2, y^3 are respectively mapped to $(0, 0), (1, 0), (0, 1)$ in the above lattice.

We use the basis v^1, v^2 for $\mathbb{R}^2$ for all calculations and equations in the remainder of the proof. See Figure 7.

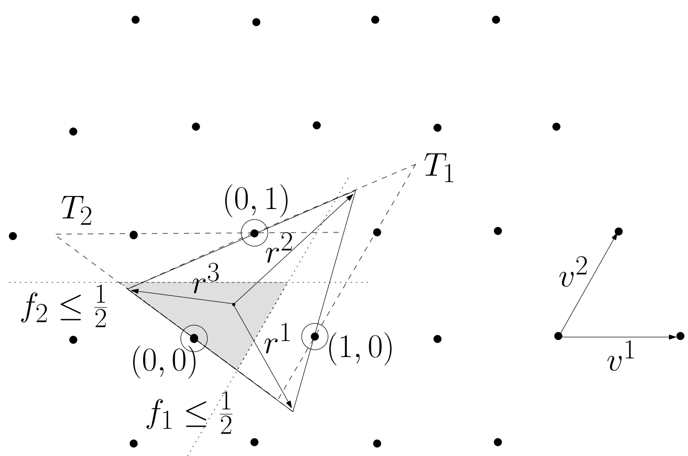

Figure 7: Approximating a Type 3 triangle inequality with Type 2 triangle inequalities. The Type 3 triangle is in solid lines. The basis vectors are $v^1 = (1, 0)$ and $v^2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

With this transformation, we can get a simple characterization for the Type 3 triangles. (See Figure 7 for an example.) We make the following claim about the relative orientations of the three sides of the Type 3 triangle.

Lemma 7.5. *Let $\mathcal{F}$ be the family of triangles formed by three lines given by:*

$$\textit{Line } 1: \quad -\frac{x_1}{t_1} + x_2 = 1 \quad \textit{with} \ \ 0 < t_1 < \infty \ ;$$
$$\textit{Line } 2: \quad t_2 \ x_1 + x_2 = 0 \quad \textit{with} \ \ 0 < t_2 < 1 \ ;$$
$$\textit{Line } 3: \quad x_1 + \frac{x_2}{t_3} = 1 \quad \textit{with} \ \ 1 < t_3 < \infty \ .$$

Any Type 3 triangle is either a triangle from $\mathcal{F}$ or a reflection of a triangle from $\mathcal{F}$ about the line $x_1 = x_2$.

Proof. Take any Type 3 triangle T. Consider the edge passing through $(1, 0)$. Since it cannot go through the interior of the equilateral triangle $(0, 0), (0, 1), (1, 0)$, there are only two choices for its orientation : a) It can go through the segment $(0, 1), (1, 1)$, or b) It can go through the segment $(0, 0), (1, -1)$.

In the first case its equation is that of Line 3. This now forces the edge of T passing through $(0, 0)$ to have the equation for Line 2. This is because the only other possibility for this edge would be for the line to pass through the segment $(-1, 1), (0, 1)$. But then the lattice point $(1, -1)$ is included in the interior of the triangle. Similarly, the third edge must now have Line 1's equation, because $(-1, 1)$ needs to be excluded from the interior.

Case b) can be mapped to Case a) by a reflection about the line $x_1 = x_2$. □

Remark 7.6. *We can choose any values for t_1, t_2, t_3 independently in the prescribed ranges, and we get a lattice free triangle. This observation shows that the family $\mathcal{F}$ defined above is exactly the family of all Type 3 triangles modulo an affine transformation.*

We now show how to bound Problem (15) and hence prove Theorem 1.7.

Consider any Type 3 triangle T. It is sufficient to consider the case where the lines supporting the edges of T have equations as stated in the Lemma 7.5.

We consider two cases for the position of the fractional point $f = (f_1, f_2)$.

(i) $f_1 \leq \frac{1}{2}$, $f_2 \leq \frac{1}{2}$;

(ii) $f_1 \leq 0$, $f_1 + f_2 \leq \frac{1}{2}$.

The two regions described above when rotated by $2\pi/3$ and $4\pi/3$ about the point $(\frac{1}{2}, \frac{1}{2})$, cover all of T. By rotational symmetry, investigating these two cases is enough.

For the first case, we relax Problem (15) by using only two inequalities from $\bar{T}_f(r^1, \ldots, r^k)$. These are derived from Type 2 triangles T_1 and T_2 (See Figure 7), which are defined as follows. T_1 has Line 1 and Line 2 supporting two of its edges and $x_1 = 1$ supporting the third one (with more than one integral point). T_2 has Line 2 and Line 3 supporting two of its edges and $x_2 = 1$ supporting the third one (with more than one integral point). Let ψ_{T_1} and ψ_{T_2} be the corresponding minimal functions derived from T_1 and T_2.

The following LP is a relaxation of Problem (15).

$$\begin{aligned} \min \quad & \sum_{i=1}^{k} s_i \\ & \sum_{i=1}^{k} \psi_{T_1}(r^i) s_i \geq 1 \quad \text{(Triangle } T_1\text{)} \\ & \sum_{i=1}^{k} \psi_{T_2}(r^i) s_i \geq 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}_+^k. \end{aligned} \tag{16}$$

Theorem 4.2 and Remark 5.2 imply that LP (16) is equivalent to

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & \psi_{T_1}(r^1)s_1 + \psi_{T_1}(r^2)s_2 + \psi_{T_1}(r^3)s_3 \geq 1 & \text{(Triangle } T_1) \\ & \psi_{T_2}(r^1)s_1 + \psi_{T_2}(r^2)s_2 + \psi_{T_2}(r^3)s_3 \geq 1 & \text{(Triangle } T_2) \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{17}$$

where r^1, r^2 and r^3 are the three corner rays (See Figure 7).

We now show that this LP has an optimal value of at least $\frac{1}{2}$.

Note that $\psi_{T_1}(r^2) = \psi_{T_1}(r^3) = 1$. $\psi_{T_1}(r^1)$ needs to be computed. First we compute r^1 and r^2 in terms of t_1, t_2, t_3, f_1 and f_2.

The intersection of Line 2 and Line 3 is given by

$$\left(\frac{t_3}{t_3 - t_2}, \frac{-t_2 t_3}{t_3 - t_2}\right) \qquad \text{and thus} \qquad r^1 = \left(\frac{t_3}{t_3 - t_2}, \frac{-t_2 t_3}{t_3 - t_2}\right) - (f_1, f_2)\ .$$

As $\psi_{T_1}(r^1) = \frac{1}{\gamma}$ where γ is such that $(f_1, f_2) + \gamma r^1$ lies on the line $x_1 = 1$, we get

$$\psi_{T_1}(r^1) = \frac{\frac{t_3}{t_3 - t_2} - f_1}{1 - f_1}\ .$$

Similarly, we only need $\psi_{T_2}(r^2)$ as $\psi_{T_2}(r^1) = \psi_{T_2}(r^3) = 1$. The intersection of Line 1 and Line 3 is

$$\frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3} \qquad \text{and thus} \qquad r^2 = \left(\frac{t_1(t_3 - 1)}{1 + t_1 t_3}, \frac{t_3(t_1 + 1)}{1 + t_1 t_3}\right) - (f_1, f_2)\ .$$

Computing the coefficient like before, we get

$$\psi_{T_2}(r^2) = \frac{\frac{t_3(t_1+1)}{1+t_1 t_3} - f_2}{1 - f_2}\ .$$

Hence LP (17) becomes

$$\begin{array}{lll} \min & s_1 + s_2 + s_3 & \\ & \frac{\frac{t_3}{t_3 - t_2} - f_1}{1 - f_1} s_1 + s_2 + s_3 \geq 1 & \text{(Triangle } T_1) \\ & s_1 + \frac{\frac{t_3(t_1+1)}{1+t_1 t_3} - f_2}{1 - f_2} s_2 + s_3 \geq 1 & \text{(Triangle } T_2) \\ & s \in \mathbb{R}^3_+. & \end{array} \tag{18}$$

As

$$\frac{\frac{t_3}{t_3 - t_2} - f_1}{1 - f_1} = \frac{\frac{t_3}{t_3 - t_2} - 1}{1 - f_1} + 1\ ,$$

the assumptions $f_1 \leq \frac{1}{2}$ and $t_2 < 1$ yield

$$\frac{\frac{t_3}{t_3 - t_2} - 1}{1 - f_1} + 1 \leq 2\frac{t_3}{t_3 - 1} - 1\ .$$

Similarly,

$$\frac{\frac{t_3(t_1+1)}{1+t_1t_3} - f_2}{1-f_2} = \frac{\frac{t_3(t_1+1)}{1+t_1t_3} - 1}{1-f_2} + 1$$

and the assumption $f_2 \le \frac{1}{2}$ gives

$$\frac{\frac{t_3(t_1+1)}{1+t_1t_3} - 1}{1-f_2} + 1 \le 2\frac{t_3(t_1+1)}{1+t_1t_3} - 1 = 2\frac{t_3-1}{1+t_1t_3} + 1 \ .$$

Under the assumption $t_1 > 0$, we obtain $2\frac{t_3-1}{1+t_1t_3} + 1 \le 2t_3 - 1$.

To get a lower bound on (18), we thus can relax its constraints to

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & (2\frac{t_3}{t_3-1} - 1)s_1 + s_2 + s_3 \ \ge 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1 + (2t_3 - 1)s_2 + s_3 \ \ge 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{19}$$

Set $\lambda = 2t_3 - 1$ and $\mu = 2\frac{t_3}{t_3-1} - 1$. Then $t_3 > 1$ implies $\lambda > 1$ and $\mu > 1$. The optimal solution of LP (19) is $s_1 = \frac{\lambda-1}{\lambda\mu-1}$, $s_2 = \frac{\mu-1}{\lambda\mu-1}$ and $s_3 = 0$ with value

$$s_1 + s_2 + s_3 = \frac{\lambda+\mu-2}{\lambda\mu-1} = \frac{t_3^2 - 2t_3 + 2}{t_3^2} \ .$$

To find the minimum over all $t_3 > 1$, we set the derivative to 0, which gives the solution $t_3 = 2$. Thus the minimum value of $s_1 + s_2 + s_3$ is $\frac{1}{2}$.

Next we consider $f_1 \le 0$ and $f_1 + f_2 \le \frac{1}{2}$, the shaded region in Figure 8. We relax Problem (15) using only two inequalities. We take T_1 as before, but T_2 is the triangle formed by the following three lines : Line 2 from Lemma 7.5, line parallel to Line 1 from Lemma 7.5 but passing through $(-1, 1)$ and the line passing through $(1, 0), (0, 1)$.

As in the previous case, we formulate the relaxation as an LP with constraints corresponding to T_1 and T_2. The only difference from LP (17) is the coefficient $\psi_{T_2}(r^2)$. This time the point $(f_1, f_2) + \gamma r^2$ lies on the line $x_1 + x_2 = 1$ (recall that $\psi_{T_2}(r^2) = \frac{1}{\gamma}$). This gives us

$$\psi_{T_2}(r^2) = \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - f_1 - f_2}{1 - f_1 - f_2} \ .$$

We then have the following LP.

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & \frac{\frac{t_3}{t_3-t_2} - f_1}{1-f_1} s_1 + s_2 + s_3 \ \ge 1 \quad \text{(Triangle } T_1\text{)} \\ & s_1 + \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - f_1 - f_2}{1 - f_1 - f_2} s_2 + s_3 \ \ge 1 \quad \text{(Triangle } T_2\text{)} \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{20}$$

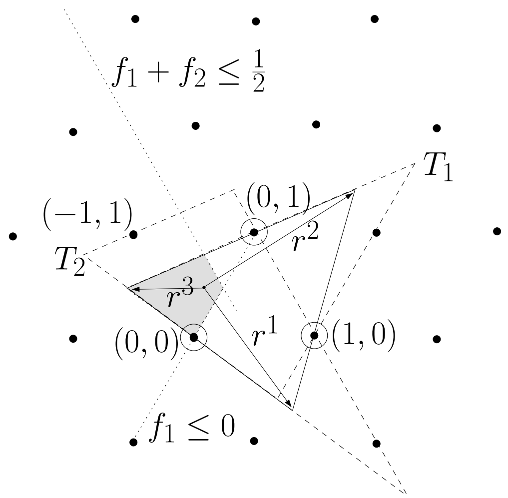

Figure 8: Approximating a Type 3 triangle inequality with Type 2 triangles inequalities - Case 2

We simplify the coefficients as earlier :

$$\frac{\frac{t_3}{t_3-t_2} - f_1}{1-f_1} = 1 + \frac{\frac{t_3}{t_3-t_2} - 1}{1-f_1} \qquad \text{and} \qquad \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - f_1 - f_2}{1-f_1-f_2} = 1 + \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - 1}{1-f_1-f_2} .$$

Using the assumptions $f_1 \leq 0, f_1 + f_2 \leq \frac{1}{2}$, we get that

$$1 + \frac{\frac{t_3}{t_3-t_2} - 1}{1-f_1} \leq \frac{t_3}{t_3-t_2} \qquad \text{and} \qquad 1 + \frac{\frac{2t_1t_3+t_3-t_1}{1+t_1t_3} - 1}{1-f_1-f_2} \leq 2(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}) - 1 .$$

We also have the conditions $t_2 < 1$ and $t_1 > 0$. $t_2 < 1$ implies $\frac{t_3}{t_3-t_2} \leq \frac{t_3}{t_3-1}$. Moreover

$$2\left(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}\right) - 1 = 2\left(2 + \frac{t_3-t_1-2}{1+t_1t_3}\right) - 1$$

decreases in value as t_1 increases. Its maximum value is less than the value for $t_1 = 0$, because of the condition $t_1 > 0$. It follows that $2(\frac{2t_1t_3+t_3-t_1}{1+t_1t_3}) - 1 \leq 2t_3 - 1$. After putting these relaxations into the constraints of LP (20), we get

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & \frac{t_3}{t_3-1}s_1 + s_2 + s_3 \quad \geq 1 \quad (\text{Triangle } T_1) \\ & s_1 + (2t_3-1)s_2 + s_3 \quad \geq 1 \quad (\text{Triangle } T_2) \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{21}$$

The optimal solution of LP (21) is

$$s_3 = 0, \quad s_1 = \frac{2(t_3-1)^2}{2t_3^2-2t_3+1}, \quad s_2 = \frac{1}{2t_3^2-2t_3+1}, \quad \text{and} \quad s_1+s_2+s_3 = \frac{2t_3^2-4t_3+3}{2t_3^2-2t_3+1} .$$

Under the condition $t_3 > 1$, the minimum value of $s_1 + s_2 + s_3$ is achieved for $t_3 = 1.70711$ with value $0.586 > \frac{1}{2}$. □

8 Split closure vs. a single triangle or quadrilateral inequality

In this section, we prove Theorem 1.7.

This is done by showing that there exist examples of integer programs (1) where the gap between the split closure $S_f(r^1, \dots, r^k)$ and a single triangle or quadrilateral inequality can be arbitrarily large. We give such examples for facets derived from triangles of Type 2 and Type 3, and from quadrilaterals.

These examples have the property that the point f lies in the relative interior of a segment joining two integral points at distance 1.

Furthermore, in these examples, the rays end on the boundary of the triangle or quadrilateral and hence the facet corresponding to it is of the form $\sum_{j=1}^k s_j \geq 1$. We show that the following LP has optimal value much less than 1.

$$\begin{aligned} z_{SPLIT} = \quad \min \quad & \sum_{j=1}^{k} s_j \\ & \sum_{j=1}^{k} \psi(r^j) s_j \quad \geq 1 \quad \text{for all splits } B_\psi \\ & s \in \mathbb{R}_+^k. \end{aligned} \tag{22}$$

Theorem 1.3 then implies Theorem 1.7.

A key step in the proof is a method for constructing a polyhedron contained in the split closure (Lemma 8.3). The resulting LP implies an upper bound on z_{SPLIT}. We then give a family of examples showing that this upper bound can be arbitrarily close to 0. We start the proof with an easy lemma.

8.1 An easy lemma

Refer to Figure 9 for an illustration of the following lemma.

Lemma 8.1. *Let r^1 and r^2 be two rays that are not multiples of each others and let H_1 and H_2 be the half-lines generated by nonnegative multiples of r^1 and r^2 respectively. Let $p := k_1\ r^1 + k_2\ r^2$ with $k_1, k_2 > 0$. Let L_1, L_2, and L_3 be three distinct lines going through p such that each of the lines intersect both H_1 and H_2 at points other than the origin. Let d_{ij} be the distance from the origin to the intersection of line L_i with the half-line H_j for $i = 1, 2, 3$ and $j = 1, 2$. Assume that $d_{11} < d_{21} < d_{31}$. Then there exists $0 < \lambda < 1$ such that*

$$\frac{1}{d_{21}} = \lambda\ \frac{1}{d_{11}} + (1-\lambda)\ \frac{1}{d_{31}} \qquad \textit{and} \qquad \frac{1}{d_{22}} = \lambda\ \frac{1}{d_{12}} + (1-\lambda)\ \frac{1}{d_{32}}.$$

Proof. Let u^i be a unit vector in the direction of r^i for $i = 1, 2$. Using $\{u^1, u^2\}$ as a base of $\mathbb{R}^2$, for $i = 1, 2, 3$, L_i has equation

$$\frac{1}{d_{i1}} x_1 + \frac{1}{d_{i2}} x_2 = 1$$

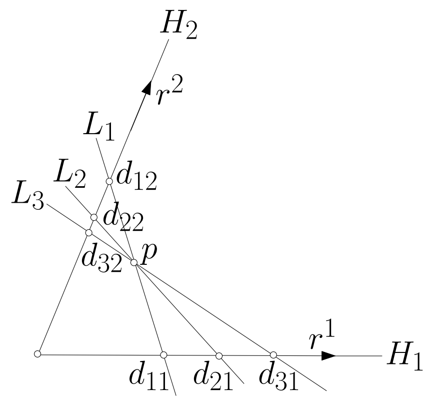

Figure 9: Illustration for Lemma 8.1

As L_2 is a convex combination of L_1 and L_3, there exists $0 < \lambda < 1$ such that $\lambda L_1 + (1-\lambda)L_3 = L_2$. The result follows. □

Corollary 8.2. *In the situation of Lemma 8.1, let L_4 be a line parallel to r^1 going through p. Let d_{42} be the distance between the origin and the intersection of H_2 with L_4. Then there exists $0 < \lambda < 1$ such that*

$$\frac{1}{d_{21}} = \lambda \, \frac{1}{d_{11}} \qquad \textit{and} \qquad \frac{1}{d_{22}} = \lambda \, \frac{1}{d_{12}} + (1-\lambda) \, \frac{1}{d_{42}}.$$

Proof. Similar to the proof of Lemma 8.1. □

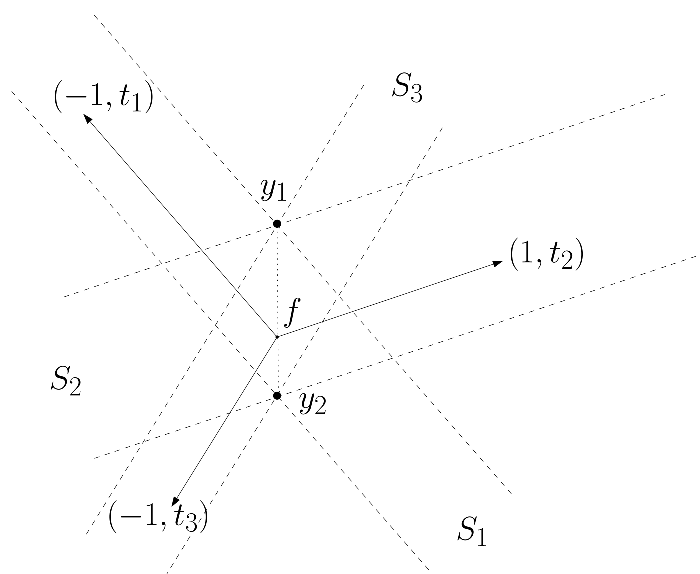

Figure 10: Dominating the Split closure with pseudo-splits

8.2 A polyhedron contained in the split closure

Our examples for proving Theorem 1.7 have the property that the point f lies in the relative interior of a segment joining two integral points y^1, y^2 at distance 1.

To obtain an upper bound on the value z_{SPLIT} of the split closure, we define some inequalities which dominate the split closure (22). A *pseudo-split* is the convex set between two distinct parallel lines passing through y^1 and y^2 respectively. The direction of the lines, called *direction* of the pseudo-split, is a parameter. Figure 10 illustrates three pseudo-splits in the directions of three rays. The *pseudo-split inequality* is derived from a pseudo-split exactly in the same way as from any maximal lattice-free convex set. Note that pseudo-splits are in general not lattice-free and hence do not generate valid inequalities for $R_f(r^1, \dots r^k)$. However, we can dominate any split inequality cutting f by an inequality derived from these convex sets. Indeed, consider any split S containing the fractional point f in its interior. Since f lies on the segment y^1y^2, both boundary lines of S pass through the segment y^1y^2. The pseudo-split with direction identical to the direction of S generates an inequality that dominates the split inequality derived from S, as the coefficient for any ray is smaller in the pseudo-split inequality.

The next lemma states that we can dominate the split closure by using only the inequalities generated by the pseudo-splits with direction parallel to the rays $r^1, \dots, r^k$ assuming mild conditions on the rays and f. We say that vectors in a given set are *not pairwise collinear* if no two of them are multiple of each other.

Lemma 8.3. *Assume that none of the rays $r^1, \dots, r^k$ has a zero first component and that at least three of them are not pairwise collinear. Assume also that $f = (0, f_2)$ with $0 < f_2 < 1$. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. Let $S_1, \dots, S_k$ be the pseudo-splits in the directions of rays $r^1, \dots, r^k$ and denote the corresponding minimal functions by $\psi_{S_1}, \dots, \psi_{S_k}$. Let S be any split with f in its interior and let S' be the corresponding pseudo-split. Then the inequality $\sum_{j=1}^k \psi_{S'}(r^j)s_j \geq 1$ corresponding to S' is dominated by a convex combination of the inequalities $\sum_{j=1}^k \psi_{S_i}(r^j)s_j$, $i = 1, \dots, k$. Therefore, the split inequality corresponding to S is dominated by a convex combination of the inequalities corresponding to $\psi_{S_1}, \dots, \psi_{S_k}$.*

Proof. As a convention, the direction of a pseudo-split forms an angle with the x_1-axis in the range of $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Without loss of generality, assume that the slope of the directions of the pseudo-splits corresponding to the rays $r^1, \dots r^k$ are monotonically non increasing. We can assume that the direction of S' is different than the direction of any of the rays in $\{r^1, \dots, r^k\}$ as otherwise the result trivially holds.

First note that, if S' has a direction with slope greater than the slope of r^1, then the inequality generated by S' is dominated by the one generated by S_1. Indeed, any ray r^j having a slope smaller than r^1 has its boundary point for S' closer to f than the one for S_1. It follows that $\psi_{S'}(r^j) \geq \psi_{S_1}(r^j)$. See Figure 11. A similar reasoning holds for the case where S' has a direction with slope smaller than the slope of r^k.

Thus we only have to consider the case where the slope of the direction of S' is strictly between the slopes of the directions of S_i and S_{i+1}, for some $1 \leq i \leq k-1$. We claim the following.

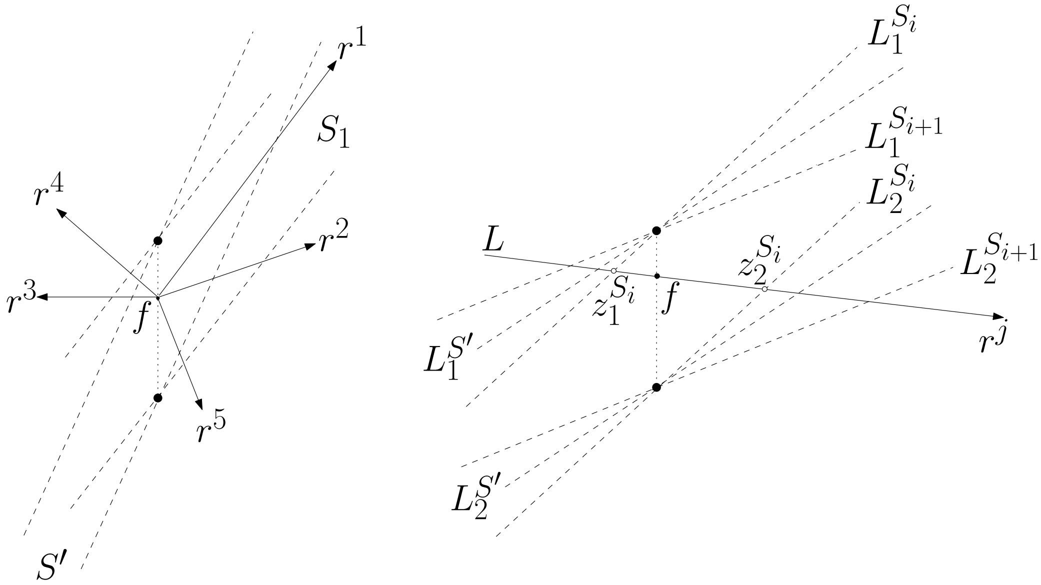

Figure 11: Bounding the split closure with a finite number of pseudo-splits

Observation 8.4. *There exists a* $0 < \lambda < 1$ *such that* $\psi_{S'}(r) = \lambda\psi_{S_i}(r) + (1-\lambda)\psi_{S_{i+1}}(r)$ *for every ray* $r \in \{r^1, \dots, r^k\}$.

Proof. For each pseudo-split $S \in \{S', S_i, S_{i+1}\}$, we denote by L_1^S its boundary line passing though $(0,1)$ and by L_2^S its boundary line passing through $(0,0)$.

Consider first any ray r^j with $j < i$ and let L^{r^j} be the half-line $f + \mu r^j$, $\mu \geq 0$. We have that L^{r^j} has a slope greater than the slope of the direction of S_i and thus L^{r^j} intersects the boundaries of S', S_i and S_{i+1} on $L_1^{S'}$, $L_1^{S_{i+1}}$ and $L_1^{S_i}$. By Lemma 8.1, there exists a $0 < \lambda_1 < 1$ such that, for all $r \in \{r^1, \dots, r^{i-1}\}$

$$\psi_{S'}(r) = \lambda_1\psi_{S_i}(r) + (1-\lambda_1)\psi_{S_{i+1}}(r) \ . \tag{23}$$

By Corollary 8.2, equation (23) also holds for $r = r^i$.

Using a similar reasoning for the rays $\{r^{i+1}, \dots, r^k\}$ and the boundary lines $L_2^{S'}$, $L_2^{S_{i+1}}$ and $L_2^{S_i}$, there exists a $0 < \lambda_2 < 1$ such that, for all $r \in \{r^{i+1}, \dots, r^k\}$

$$\psi_{S'}(r) = \lambda_2\psi_{S_i}(r) + (1-\lambda_2)\psi_{S_{i+1}}(r) \ . \tag{24}$$

It remains to show that $\lambda_1 = \lambda_2$. Consider any ray r^j that is not collinear with r^i or r^{i+1} and let L be the line passing through f with direction r^j. For each $S \in \{S', S_i, S_{i+1}\}$, let z_1^S (resp. z_2^S) be the intersection of L with L_1^S (resp. L_2^S) and let d_1^S (resp. d_2^S) be the distance from f to z_1^S (resp. z_2^S). See Figure 11. By Lemma 8.1

$$\frac{1}{d_1^{S'}} = \lambda_1 \ \frac{1}{d_1^{S_i}} + (1-\lambda_1) \ \frac{1}{d_1^{S_{i+1}}} \qquad \text{and} \qquad \frac{1}{d_2^{S'}} = \lambda_2 \ \frac{1}{d_2^{S_i}} + (1-\lambda_2) \ \frac{1}{d_2^{S_{i+1}}}. \tag{25}$$

The length of segments fy^1, fz_1^S, fy^2 and fz_2^S are respectively $1 - f_2, d_1^S, f_2$ and d_2^S. Observe that the triangles $fz_1^S y^1$ and $fz_2^S y^2$ are homothetic with homothetic ratio $t = \frac{1-f_2}{f_2}$. It follows that $\frac{d_1^S}{d_2^S} = t$. Substituting d_1^S by $t \cdot d_2^S$ in (25) yields $\lambda_1 = \lambda_2$. □

This observation proves the lemma.

□

Using the above lemma, we can bound the split closure for three rays, assuming that none of the rays has a zero first component and that they are not pairwise collinear. The latter condition is always verified if we assume that nonnegative combinations of the three rays generate $\mathbb{R}^2$, the situation we will consider in the remainder of this section. Without loss of generality, we make the following assumptions. The rays are $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i's are rational numbers in the range $[-\infty, \infty]$, with $t_1 > t_3$ and μ_i's are scaling factors with $\mu_i > 0$. Any configuration of three rays satisfying the above assumptions either fits this description or is a reflection of it about the segment $(0,0), (0,1)$. In addition, we must have $-t_1 < t_2 < -t_3$. See Figure 10 for an illustration.

Theorem 8.5. *Assume that $f = (0, f_2)$ with $0 < f_2 < 1$. Consider rays $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i's are rational numbers with $-t_1 < t_2 < -t_3$ and $\mu_i > 0$. Then*

$$z_{SPLIT} \leq \frac{1}{t_1 - t_3} \left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3} \right) .$$

Proof. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. By Lemma 8.3, we know that the three pseudo-splits S_1, S_2, S_3 corresponding to the directions of r^1, r^2, r^3 dominate the entire split closure. More formally, the following LP is a strengthening of (22) in this example of three rays.

$$\begin{array}{ll} \min & s_1 + s_2 + s_3 \\ & \psi_{S_1}(r^1)s_1 + \psi_{S_1}(r^2)s_2 + \psi_{S_1}(r^3)s_3 \quad \geq 1 \\ & \psi_{S_2}(r^1)s_1 + \psi_{S_2}(r^2)s_2 + \psi_{S_2}(r^3)s_3 \quad \geq 1 \\ & \psi_{S_3}(r^1)s_1 + \psi_{S_3}(r^2)s_2 + \psi_{S_3}(r^3)s_3 \quad \geq 1 \\ & s \in \mathbb{R}^3_+. \end{array} \tag{26}$$

It is fairly straightforward to compute the coefficients in the above inequalities. We give the calculations for S_1; the coefficients for the other two follow along similar lines.

$\psi_{S_1}(r^1)$ is 0, since r^1 is parallel to the direction of S_1.

Consider r^2 and let its boundary point p for S_1 be $(0, f_2) + \gamma\mu_2(1, t_2)$, for some $\gamma \geq 0$. Then $\psi_{S_1}(r^2)$ is $\frac{1}{\gamma}$. To compute γ, we observe that p is on boundary 1 of S_1, by assumption of $t_2 > -t_1$. Hence, the slope of the line connecting p and $(0, 1)$ is $-t_1$. Therefore,

$$\frac{f_2 + \gamma\mu_2 t_2 - 1}{0 + \gamma\mu_2} = -t_1$$

which yields $\gamma = \frac{1-f_2}{\mu_2(t_1+t_2)}$. Hence $\psi_{S_1}(r^2) = \frac{\mu_2(t_1+t_2)}{1-f_2}$.

Now consider r^3. As before, let its boundary point p' for S_1 be $(0, f_2) + \gamma'\mu_3(-1, t_3)$, for some $\gamma' \geq 0$. This time note that the ray intersects boundary 2 (by the assumption $t_3 < t_1$). Equating slopes, we get

$$\frac{f_2 + \gamma'\mu_3 t_3}{0 - \gamma'\mu_3} = -t_1$$

which yields $\gamma' = \frac{f_2}{\mu_3(t_1 - t_3)}$. Hence $\psi_{S_1}(r^3) = \frac{\mu_3(t_1-t_3)}{f_2}$.

So we have that the inequality corresponding to S_1 is

$$0 \cdot s_1 + \frac{\mu_2(t_1+t_2)}{1-f_2} s_2 + \frac{\mu_3(t_1-t_3)}{f_2} s_3 \geq 1 \ .$$

By very similar calculations, we can get the inequalities corresponding to ψ_{S_2} and ψ_{S_3}. LP (26) becomes

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & 0 \cdot s_1 + \frac{\mu_2(t_1+t_2)}{1-f_2} s_2 + \frac{\mu_3(t_1-t_3)}{f_2} s_3 && \geq 1 \\ & \frac{\mu_1(t_1+t_2)}{1-f_2} s_1 + 0 \cdot s_2 + \frac{\mu_3(-t_3-t_2)}{f_2} s_3 && \geq 1 \\ & \frac{\mu_1(t_1-t_3)}{1-f_2} s_1 + \frac{\mu_2(-t_3-t_2)}{f_2} s_2 + 0 \cdot s_3 && \geq 1 \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{27}$$

As a sanity check, note that the assumption $-t_1 < t_2 < -t_3$ implies that all the coefficients are non-negative.

The following solution is feasible for LP (27):

$$s_1 = \frac{1-f_2}{\mu_1(t_1-t_3)}, \quad s_2 = 0, \quad s_3 = \frac{f_2}{\mu_3(t_1-t_3)} \quad \text{and} \quad s_1 + s_2 + s_3 = \frac{1}{t_1-t_3}\left(\frac{1-f_2}{\mu_1} + \frac{f_2}{\mu_3}\right) \ .$$

Since the above LP was a strengthening of (22), we obtain

$$z_{SPLIT} \leq s_1 + s_2 + s_3 = \frac{1}{t_1-t_3}\left(\frac{1-f_2}{\mu_1} + \frac{f_2}{\mu_3}\right) \ .$$

□

If the rays are such that $\mu_1 = \mu_3 = 1$, then the above expression is $\frac{1}{t_1-t_3}$. This implies that in this case if we have rays such that $(t_1 - t_3)$ tends to infinity, then z_{SPLIT} tends to 0.

8.3 Type 2 triangles that do much better than the split closure

In Section 8.2, we showed that we can bound the value of the split closure under mild conditions on f and the rays. In particular, we showed that as $t_1 - t_3$ increases in value, the split closure does arbitrarily badly. In this section, we consider an infinite family of Type 2 triangles with rays pointing to its corners which satisfy these conditions.

Consider the same situation as in section 8.2 and consider the Type 2 triangle T with the following three edges. The line parallel to the x_2-axis and passing through $(-1, 0)$ supports

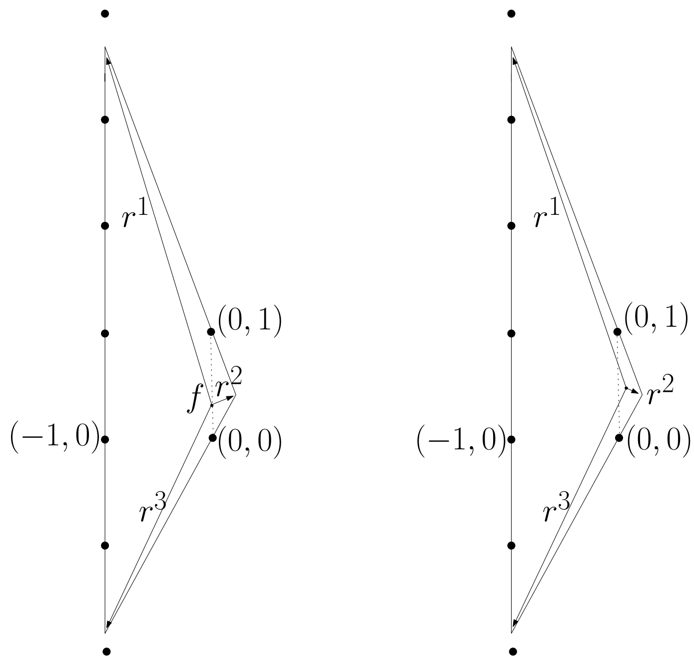

Figure 12: Facets from Type 2 triangles with large gap versus the split closure

one of the edges, and the other two edges are supported by lines passing through $(0,1)$ and $(0,0)$ respectively. See left part of Figure 12. Note that in this example, the rays are of the form $r^1 = (-1, t_1), r^2 = \mu(1, t_2), r^3 = (-1, t_3)$. In the notation of Section 8.2, $\mu_1 = \mu_3 = 1$.

Theorem 8.6. *Given any $\alpha > 1$, there exists a Type 2 triangle T as shown in Figure 12 such that for any point f in the relative interior of the segment joining $(0,0)$ to $(0,1)$, LP (22) has value $z_{SPLIT} \leq \frac{1}{\alpha}$.*

Proof. Let $M = \lceil \alpha \rceil$. When the fractional point f is on the segment connecting $(0,0)$ and $(0,1)$, consider the triangle T with M integral points in the interior of the vertical edge (the triangle on the left in Figure 12). This implies $t_1 - t_3 \geq M$. Therefore, from the result of Section 8.2, $\mu_1 = \mu_3 = 1$ implies that $z_{SPLIT} \leq \frac{1}{t_1 - t_3} \leq \frac{1}{\alpha}$.

□

In this example, for any large constant α, optimizing over the split closure in the direction of the facet defined by these Type 2 triangles yields at most $\frac{1}{\alpha}$. This implies Theorem 1.7.

8.4 More bad examples

The examples of Section 8.3 can be modified in various ways while keeping the property that the split closure is arbitrarily bad. The proofs are similar to that of Theorem 1.7.

8.4.1 Type 2 triangles when f is not on the segment joining $(0,0)$ to $(0,1)$

The example of Section 8.3 can be generalized to the case where f is not on the segment connecting the points $(0,0)$ and $(0,1)$ as follows. Let T be a Type 2 triangle as shown on the right part of Figure 12. Let Δ be the triangle with vertices $(0,0)$, $(0,1)$ and the vertex x^2 of T with positive first coordinate. When the fractional point f is in the interior of triangle Δ, and triangle T has $2M$ integral points on its vertical edge, one can show that $z_{SPLIT} \leq \frac{1}{M}$.

However, such bad examples cannot be constructed for any position of point f in the triangle T. In particular, define the triangle Δ' obtained from Δ by a homothetic transformation with center x^2 and factor 2 (so one vertex of Δ' is x^2 and points $(0,0)$ and $(0,1)$ become the middle points of the two edges of Δ' with endpoint x^2). When f is an interior point of T outside Δ', it is easy to see that the split inequality obtained from the split parallel to the x_2-axis $-1 \leq x_1 \leq 0$ approximates the triangle inequality defined by T to within a factor at most 2. Indeed the linear program is

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & \frac{1+u-f_1}{1-f_1} s_1 + s_2 + s_3 \quad \geq 1 \\ & s \in \mathbb{R}^3_+, \end{aligned} \tag{28}$$

where u is the first coordinate of x^2. The optimal solution is $s_1 = \frac{1-f_1}{1+u-f_1}$, $s_2 = s_3 = 0$. Thus $s_1 + s_2 + s_3 = \frac{1-f_1}{1+u-f_1} \geq \frac{1}{2}$ since $1 - f_1 \geq u$ for any $f \in T \setminus \Delta'$. This implies that the split inequality approximates the triangle inequality by a factor at most 2 when f is outside Δ'.

8.4.2 Triangles of Type 3 and quadrilaterals

We now show how to modify the construction of Section 8.3 to get examples of Type 3 triangles and quadrilaterals that do arbitrarily better than the Split Closure.

To get a Type 3 triangle, we tilt the vertical edge of the triangle in Figure 12 around its integral point with minimum x_2-value. See Figure 13. The same bound on z_{SPLIT} is then achieved.

Similarly, quadrilaterals can be constructed by breaking the vertical edge in Figure 12 into two edges of the quadrilateral. See Figure 13. By very similar arguments as the previous section, we can show that z_{SPLIT} tends to 0.

9 Conclusion

In this paper we gave examples of integer programs with two equality constraints, two free integer variables and nonnegative integer variables where the fraction of the integrality gap closed by the split closure is arbitrarily small. On the other hand we showed that the triangle closure always closes at least half of the integrality gap. Similarly, the quadrilateral closure always closes at least half of the integrality gap.

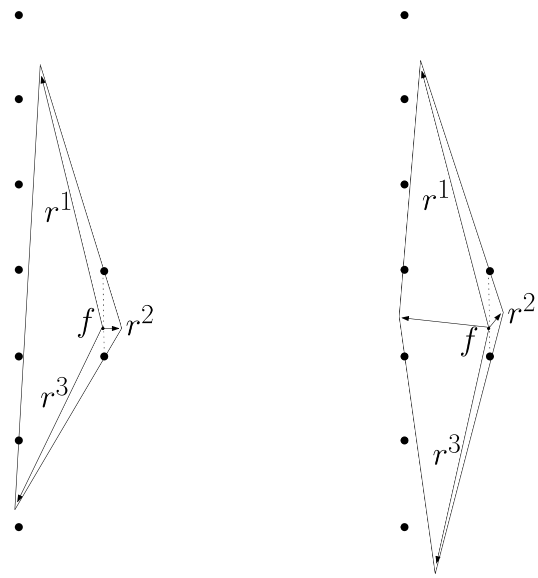

Figure 13: Facets from Type 3 triangles and quadrilaterals on which the split closure does poorly

References

[1] K. Andersen, Q. Louveaux, R. Weismantel and L. Wolsey, Cutting Planes from Two Rows of a Simplex Tableau, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 1–15.

[2] E. Balas, Intersection Cuts - A New Type of Cutting Planes for Integer Programming, *Operations Research 19* (1971) 19–39.

[3] E. Balas, S. Ceria and G. Cornuéjols, A Lift-and-project Cutting Plane Algorithm for Mixed 0-1 Programs, *Mathematical Programming 58* (1993) 295–324.

[4] E. Balas and A. Saxena, Optimizing over the Split Closure, *Mathematical Programming A 113* (2008) 219–240.

[5] V. Borozan and G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, technical report (July 2007).

[6] W. Cook, R. Kannan and A. Schrijver, Chvátal Closures for Mixed Integer Programming Problems, *Mathematical Programming 47* (1990) 155–174.

[7] G. Cornuéjols and F. Margot, On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints, to appear in *Mathematical Programming.*

[8] H. Crowder, E.L. Johnson, M. Padberg, Solving Large-Scale Zero-One Linear Programming Problems, *Operations Research 31* (1983) 803–834.

[9] S. Dash, O. Günlük, A. Lodi, On the MIR Closure of Polyhedra, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 337–351.

[10] S.S. Dey and L.A. Wolsey, Lifting Integer Variables in Minimal Inequalities Corresponding to Lattice-Free Triangles, *IPCO 2008*, Bertinoro, Italy (June 2008), Lecture Notes in Computer Science 5035, 463–476.

[11] D. Espinoza, Computing with multi-row Gomory cuts, *IPCO 2008*, Bertinoro, Italy (June 2008), Lecture Notes in Computer Science 5035, 214–224.

[12] M.X. Goemans, Worst-case Comparison of Valid Inequalities for the TSP, *Mathematical Programming 69* (1995) 335–349.

[13] R.E. Gomory, An Algorithm for Integer Solutions to Linear Programs, *Recent Advances in Mathematical Programming*, R.L. Graves and P. Wolfe eds., McGraw-Hill, New York (1963) 269–302.

[14] R.E. Gomory, Thoughts about Integer Programming, 50th Anniversary Symposium of OR, University of Montreal, January 2007, and Corner Polyhedra and Two-Equation Cutting Planes, George Nemhauser Symposium, Atlanta, July 2007.

[15] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989) 177–210.

[16] H. Marchand and L.A. Wolsey, Aggregation and Mixed Integer Rounding to Solve MIPs, *Operations Research 49* (2001) 363–371.

[17] R.R. Meyer, On the Existence of Optimal Solutions to Integer and Mixed Integer Programming Problems, *Mathematical Programming 7* (1974) 223–235.

[18] G.L. Nemhauser and L.A. Wolsey, A Recursive Procedure to Generate All Cuts for 0-1 Mixed Integer Programs, *Mathematical Programming 46* (1990) 379–390.

[19] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, (1986) p.114.