Bicolorings and Equitable Bicolorings of Matrices

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dedicated to Manfred Padberg

Abstract

Two classical theorems of Ghouila-Houri and Berge characterize total unimodularity and balancedness in terms of equitable bicolorings and bicolorings, respectively. In this paper, we prove a bicoloring result that provides a common generalization of these two theorems.

A $0, \pm 1$ matrix is *balanced* if it does not contain a square submatrix with exactly two nonzero entries per row and per column such that the sum of all the entries is congruent to 2 modulo 4. This notion was introduced by Berge [1] for $0, 1$ matrices and generalized by Truemper [15] to $0, \pm 1$ matrices.
A $0, \pm 1$ matrix is *bicolorable* if its columns can be partitioned into blue columns and red columns so that every row with at least two nonzero entries contains either two nonzero entries of opposite sign in columns of the same color or two nonzero entries of the same sign in columns of different colors. Berge [1] showed that a $0, 1$ matrix A is balanced if and only if every submatrix of A is bicolorable. Conforti and Cornuéjols [6] extended this result to $0, \pm 1$ matrices. Cameron and Edmonds [3] gave a simple greedy algorithm to find a bicoloring of a balanced matrix. In fact, given any $0, \pm 1$ matrix A, their algorithm finds either a bicoloring of A or a square submatrix of A with exactly two nonzero entries per row and per column such that the sum of all the entries is congruent to 2 modulo 4. Does this algorithm provide an easy test for balancedness? The answer is no, because the algorithm may find a bicoloring of A even when A is not balanced.

A real matrix is *totally unimodular* (t.u.) if every nonsingular square submatrix has determinant ± 1 (note that every t.u. matrix must be a $0, \pm 1$ matrix).
A $0, \pm 1$ matrix A has an *equitable bicoloring* if its columns can be partitioned into red

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and blue columns so that, for every row of A, the sum of the entries in the red columns differs by at most one from the sum of the entries in the blue columns. Ghouila-Houri [9] showed that a $0, \pm 1$ matrix is totally unimodular if and only if every submatrix of A has an equitable bicoloring.

A $0, \pm 1$ matrix which is not totally unimodular but whose submatrices are all totally unimodular is said *almost totally unimodular*. Camion [4] proved the following:

Theorem 1 (Camion [4] and Gomory [cited in [4]]) *Let A be an almost totally unimodular $0, \pm 1$ matrix. Then A is square, $\det A = \pm 2$ and A^{-1} has only $\pm\frac{1}{2}$ entries. Furthermore, each row and each column of A has an even number of nonzero entries and the sum of all entries in A equals 2 modulo 4.*

A nice proof of this result can be found in Padberg [12], [13]. Note that a matrix is balanced if and only if it does not contain any almost totally unimodular matrix with two nonzero entries in each row. For any positive integer k, we say that a $0, \pm 1$ matrix A is *k-balanced* if it does not contain any almost totally unimodular submatrix with at most $2k$ nonzero entries in each row. Obviously, an $m \times n$ $0, \pm 1$ matrix A is balanced if and only if it is 1-balanced, while A is totally unimodular if and only if A is k-balanced for some $k \geq \lfloor n/2 \rfloor$. The class of k-balanced matrices was introduced by Conforti, Cornuéjols and Truemper in [7], where they proved the following:

For any integer k, we denote by $\mathbf{k}$ a vector with all entries equal to k. For any $m \times n$ $0, \pm 1$ matrix A, we denote by $n(A)$ the vector with m components whose ith component is the number of -1's in the ith row of A.

Theorem 2 *Let A be an $m \times n$ k-balanced $0, \pm 1$ matrix with rows a^i, $i \in [m]$, b be a vector with entries b_i, $i \in [m]$, and let S_1, S_2, S_3 be a partition of $[m]$. Then*

$$\begin{aligned} P(A,b) = \{x \in \mathbb{R}^n \;\; : \;\; & a^i x \leq b_i \textit{ for } i \in S_1 \\ & a^i x = b_i \textit{ for } i \in S_2 \\ & a^i x \geq b_i \textit{ for } i \in S_3 \\ & \mathbf{0} \leq x \leq \mathbf{1}\} \end{aligned}$$

is an integral polytope for all integral vectors b such that $-n(A) \leq b \leq \mathbf{k} - n(A)$.

This theorem generalizes previous results by Hoffman and Kruskal [10] for totally unimodular matrices, Berge [2] for $0, 1$ balanced matrices, Conforti and Cornuéjols [6] for $0, \pm 1$ balanced matrices, and Truemper and Chandrasekaran [16] for k-balanced 0,1 matrices.

A $0, \pm 1$ matrix A has a *k-equitable bicoloring* if its columns can be partitioned into blue columns and red columns so that:

- the bicoloring is equitable for the row submatrix A' determined by the rows of A with at most $2k$ nonzero entries,

- every row with more than $2k$ nonzero entries contains k pairwise disjoint pairs of nonzero entries such that each pair contains either entries of opposite sign in columns of the same color or entries of the same sign in columns of different colors.

Obviously, an $m \times n$ $0, \pm 1$ matrix A is bicolorable if and only if A has a 1-equitable bicoloring, while A has an equitable bicoloring if and only if A has a k-equitable bicoloring for $k \geq \lfloor n/2 \rfloor$. The following theorem provides a new characterization of the class of k-balanced matrices, which generalizes the bicoloring results mentioned above for balanced and totally unimodular matrices.

Theorem 3 *A $0, \pm 1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable bicoloring.*

Proof. Assume first that A is k-balanced and let B be any submatrix of A. Assume, up to row permutation, that

$$B = \begin{pmatrix} B' \\ B'' \end{pmatrix}$$

where B' is the row submatrix of B determined by the rows of B with $2k$ or fewer nonzero entries. Consider the system

$$\begin{array}{rcl} B'x & \geq & \left\lfloor \dfrac{B'\mathbf{1}}{2} \right\rfloor \\ -B'x & \geq & -\left\lceil \dfrac{B'\mathbf{1}}{2} \right\rceil \\ B''x & \geq & \mathbf{k} - n(B'') \\ -B''x & \geq & \mathbf{k} - n(-B'') \\ \mathbf{0} \leq x & \leq & \mathbf{1} \end{array} \tag{1}$$

Since B is k-balanced, also $\begin{pmatrix} B \\ -B \end{pmatrix}$ is k-balanced. Therefore the constraint matrix of system (1) above is k-balanced. One can readily verify that $-n(B') \leq \left\lfloor \frac{B'\mathbf{1}}{2} \right\rfloor \leq \mathbf{k} - n(B')$ and $-n(-B') \leq -\left\lceil \frac{B'\mathbf{1}}{2} \right\rceil \leq \mathbf{k} - n(-B')$. Therefore, by Theorem 2 applied with $S_1 = S_2 = \emptyset$, system (1) defines an integral polytope. Since the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a solution for (1), the polytope is nonempty and contains a $0,1$ point $\bar{x}$. Color a column i of B blue if $\bar{x}_i = 1$, red otherwise. It can be easily verified that such a bicoloring is, in fact, k-equitable.

Conversely, assume that A is not k-balanced. Then A contains an almost totally unimodular matrix B with at most $2k$ nonzero elements per row. Suppose that B has a k-equitable bicoloring, then such a bicoloring must be equitable since each row has, at most, $2k$ nonzero elements. By Theorem 1, B has an even number of nonzero elements in each row. Therefore the sum of the columns colored blue equals the sum of the columns colored red, therefore B is a singular matrix, a contradiction. $\Box$

Given a $0, \pm 1$ matrix A and positive integer k, one can find in polynomial time a k-equitable bicoloring of A or a certificate that A is not k-balanced as follows:

Find a basic feasible solution of (1). If the solution is not integral, A is not k-balanced by Theorem 2. If the solution is a 0,1 vector, it yields a k-equitable bicoloring as in the proof of Theorem 3.

Note that, as with the algorithm of Cameron and Edmonds [3], a 0,1 vector may be found even when the matrix A is not k-balanced.

Using the fact that the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a feasible solution of (1), a basic feasible solution of (1) can actually be derived in strongly polynomial time using an algorithm of Megiddo [11].

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