# Split Closure and Intersection Cuts *

Kent Andersen Gérard Cornuéjols Yanjun Li

Graduate School of Industrial Administration
Carnegie Mellon University, Pittsburgh, PA, USA
{kha,gc0v,yanjun}@andrew.cmu.edu

**Abstract.** In the seventies, Balas introduced intersection cuts for a Mixed Integer Linear Program (MILP), and showed that these cuts can be obtained by a closed form formula from a basis of the standard linear programming relaxation. In the early nineties, Cook, Kannan and Schrijver introduced the split closure of an MILP, and showed that the split closure is a polyhedron. In this paper, we show that the split closure can be obtained using only intersection cuts. We give two different proofs of this result, one geometric and one algebraic. Furthermore, the result is used to provide a new proof of the fact that the split closure is a polyhedron. Finally, we extend the result to more general two-term disjunctions.

## 1 Introduction

In the seventies, Balas showed how a cone and a disjunction can be used to derive a cut [1] for a Mixed Integer Linear Program (MILP). In that paper, the cone was obtained from an optimal basis to the standard linear programming relaxation. The cut was obtained by a closed form formula and was called the intersection cut.

Later in the seventies, Balas generalized the idea to polyhedra [2]. It was demonstrated that, given a polyhedron, and a valid but violated disjunction, a cut could be obtained by solving a linear program. The idea was further expanded in the early nineties, where Cook, Kannan and Schrijver [4] studied split cuts obtained from two-term disjunctions that are easily seen to be valid for an MILP. The intersection of all split cuts is called the split closure of an MILP. Cook, Kannan and Schrijver proved that the split closure of an MILP is a polyhedron.

Any basis of the constraint matrix describing the polyhedron can be used, together with a disjunction, to derive an intersection cut, i.e. the basis used does not have to be optimal or even feasible. A natural question is how intersection cuts relate to disjunctive cuts obtained from polyhedra. This question was answered by Balas and Perregaard [3] for the 0-1 disjunction for Mixed Binary Linear Programs. The conclusion was that any disjunctive cut obtained from a polyhedron and the 0-1 disjunction is identical to, or dominated by, intersection cuts obtained from the 0-1 disjunction and bases of the constraint matrix describing the polyhedron. We generalize this result from 0-1 disjunctions to more

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general two-term disjunctions. This is the main result of this paper. We provide two different proofs, one geometric and one algebraic. A consequence is a new proof of the fact that the split closure is a polyhedron.

We consider the Mixed Integer Linear Program (MILP):

$$\text{(MILP)} \qquad min\{c^T x : Ax \leq b, x_j \text{ integer}, j \in N_I\},$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $N_I \subseteq N := \{1, 2, \ldots, n\}$ and $A$ is an $m \times n$ matrix. $r$ denotes the rank of $A$. LP is the Linear Programming problem obtained from MILP by dropping the integrality conditions on $x_j$, $j \in N_I$. $P_I$ and $P$ denote the sets of feasible solutions to MILP and LP respectively. $M := \{1, 2, \ldots, m\}$ is used to index the rows of $A$. $a_{i.}$, for $i \in M$, denotes the $i^{th}$ row of $A$. We assume $a_{i.} \neq 0_n$ for all $i \in M$. Given $S \subseteq M$, $r(S)$ denotes the rank of the sub-matrix of $A$ induced by the rows in $S$ ($r(M) = r$). Furthermore, for $S \subseteq M$, the relaxation of $P$ obtained by dropping the constraints indexed by $M \setminus S$ from the description of $P$, is denoted $P(S)$, i.e. $P(S) := \{x \in \mathbb{R}^n : (a_{i.})^T x \leq b_i, \forall i \in S\}$ ($P(M) = P$). A *basis* of $A$ is an $n$-subset $B$ of $M$, such that the vectors $a_{i.}$, $i \in B$, are linearly independent. Observe that, if $r < n$, $A$ does not have bases. $\mathcal{B}_k^*$, where $k$ is a positive integer, denotes the set of $k$-subsets $S$ of $M$, such that the vectors $a_{i.}$, $i \in S$, are linearly independent ($\mathcal{B}_n^*$ denotes the set of bases of $A$).

The most general two-term disjunction considered in this paper is an expression $D$ of the form $D^1 x \leq d^1 \vee D^2 x \leq d^2$, where $D^1 : m_1 \times n$, $D^2 : m_2 \times n$, $d^1 : m_1 \times 1$ and $d^2 : m_2 \times 1$. The set of points in $\mathbb{R}^n$ satisfying $D$ is denoted $F_D$. The set $conv(P \cap F_D)$ is called the *disjunctive set* defined by $P$ and $D$ in the remainder. In addition, given a subset $S$ of the constraints, the set $conv(P(S) \cap F_D)$ is called the disjunctive set defined by $S$ and $D$. Finally, given a basis $B$ in $\mathcal{B}_r^*$, the set $conv(P(B) \cap F_D)$ is called a *basic* disjunctive set.

An important two-term disjunction, in the context of an MILP, is the *split disjunction* $D(\pi, \pi_0)$ of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\pi_j = 0$ for all $j \notin N_I$. For the split disjunction, a complete description of the basic disjunctive sets is available as follows. Given a set $B$ in $\mathcal{B}_r^*$, the basic disjunctive set defined by $B$ and $D(\pi, \pi_0)$ is the set of points in $P(B)$ that satisfy the intersection cut derived from $B$ and $D(\pi, \pi_0)$ (Lemma 1). Let $\Pi^n(N_I) := \{(\pi, \pi_0) \in \mathbb{Z}^{n+1} : \pi_j = 0, j \notin N_I\}$. The *split closure* of MILP, denoted by SC, is defined to be the intersection of the disjunctive sets defined by $P$ and $D(\pi, \pi_0)$ over all disjunctions $(\pi, \pi_0)$ in $\Pi^n(N_I)$. Similarly, given $S \subseteq M$, SC$(S)$, is defined to be the intersection of the disjunctive sets defined by $P(S)$ and $D(\pi, \pi_0)$ over all disjunctions $(\pi, \pi_0)$ in $\Pi^n(N_I)$. A *split cut* is a valid inequality for SC.

The first contribution in this paper is a theorem (Theorem 1) stating that the split closure of MILP can be written as the intersection of the split closures of the sets $P(B)$ over all sets $B$ in $\mathcal{B}_r^*$. (i.e. $SC = \bigcap_{B \in \mathcal{B}_r^*} SC(B)$). We prove this theorem by proving that the disjunctive set defined by $P$ and $D(\pi, \pi_0)$, for a split disjunction $D(\pi, \pi_0)$, can be written as the intersection of the basic disjunctive sets (i.e. $conv(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{B \in \mathcal{B}_r^*} conv(P(B) \cap F_{D(\pi,\pi_0)})$). We provide both a geometric and an algebraic proof of this result. The result implies that both the disjunctive set defined by $P$ and $D(\pi, \pi_0)$ and the split closure of MILP can be obtained using only intersection cuts. This generalizes a result of Balas and Perregaard showing the same result for the disjunction $x_j \le 0 \vee x_j \ge 1$ for Mixed Binary Linear Programs [3]. (In fact, in that paper, it was assumed that $r = n$, whereas the theorem presented here does not have this assumption). Furthermore, the result leads to a new proof of the fact that the split closure is a polyhedron (Theorem 3).

The second contribution in this paper is a theorem (Theorem 6) stating that the disjunctive set defined by $P$ and a general two-term disjunction $D$ can be written as the intersection of disjunctive sets defined by $D$ and $(r+1)$-subsets $S$ of $M$ (i.e. $conv(P \cap F_D) = \bigcap_{S \in \mathcal{C}_1^*} conv(P(S) \cap F_D)$, where $\mathcal{C}_1^*$ is some family of $(r+1)$-subsets of $M$ to be defined later). The theorem implies that any valid inequality for the disjunctive set defined by $P$ and $D$ is identical to, or dominated by, inequalities derived from the disjunction $D$ and $(r+1)$-subsets of the constraints describing the polyhedron. Furthermore, in the special case where $r = n$, we show that it is enough to consider a certain family $\mathcal{C}_2^*$ of $n$-subsets of the constraints describing the polyhedron.

The rest of this paper is organized as follows. In section 2, we consider basic disjunctive sets for split disjunctions. In section 3, the characterization of the split closure in terms of intersection cuts is presented, and a geometric argument for the validity of the result is presented. In section 4 we give a new proof of the fact that the split closure is a polyhedron. In section 5, we generalize the results of section 3 to more general two-term disjunctions. We also give an example showing that other extensions are incorrect. The arguments used in this section are mostly algebraic. In fact, section 3 and section 5 could be read independently.

## 2 A complete description of the basic disjunctive set for split disjunction

In this section, we describe the set $P(B)$ for $B$ in $B_r^*$ as the translate of a cone, and use this cone together with a split disjunction to derive an intersection cut. The intersection cut is then used to characterize the basic disjunctive set obtained from the split disjunction.

Let $B \in \mathcal{B}_r^*$ be arbitrary. The set $P(B)$ was defined using half-spaces in the introduction. We now give an alternative description. Let $\bar{x}(B)$ satisfy $a_{i.}^T \bar{x}(B) = b_i$ for all $i \in B$. Furthermore, let $L(B) := \{x \in \mathbb{R}^n : a_{i.}^T x = 0, \forall i \in B\}$. Finally, let $r^i(B)$, where $i \in B$, be a solution to the system $(a_{k.})^T r^i(B) = 0$, for $k \in B \setminus \{i\}$, and $(a_{i.})^T r^i(B) = -1$. We have:

$$P(B) = \bar{x}(B) + L(B) + Cone(\{r^i(B) : i \in B\}), \tag{1}$$

where $Cone(\{r^i(B) : i \in B\}) := \{x \in \mathbb{R}^n : x = \sum_{i \in B} \lambda_i r^i(B),\ \lambda_i \ge 0, i \in B\}$ denotes the cone generated by the vectors $r^i(B)$, $i \in B$. Observe that the vectors $r^i(B)$, $i \in B$, are linearly independent.

Let $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. Assume that all points $y$ in $\bar{x}(B) + L(B)$ violate the disjunction $D(\pi, \pi_0)$ (Lemma 1 below shows that this is the only interesting case).

Observe that this implies that the linear function $\pi^T x$ is constant on $\bar{x}(B)+L(B)$. The intersection cut defined by $B$ and $D(\pi,\pi_0)$ can now be described. Define $\epsilon(\pi,B) := \pi^T\bar{x}(B) - \pi_0$ to be the amount by which the points in $\bar{x}(B)+L(B)$ violate the first term in the disjunction. Also, for $i$ in $B$, define:

$$\alpha_i(\pi,B) := \begin{cases} -\epsilon(\pi,B)/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) < 0, \\ (1-\epsilon(\pi,B))/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

The interpretation of the numbers $\alpha_i(\pi,B)$, for $i \in B$, is the following. Let $x^i(\alpha,B) := \bar{x}(B) + \alpha r^i(B)$, where $\alpha \in \mathbb{R}_+$, denote the half-line starting in $\bar{x}(B)$ in the direction $r^i(B)$. The value $\alpha_i(\pi,B)$ is the smallest value of $\alpha \in \mathbb{R}_+$, such that $x^i(\alpha,B)$ satisfies the disjunction $D(\pi,\pi_0)$, i.e. $\alpha_i(\pi,B) = \inf\{\alpha \geq 0 : x^i(\alpha,B) \in F_{D(\pi,\pi_0)}\}$. Given the numbers $\alpha_i(\pi,B)$ for $i \in B$, the intersection cut associated with $B$ and $D(\pi,\pi_0)$ is given by:

$$\sum\nolimits_{i\in B} (b_i - a_{i.}^T x)/\alpha_i(\pi,B) \geq 1. \tag{3}$$

The validity of this inequality for $Conv(P(B)\cap F_{D(\pi,\pi_0)})$ was proven by Balas [1]. In fact, we have:

**Lemma 1** *Let $B \in \mathcal{B}_r^*$ and $D(\pi,\pi_0)$ be a split disjunction, where $(\pi,\pi_0) \in \mathbb{Z}^{n+1}$.*

*(i) If $\pi^T x \notin ]\pi_0,\pi_0+1[$, for some $x \in \bar{x}(B)+L(B)$, then $Conv(P(B)\cap F_{D(\pi,\pi_0)}) = P(B)$.*
*(ii) If $\pi^T x \in ]\pi_0,\pi_0+1[$, for all $x \in \bar{x}(B)+L(B)$, then $Conv(P(B)\cap F_{D(\pi,\pi_0)}) = \{x \in P(B) : (3)\}$.*

## 3 Split closure characterization

In this section, we give a geometric proof of the following theorem, which characterizes the split closure in terms of certain basic subsets of the constraints.

**Theorem 1**

$$SC = \bigcap_{B\in\mathcal{B}_r^*} SC(B). \tag{4}$$

We prove this result in the following lemmas and corollaries. Let $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$ be inequalities, where $\alpha \in \mathbb{R}^n$ and $\beta < \psi$. When $\alpha \neq 0_n$, $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$ represent two non-intersecting half-spaces. We have the following key lemma:

**Lemma 2** *Assume $P$ is full-dimensional. Then:*

$$Conv((P\cap\{x:\alpha^T x\leq\beta\})\cup(P\cap\{x:\alpha^T x\geq\psi\})) = \bigcap_{B\in\mathcal{B}_r^*} Conv((P(B)\cap\{x:\alpha^T x\leq\beta\})\cup(P(B)\cap\{x:\alpha^T x\geq\psi\})). \tag{5}$$

*Proof.* The following notation will be convenient. Define $P_1 := P \cap \{x : \alpha^T x \leq \beta\}$ and $P_2 := P \cap \{x : \alpha^T x \geq \psi\}$. Furthermore, given a set $B \in \mathcal{B}_r^*$, let $P_1(B) := P(B) \cap \{x : \alpha^T x \leq \beta\}$ and $P_2(B) := P(B) \cap \{x : \alpha^T x \geq \psi\}$.

When $\alpha = 0_n$, no matter what the values of $\beta$ and $\psi$ are, at least one of $P_1$ and $P_2$ is empty (Notice that we always have $\beta < \psi$). If both are empty, then the lemma holds trivially by $\emptyset = \emptyset$. If one is not empty, then $Conv(P_1 \cup P_2) = P$ and, similarly, $\bigcap_{B \in \mathcal{B}_r^*} Conv((P_1(B) \cup P_2(B)) = \bigcap_{B \in \mathcal{B}_r^*} P(B) = P$. The last equality is due to the assumption $a_{i.} \neq 0_n$ for $i \in M$. Therefore, we assume $\alpha \neq 0_n$ in the rest of the proof.

Because $P \subseteq P(B)$ for any $B \in \mathcal{B}_r^*$, it is clear that $Conv(P_1 \cup P_2) \subseteq \bigcap_{B \in \mathcal{B}_r^*} Conv((P_1(B) \cup P_2(B))$. Therefore, we only need to show the other direction of the inclusion.

Observe that it suffices to show that any valid inequality for $Conv(P_1 \cup P_2)$ is valid for $Conv(P_1(B) \cup P_2(B))$ for at least one $B \in \mathcal{B}_r^*$. Now, let $\delta^T x \leq \delta_0$ be a valid inequality for $Conv(P_1 \cup P_2)$. Clearly, we can assume $\delta^T x \leq \delta_0$ is facet-defining for $Conv(P_1 \cup P_2)$. This is clearly true for the valid inequalities of $P$, since we can always choose a $B \in \mathcal{B}_r^*$, by applying the techniques of linear programming, such that the valid inequality of $P$ is valid for $P(B)$. So we may assume that the inequality is valid for $Conv(P_1 \cup P_2)$ but not valid for $P$.

**Case 1.** $P_2 = \emptyset$.

Since $P_2 = \emptyset$, $Conv(P_1 \cup P_2) = P_1 = P \cap \{x : \alpha^T x \leq \beta\}$. Hence $\alpha^T x \leq \beta$ is a valid inequality for $Conv(P_1 \cup P_2)$. We just want to show that it is also valid for $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$ for some $B \in \mathcal{B}_r^*$. Because $P_2 = \emptyset$ and $P$ is full-dimensional, applying the techniques of linear programming shows that the value $\tilde{\gamma} = \max\{\gamma : P \cap \{x : \alpha^T x = \gamma\} \neq \emptyset\}$ specifies $B \in \mathcal{B}_r^*$ such that $P(B) \cap \{x : \alpha^T x \geq \psi\} = \emptyset$ and $\tilde{\gamma} = \max\{\gamma : P(B) \cap \{x : \alpha^T x = \gamma\} \neq \emptyset\}$, where $\tilde{\gamma} < \psi$. We have $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\})) = P(B) \cap \{x : \alpha^T x \leq \beta\}$ for this particular $B$. Therefore, $\alpha^T x \leq \beta$ is valid for $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$.

**Case 2.** $P_1$ and $P_2$ are full-dimensional.

Consider an arbitrary facet $F$ of $Conv(P_1 \cup P_2)$ which does not induce a valid inequality for $P$. We are going to use the corresponding $F$-defining inequality (half-space) and $F$-defining equality (hyperplane). Our goal is to show that the $F$-defining inequality is valid for $Conv(P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$ for some $B \in \mathcal{B}_r^*$.

Let $F_1 := F \cap P_1$ and $F_2 := F \cap P_2$. Since the $F$-defining inequality is valid for $P_1$ and $P_2$ but not valid for $P$, we can deduce $F_1 \subseteq \{x \in \mathbb{R}^n | \alpha^T x = \beta\}$ and $F_2 \subseteq \{x \in \mathbb{R}^n : \alpha^T x = \psi\}$. $F$ is the convex combination of $F_1$ and $F_2$, where $F_1$ is a $k$-dimensional face of $P_1$ and $F_2$ is an $m$-dimensional face of $P_2$. Since $F$ is of dimension $n-1$, we have $0 \leq k \leq n-2$ and $n-k-2 \leq m \leq n-2$.

The intersection of the $F$-defining hyperplane with $\alpha^T x = \beta$ (or $\alpha^T x = \psi$) defines a $(n-2)$-dimensional affine subspace which contains $F_1$ (or $F_2$). Therefore, $\mathit{Aff}(F_2)$ contains two affine subspaces $S_1$ and $S_2$ of dimensions $n-k-2$ and $m-(n-k-2)$ respectively, where $S_1$ is orthogonal to $\mathit{Aff}(F_1)$ and $S_2$ is parallel to $\mathit{Aff}(F_1)$. In other words, $\mathit{Aff}(F_2)$ can be orthogonally decomposed into two affine subspaces $S_1$ and $S_2$ such that $S_1 \cap S_2$ has an unique point $x_0$, $(x_1 - x_0)^T(x_2 - x_0) = 0$ for any $x_1 \in S_1$ and $x_2 \in S_2$, and for some $x_3 \in \mathit{Aff}(F_1)$ we have $\{x_i - x_0 : x_i \in S_2\} \subseteq \{x_i - x_3 : x_i \in \mathit{Aff}(F_1)\}$ and $(x_1 - x_0)^T(x_4 - x_3) = 0$ for any $x_1 \in S_1$ and $x_4 \in \mathit{Aff}(F_1)$.

There exist $n - k - 1$ constraints of $P$ such that the corresponding hyperplanes, together with $\alpha^T x = \beta$, define $\mathit{Aff}(F_1)$. Let these hyperplanes be $A_{(n-k-1)\times n}x = b_1$. Similarly, there are $n - m - 1$ constraints of $P$ such that the corresponding hyperplanes, together with $\alpha^T x = \psi$, define $\mathit{Aff}(F_2)$. Let them be $A_{(n-m-1)\times n}x = b_2$. From the discussion in the previous paragraphs, one can easily see that the equations $A_{(n-k-1)\times n}x = 0$, $A_{(n-m-1)\times n}x = 0$ have solution space $\{x_i - x_0 : x_i \in S_2\}$ with dimension $m - (n - k - 2)$. Since $m - (n - k - 2) \equiv n - [(n - m - 1) + (n - k - 1)]$, the matrix $\binom{A_{(n-k-1)\times n}}{A_{(n-m-1)\times n}}$ is full row-rank. Because the rank of $A$ is $r$, $(n - k - 1) + (n - m - 1) \leq r$. This allows us to choose another $r - (n - k - 1) - (n - m - 1)$ constraints $A_{[r-(n-k-1)-(n-m-1)]\times n}x \leq b_3$ from $Ax \leq b$, together with $A_{(n-k-1)\times n}x \leq b_1$ and $A_{(n-m-1)\times n}x \leq b_2$, to construct a $B \in \mathcal{B}_r^*$ such that the $F$-defining inequality is valid for $\mathit{Conv}(P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$.

**Case 3.** $P_1 \neq \emptyset$, $P_2 \neq \emptyset$, and some of $P_1$ and $P_2$ is not full-dimensional.

Instead of considering the inequalities $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$, we construct two inequalities $\alpha^T x \leq \beta + \epsilon$ and $\alpha^T x \geq \psi - \epsilon$, where $\epsilon$ is an arbitrarily small positive number satisfying $\beta + \epsilon < \psi - \epsilon$. Let $P_1^\epsilon := P \cap \{x : \alpha^T x \leq \beta + \epsilon\}$ and $P_2^\epsilon := P \cap \{x : \alpha^T x \geq \psi - \epsilon\}$. Since $P_1 \neq \emptyset$ and $P_2 \neq \emptyset$, $P_1^\epsilon$ and $P_2^\epsilon$ are full-dimensional polyhedra.

Because $|\mathcal{B}_r^*|$ is finite and $B$ and $P$ are closed sets in $\mathbb{R}^n$, by the definition of the $\mathit{Conv}$ operation and the result proved in Case 2, we have

$$\begin{aligned}
&\bigcap_{B\in\mathcal{B}_r^*} \mathit{Conv}((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))\\
&= \bigcap_{B\in\mathcal{B}_r^*} \lim_{\epsilon\to 0^+} \mathit{Conv}((P(B) \cap \{x : \alpha^T x \leq \beta + \epsilon\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi - \epsilon\}))\\
&= \lim_{\epsilon\to 0^+} \bigcap_{B\in\mathcal{B}_r^*} \mathit{Conv}((P(B) \cap \{x : \alpha^T x \leq \beta + \epsilon\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi - \epsilon\}))\\
&= \lim_{\epsilon\to 0^+} \mathit{Conv}((P \cap \{x : \alpha^T x \leq \beta + \epsilon\}) \cup (P \cap \{x : \alpha^T x \geq \psi - \epsilon\}))\\
&= \mathit{Conv}((P \cap \{x : \alpha^T x \leq \beta\}) \cup (P \cap \{x : \alpha^T x \geq \psi\})).
\end{aligned}$$

□

From Lemma 2 we immediately have the following:

**Corollary 1** *Let $S \subset \mathbb{R}^{n+2}$ be a set of $(\alpha, \beta, \psi) \in \mathbb{R}^{n+2}$ such that $\alpha \in \mathbb{R}^n$ and $\beta < \psi$. When $P$ is full-dimensional,*

$$\bigcap_{(\alpha,\beta,\psi)\in S} \mathit{Conv}((P \cap \{x : \alpha^T x \leq \beta\}) \cup (P \cap \{x : \alpha^T x \geq \psi\})) =$$
$$\bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\alpha,\beta,\psi)\in S} \mathit{Conv}((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\})). \quad (6)$$

By choosing $S = \{(\pi, \pi_0, \pi_0 + 1) : (\pi, \pi_0) \in Z^{n+1}\}$ we get:

**Corollary 2** *Equation* (4) *holds when* $P$ *is full-dimensional.*

Now assume that $P$ is not full-dimensional. We first consider the case when $P$ is empty:

**Lemma 3** *Equation* (4) *holds when* $P$ *is empty.*

*Proof.* There always exist $\bar{S} \subseteq M$ and $\bar{i} \in M$, where $\bar{i} \notin \bar{S}$, such that $\bar{S}$ contains $\bar{B} \in \mathcal{B}_r^*$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{S}\} \neq \emptyset$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{S} \cup \{\bar{i}\}\} = \emptyset$. Actually, $\bar{S}$ and $\bar{i}$ can be chosen by an iterative procedure. In iteration $k$ ($k \geq 0$), an $i_k \in M_k \subseteq M$ is chosen such that $M_k \backslash \{i_k\}$ contains a $B \in \mathcal{B}_r^*$, where $M_0 := M$. If $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in M_k \backslash \{i_k\}\} \neq \emptyset$, then $\bar{S} := M_k \backslash \{i_k\}$ and $\bar{i} := i_k$. Otherwise, let $M_{k+1} := M_k \backslash \{i_k\}$ and proceed until finally we obtain $\bar{S}$ and $\bar{i}$. The fact that $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}\} \neq \emptyset$ for any $\bar{B} \in \mathcal{B}_r^*$ ensures the availability of $\bar{S}$ and $\bar{i}$.

By applying the techniques of linear programming, we see that $\bar{S}$ contains $\bar{B}^* \in \mathcal{B}_r^*$ such that $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^* \cup \{\bar{i}\}\} = \emptyset$. It is possible to choose $\tilde{i} \in \bar{B}^*$ such that $\tilde{B}^* := (\bar{B}^* \backslash \{\tilde{i}\}) \cup \{\bar{i}\} \in \mathcal{B}_r^*$. Then $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\} \cap \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\} = \emptyset$.

Because $P = \emptyset$, $\text{SC} = \emptyset$. $\text{SC}(\bar{B}^*) \cap \text{SC}(\tilde{B}^*) = \emptyset$ follows $\text{SC}(\bar{B}^*) \subseteq \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\}$, $\text{SC}(\tilde{B}^*) \subseteq \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\}$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\} \cap \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\} = \emptyset$. Therefore, $\text{SC} = \emptyset = \text{SC}(\bar{B}^*) \cap \text{SC}(\tilde{B}^*) = \bigcap_{B \in \mathcal{B}_r^*} \text{SC}(B)$. □

In the remainder of this section, we assume that $P$ is non-empty and not full-dimensional. Let $\mathit{Aff}(P)$ denote the affine hull of $P$. When $P$ is not full-dimensional, it is full-dimensional in $\mathit{Aff}(P)$. Let $l := \dim(\mathit{Aff}(P)) < n$ denote the dimension of $P$. Also, $M^= := \{i \in M : a_{i.}^T x = b_i,\ \forall x \in P\}$ denotes the constraints describing $P$ satisfied with equality by all points in $P$. Since $\dim(P) = l$, there exists a set $B \in \mathcal{B}_{n-l}^*$, such that $\mathit{Aff}(P) = \{x \in \mathbb{R}^n : a_{i.}^T x = b_i, i \in B\}$. $\mathcal{S}^= := \{B \in \mathcal{B}_{n-l}^* : B \subseteq M^=\}$ denotes the set of all such sets $B$. The following properties are needed:

**Lemma 4** *Assume* $P$ *is non-empty and not full-dimensional. Then:*

($i$) $\mathit{Aff}(P) = P(M^=)$.

($ii$) *Let* $i^* \in M^=$ *be arbitrary. The linear program* $\min\{a_{i^*.}^T x : x \in P(M^= \backslash \{i^*\})\}$ *is bounded, the optimal objective value is* $b_{i^*}$ *and the set of optimal solutions is* $\mathit{Aff}(P)$.

($iii$) *There exist* $B' \in \mathcal{S}^=$ *and* $i' \in M^= \setminus B'$ *such that* $\mathit{Aff}(P) = P(B' \cup \{i'\})$.

($iv$) *Let* $i'$ *and* $B'$ *be as in* ($iii$). *There exists* $i'' \in B'$ *such that* $B'' := (B' \backslash \{i''\}) \cup \{i'\} \in \mathcal{S}^=$ *and* $\mathit{Aff}(P) = P(B') \cap P(B'')$.

*Proof.* The correctness of ($i$), ($ii$) and ($iv$) is easy to check. So next we just prove ($iii$).

Because of ($i$), we have $|M^=| \geq n - l + 1$. Choose $i' \in M^=$ such that $(M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^* \neq \emptyset$. So ($ii$) is true for $i^* = i'$. Since $(M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^* \neq \emptyset$, the optimal dual solution of $\min\{a_{i'.}^T x : a_{i.}^T \leq b_i,\ i \in M^= \backslash \{i'\}\}$ specifies a $B' \in (M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^*$ such that $\min\{a_{i'.}^T x : a_{i.}^T \leq b_i,\ i \in B'\} = b_{i'}$ with optimal solution set $\mathit{Aff}(P)$. Therefore, $\mathit{Aff}(P) = \{x \in \mathbb{R}^n : a_{i.}^T \leq b_i,\ i \in B' \cup \{i'\}\}$. □

Let $B'$ and $B''$ be as in Lemma 4. The sets $B'$ and $B''$ might not be of cardinality $r$, i.e. $B'$ ($B''$) might not be a maximal subset of $M$ such that the vectors $a_{i.}$, for $i \in B'$ ($i \in B''$), are linearly independent. Define $\gamma := r-(n-l)$. It follows that $0 \leq \gamma \leq l$. $\mathcal{B}_\gamma^A := \{B \in \mathcal{B}_\gamma^* : B \cup B' \in \mathcal{B}_r^*\}$ ($= \{B \in \mathcal{B}_\gamma^* : B \cup B'' \in \mathcal{B}_r^*\}$) denotes the family of $\gamma$-subsets $B$ in $\mathcal{B}_\gamma^*$ such that $B' \cup B$ ($B'' \cup B$) is an $r$-subset in $\mathcal{B}_r^*$. Also, $\mathcal{B'}_r^* := \{B \in \mathcal{B}_r^* : B \supseteq B'\}$ and $\mathcal{B''}_r^* := \{B \in \mathcal{B}_r^* : B \supseteq B''\}$ denotes the families of $r$-subsets in $\mathcal{B}_r^*$ containing $B'$ and $B''$ respectively. The following is immediate from the definitions of $\mathcal{B}_\gamma^A$, $\mathcal{B'}_r^*$ and $\mathcal{B''}_r^*$:

**Lemma 5** *There is a one-to-one mapping from $\mathcal{B}_\gamma^A$ to $\mathcal{B'}_r^*$, and there is a one-to-one mapping from $\mathcal{B}_\gamma^A$ to $\mathcal{B''}_r^*$.*

We are now able to finish the proof of Theorem 1:

**Lemma 6** *Equation* (4) *holds when $P$ is non-empty and not full-dimensional in $\mathbb{R}^n$.*

*Proof.* $P$ is full-dimensional in $\mathit{Aff}(P)$. If there exists $(\pi,\pi_0) \in \mathbb{Z}^{n+1}$ such that $\mathit{Aff}(P)$ is between the hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$ and $\mathit{Aff}(P)$ does not intersect them, then Lemma 6 is trivially true with $\emptyset = \emptyset$. Otherwise, we only need to consider $(\pi,\pi_0) \in \mathbb{Z}^{n+1}$ such that both hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$ intersect $\mathit{Aff}(P)$ and neither of them contains $\mathit{Aff}(P)$. Denote the set of these $(\pi,\pi_0)$ by $S_A$.

Now we have $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{(\pi,\pi_0)\in S_A} \mathit{Conv}((P \cap \{x \in \mathit{Aff}(P) : \pi^T x \leq \pi_0\}) \cup (P \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\}))$. Applying Corollary 1 to the affine subspace $\mathit{Aff}(P)$, we see that the latter is equal to $\bigcap_{B\in\mathcal{B}_\gamma^A} \bigcap_{(\pi,\pi_0)\in S_A} \mathit{Conv}(((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \leq \pi_0\}) \cup ((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\}))$. By Lemma 4(iv) and Lemma 5, for any $P(B)$, where $B \in \mathcal{B}_\gamma^A$, there always exist $P(\tilde{B}')$ and $P(\tilde{B}'')$, where $\tilde{B}' \in \mathcal{B'}_r^*$ and $\tilde{B}'' \in \mathcal{B''}_r^*$, such that $\tilde{B}' = B' \cup B$, $\tilde{B}'' = B'' \cup B$ and $P(B) \cap \mathit{Aff}(P) = P(\tilde{B}') \cap P(\tilde{B}'')$. Therefore, $\bigcap_{B\in\mathcal{B}_\gamma^A} \bigcap_{(\pi,\pi_0)\in S_A} \mathit{Conv}(((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \leq \pi_0\}) \cup ((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\})) \supseteq \bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$, which implies $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) \supseteq \bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$. Because $P \subseteq P(B)$ for any $B \in \mathcal{B}_r^*$, it is easy to obtain $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) \subseteq \bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$. The lemma is proved. □

Theorem 1 is implied by Corollary 2, Lemma 3 and Lemma 6. In fact, the proofs allow us to extend Theorem 1 to arbitrary subsets of $\Pi^n(N_I)$:

**Theorem 2** *Assume $S \subseteq \Pi^n(N_I)$. Then:*

$$\bigcap_{(\pi,\pi_0)\in S} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\pi,\pi_0)\in S} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}). \tag{7}$$

## 4 Polyhedrality of split closure

In this section, we assume $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^n$, i.e. that $P$ is a rational polyhedron. Cook, Kannan and Schrijver proved the following result [4]:

**Theorem 3** *The split closure of MILP is a polyhedron.*

We will give a new proof of this result using the characterization of the split closure obtained in the previous section. Let $\tilde{C} := \tilde{x} + Cone(\{\tilde{r}_i : i = 1, 2, \ldots, q\})$ be (the translate of) a cone with apex $\tilde{x} \in \mathbb{Q}^n$ and $q$ linearly independent extreme ray vectors $\{\tilde{r}_i\}_{i=1}^q$, where $q \le n$ and $\tilde{r}_i \in \mathbb{Z}^n$ for $1 \le i \le q$. The following lemma plays a key role in proving Theorem 3.

**Lemma 7** *The split closure of $\tilde{C}$ is a polyhedron.*

*Proof.* Suppose that the disjunction $\pi^T x \le \pi_0$ and $\pi^T x \ge \pi_0 + 1$ induces a split cut that is not valid for some part of $C$. Then it must be not valid for $\tilde{x}$ either. So we know $\pi_0 < \pi^T \tilde{x} < \pi_0 + 1$, i.e. the point $\tilde{x}$ is between the two hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$.

Choose an extreme ray generated by vector $\tilde{r}_i$ and assume that the hyperplane $\pi^T x = \pi_0$ intersects the extreme ray at $\tilde{x} + \tilde{\alpha}_i \tilde{r}_i$, where $\tilde{\alpha}_i > 0$ ($\tilde{\alpha}_i = +\infty$ is allowed). Then $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i}$ can be easily calculated, where $\pi^T \tilde{r}_i \le 0$, $\epsilon := \pi^T \tilde{x} - \pi_0$ and $0 < \epsilon < 1$.

We claim that $\tilde{\alpha}_i$ is either $+\infty$ or bounded above by 1. When $\pi^T \tilde{r}_i = 0$, $\tilde{\alpha}_i = +\infty$, in which case the hyperplane $\pi^T x = \pi_0$ is parallel to the vector $\tilde{r}_i$. When $\pi^T \tilde{r}_i < 0$, $0 < \tilde{\alpha}_i < +\infty$, which means that the hyperplane intersects the ray at some finite point. In this case, because $\pi$ and $\tilde{r}_i$ are in $\mathbb{Z}^n$, we know $-\pi \tilde{r}_i \ge 1$. Hence, $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i} \le \epsilon < 1$.

Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^T \in Q^n$. Let $g$ be the least common multiple of all the denominators of $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$. Noticing the fact that $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i}$, it follows that $\tilde{\alpha}_i$ can be expressed as $\frac{p}{wg}$, where $p, w \in \mathbb{Z}_+$ and $0 < p < g$.

By the following claim, we actually prove the lemma.

*Claim.* There is only a finite number of undominated split cuts for cone $C$.

*Proof of claim.* By induction on $m$, the number of extreme rays of $C$.

When $q = 1$, $C = \{x \in \mathbb{R}^n : x = \tilde{x} + \alpha_1 \tilde{r}_1,\ \alpha_1 \ge 0\}$. The case $\tilde{\alpha}_1 = +\infty$ does not yield a split cut, so $\tilde{\alpha}_1$ is bounded above by 1 for every split cut. Note that the maximum value of $\tilde{\alpha}_1$ is reachable, because $\tilde{\alpha}_1$ has the form of $\frac{p}{wg}$ as mentioned above. Let $\pi^* x = \pi_0^*$ be the hyperplane for which $\tilde{\alpha}_1$ reaches its maximum. Then the split cut $\pi^* x \le \pi_0^*$ is an undominated split cut of $C$.

Assume that the claim is true for $q = k < n$. Let us consider the case of $q = k + 1$.

Let $C_i := \{x \in \mathbb{R}^n : x = \tilde{x} + \sum_{j \ne i} \alpha_j \tilde{r}_j,\ \alpha_j \ge 0,\ 1 \le j \le k+1,\ j \ne i\}$, where $1 \le i \le k + 1$. Each $C_i$ is a polyhedral cone with apex $\tilde{x}$ and $k$ linearly independent extreme rays. By induction hypothesis, there is only a finite number of undominated split cuts for each cone $C_i$. Among those points obtained by intersecting the undominated split cuts for $C_i$ with the extreme ray generated by $\tilde{r}_i$, let $z_i$ be the closest point to $\tilde{x}$.

Now we claim that any undominated split cut of $C$ cannot intersect the extreme ray generated by $\tilde{r}_i$ $(1 \le i \le k+1)$ at a point which is closer to $\tilde{x}$ than $z_i$. Otherwise, let us assume that there is an undominated split cut $\mathcal{H}$ of $C$ which intersects the extreme ray generated by $\tilde{r}_i$ at a point $\tilde{z}_i$ that is between $\tilde{x}$ and $z_i$. By the definition of $z_i$, the cut (when restricted to $C_i$) must be dominated by a cut of $C_i$, say $\mathcal{H}'$. Wlog, assume that the cut $\mathcal{H}'$ is an undominated cut for $C_i$. So $\mathcal{H}'$ must intersect the extreme ray generated by $\tilde{r}_i$ at $z'_i$ that is not between $\tilde{x}$ and $z_i$. But now $\mathcal{H}'$ dominates $\mathcal{H}$, a contradiction to the choice of $\mathcal{H}$.

We know that the intersection point of any undominated split cut with the extreme ray of $\tilde{r}_i$ $(1 \le i \le k+1)$ is either at infinity or between $z_i$ and $\tilde{x}+\tilde{r}_i$. Since $\tilde{\alpha}_i = \frac{p}{wg}$, there are only finitely many points between $z_i$ and $\tilde{x}+\tilde{r}_i$ that could be the intersections of the split cuts with the extreme ray. Therefore, we see that the claim is true when $q = k+1$. □

Let $B \in \mathcal{B}_r^*$ be arbitrary. From section 2 we know that $P(B)$ can be written as $P(B) = \bar{x}(B) + L(B) + C(B)$, where $C(B) := \mathit{Cone}(\{r^i(B) : i \in B\})$ and $\{r^i(B) : i \in B\} \subseteq \mathbb{Z}^n$ are linearly independent (by scaling). The following lemmas are straightforward:

**Lemma 8** *Let $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $B \in \mathcal{B}_r^*$ be arbitrary. If an inequality is valid for $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$ but not valid for $P(B)$, then $\pi^T = \sum_{i \in B} \alpha_i a_{i.}^T$, where $\alpha_i \in \mathbb{R}$ $(i \in B)$.*

Define $S_B := \{\pi \in \mathbb{Z}^n : \pi^T = \sum_{i \in B} \alpha_i a_{i.}^T,\ \alpha_i \in \mathbb{R},\ i \in B\}$. From Lemma 8 we have:

**Lemma 9** *Let $B \in \mathcal{B}_r^*$, $(\pi, \pi_0) \in S_B \times \mathbb{Z}$.*

*(i) Assume there exists a facet $F$ of $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $P(B)$. Then $F$ is unique and there exists a unique facet $\tilde{F}$ of $\mathit{Conv}((\bar{x}(B) + C(B)) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $\bar{x}(B) + C(B)$. Furthermore $F = L(B) + \tilde{F}$.*

*(ii) Assume there exists a facet $\tilde{F}$ of $\mathit{Conv}((\bar{x}(B) + C(B)) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $\bar{x}(B) + C(B)$. Then $\tilde{F}$ is unique and there exists a unique facet $F$ of $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $P(B)$. Furthermore $\tilde{F} = F \cap (\bar{x}(B) + C(B))$.*

The following result is implied by Lemma 7 and Lemma 9:

**Lemma 10** *$SC(B)$, where $B \in \mathcal{B}_r^*$, is a polyhedron.*

Now, Theorem 3 follows from Theorem 1 and Lemma 10.

## 5 Disjunctive sets derived from polyhedra and two-term disjunctions

In this section, two decomposition results for the set $conv(P \cap F_D)$ are presented. The first decomposition result (Theorem 6) states that $conv(P \cap F_D)$ can be written as the intersection of sets $conv(P(T) \cap F_D)$ over sets $T \in C_1^*$, where $C_1^*$ is some family of $(r+1)$-subsets of $M$. Furthermore, in the special case where $r = n$, we show that it is enough to consider $r$-subsets $T$ of $M$. The second result strengthens the first result for split disjunctions $D(\pi, \pi_0)$, and states that the set $conv(P \cap F_{D(\pi,\pi_0)})$ can be written as the intersection of the sets $conv(P(B) \cap F_{D(\pi,\pi_0)})$ for $B \in \mathcal{B}_r^*$. We start by proving the following:

**Theorem 4** *Let $S \subseteq M$ be non-empty. If $S$ satisfies $|S| \geq r(S) + 2$, then*

$$conv(P(S) \cap F_D) = \bigcap_{i \in S} conv(P(S \setminus \{i\}) \cap F_D). \tag{8}$$

*Furthermore,* (8) *remains true if $r(S) = n$ and $|S| = n + 1$.*

One direction is easy to prove:

**Lemma 11** *Let $S \subseteq M$ be non-empty. Then,*

$$conv(P(S) \cap F_D) \subseteq \bigcap_{i \in S} conv(P(S \setminus \{i\}) \cap F_D). \tag{9}$$

*Proof.* Clearly $P(S) \subseteq P(S \setminus \{i\})$ for all $i \in S$. Intersecting with $F_D$ on both sides gives $P(S) \cap F_D \subseteq P(S \setminus \{i\}) \cap F_D$ for all $i \in S$. Convexifying both sides results in $Conv(P(S) \cap F_D) \subseteq Conv(P(S \setminus \{i\}) \cap F_D)$ for all $i \in S$, and finally, since this holds for all $i \in S$, the result follows. □

The proof of the other direction involves the idea introduced by Balas [2] of lifting the set $Conv(P(S) \cap F_D)$ onto a higher dimensional space. Specifically, $Conv(P(S) \cap F_D)$ can be described as the projection of the set, described by the following constraints, onto the space of $x$-variables:

$$x = x^1 + x^2, \tag{10}$$
$$a_{i.}^T x^1 \leq b_i \lambda^1, \qquad \forall i \in S, \tag{11}$$
$$a_{i.}^T x^2 \leq b_i \lambda^2, \qquad \forall i \in S, \tag{12}$$
$$\lambda^1 + \lambda^2 = 1, \tag{13}$$
$$D^1 x^1 \leq d^1 \lambda^1, \tag{14}$$
$$D^2 x^2 \leq d^2 \lambda^2, \tag{15}$$
$$\lambda^1, \lambda^2 \geq 0. \tag{16}$$

The description (10)-(16) can be projected onto the $(x, x^1, \lambda^1)$-space by using constraints (10) and (13). By doing this, we arrive at the following characterization of $Conv(P(S) \cap F_D)$. Later, the constraints below will be used in the formulation of an LP problem. Therefore, we have written the names of the corresponding dual variables next to the constraints:

$$-\lambda^1 b_i + a_{i.}^T x^1 \leq 0, \qquad \forall i \in S, \qquad (u_i) \tag{17}$$
$$-\lambda^1 b_i + a_{i.}^T x^1 \leq b_i - a_{i.}^T x, \qquad \forall i \in S, \qquad (v_i) \tag{18}$$
$$\lambda^1 \leq 1, \qquad (w_0) \tag{19}$$
$$-\lambda^1 d^1 + D^1 x^1 \leq 0_{m_1}, \qquad (u^0) \tag{20}$$
$$\lambda^1 d^2 - D^2 x^1 \leq d^2 - D^2 x, \qquad (v^0) \tag{21}$$
$$\lambda^1 \geq 0. \qquad (t_0) \tag{22}$$

Consider now relaxing constraints (20) and (21), i.e. replacing (20) and (21) by the following constraints:

$$-\lambda^1 d^1 + D^1 x^1 - s 1_{m_1} \leq 0_{m_1}, \qquad (u^0) \tag{23}$$
$$\lambda^1 d^2 - D^2 x^1 - s 1_{m_2} \leq d^2 - D^2 x, \qquad (v^0) \tag{24}$$
$$s \geq 0. \qquad (t_1) \tag{25}$$

Now, the problem of deciding whether or not a given vector $x \in \mathbb{R}^n$ belongs to $Conv(P(S) \cap F_D)$ can be decided by solving the following linear program, which will be called $P_{LP}(x, S)$ in the following:

$$\begin{aligned} &\max \; -s \\ s.t. \quad &(17)-(19),\ (22) \text{ and } (23)-(25). \end{aligned} \qquad (P_{LP}(x,S))$$

Observe that $P_{LP}(x, S)$ is feasible if and only if $x \in P(S)$, and that $P_{LP}(x, S)$ is always bounded above by zero. Finally, note that $x \in Conv(P(S) \cap F_D)$ if and only if $P_{LP}(x, S)$ is feasible and bounded, and there exists an optimal solution in which the variable $s$ has the value zero.

The other direction is proved with the aid of the problem $P_{LP}(x, S)$ and its dual $D_{LP}(x, S)$. Suppose $S \subseteq M$, satisfies $S \neq \emptyset$, $|S| \geq r(S) + 1$ and that $\bar{x} \in \bigcap_{i \in S} Conv(P(S \setminus \{\bar{i}\}) \cap F_D)$. Then $\bar{x} \in \bigcap_{i \in S} P(S \setminus \{i\})$, and since $|S| \geq 2$, we have $\bar{x} \in P(S)$. Hence $P_{LP}(\bar{x}, S)$ is feasible and bounded if $S$ satisfies $|S| \geq r(S) + 1$ and $\bar{x} \in \bigcap_{i \in S} Conv(P(S \setminus \{\bar{i}\}) \cap F_D)$. In the case where $P_{LP}(\bar{x}, S)$ is feasible and bounded, $(\bar{x}^1, \bar{\lambda}^1, \bar{s})$ denotes an optimal basic feasible solution to $P_{LP}(\bar{x}, S)$ and $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ denotes a corresponding optimal basic feasible solution to $D_{LP}(\bar{x}, S)$.

For $u^0 \geq 0_{m_1}$, $u_i \geq 0$ for $i \in S$ and $j \in N$, define the quantities $\alpha_j^1(S, u, u^0) := \sum_{i \in S} u_i a_{i,j} + (u^0)^T D^1_{.j}$ and $\beta^1(S, u, u^0) := \sum_{i \in S} u_i b_i + (u^0)^T d^1$, where $D^1_{.j}$ denotes the $j^{th}$ column of $D^1$. The inequality $(\alpha^1(S, u, u^0))^T x \leq \beta^1(S, u, u^0)$ is valid for $\{x \in P(S) : D^1 x \leq d^1\}$. Similarly, for $v^0 \geq 0_{m_2}$, $v_i \geq 0$ for $i \in S$ and $j \in N$, defining the quantities $\alpha_j^2(S, v, v^0) := \sum_{i \in S} v_i a_{i,j} + (v^0)^T D^2_{.j}$ and $\beta^2(S, v, v^0) := \sum_{i \in S} v_i b_i + (v^0)^T d^2$, gives the inequality $(\alpha^2(S, v, v^0))^T x \leq \beta^2(S, v, v^0)$, which is valid for $\{x \in P(S) : D^2 x \leq d^2\}$. With these quantities, the dual $D_{LP}(\bar{x}, S)$

of $P_{LP}(\bar{x}, S)$ can be formulated as follows:

$$\begin{aligned}
\min \quad & \beta^2(S,v,v^0) - (\alpha^2(S,v,v^0))^T\bar{x} + w_0 & & \\
\text{s.t.} \quad & \alpha_j^1(S,u,u^0) - \alpha_j^2(S,v,v^0) = 0, \qquad \forall j \in N, & (x_j^1) \quad & (26)\\
& \beta^2(S,v,v^0) - \beta^1(S,u,u^0) + w_0 - t_0 = 0, & (\lambda^1) \quad & (27)\\
& 1_{m_1}^T u^0 + 1_{m_2}^T v^0 + t_1 = 1, & (\text{s}) \quad & (28)\\
& u^0 \geq 0_{m_1}, & & (29)\\
& v^0 \geq 0_{m_2}, & & (30)\\
& w_0, t_0, t_1 \geq 0, & & (31)\\
& u_i, v_i \geq 0, \qquad \forall i \in S. & & (32)
\end{aligned}$$

**Lemma 12** *Let $S \subseteq M$ be arbitrary. Suppose $\bar{x} \in P(S) \setminus Conv(P(S) \cap F_D)$. Then $\bar{u}^0 \neq 0_{m_1}$ and $\bar{v}^0 \neq 0_{m_2}$.*

*Proof.* Let $\bar{x}$ be as stated, and suppose first that $\bar{v}^0 = 0_{m_2}$. The inequality $(\alpha^2(S,\bar{v},0_{m_2}))^T x \leq \beta^2(S,\bar{v},0_{m_2})$ is valid for $P(S)$. However, since the optimal objective value to $D_{LP}(\bar{x},S)$ is negative, and $\bar{w}_0 \geq 0$, we have $\beta^2(S,\bar{v},0_{m_2}) - (\alpha^2(S,\bar{v},0_{m_2}))^T\bar{x} < 0$, which contradicts the assumption that $\bar{x} \in P(S)$.

Now suppose $\bar{u}_0 = 0_{m_1}$. The inequality $(\alpha^1(S,\bar{u},0_{m_1}))^T x \leq \beta^1(S,\bar{u},0_{m_1})$ is valid for $P(S)$, but $\beta^1(S,\bar{u},0_{m_1}) - (\alpha^1(S,\bar{u},0_{m_1}))^T\bar{x} \leq \beta^2(S,\bar{v},\bar{v}^0) + w_0 - (\alpha^2(S,\bar{v},\bar{v}^0))^T\bar{x} < 0$, where we have used (20) and (21). This contradicts the assumption that $\bar{x} \in P(S)$. □

The next lemma is essential for the proof of the converse direction of Theorem 4:

**Lemma 13** *Let $S \subseteq M$, and let $T \subseteq S$ satisfy $|T| \geq 2$. Also, suppose $\bar{x} \in \bigcap_{i \in T} Conv(P(S \setminus \{i\}) \cap F_D)$ and $\bar{x} \notin Conv(P(S) \cap F_D)$. Then $D_{LP}(\bar{x},S)$ is feasible and bounded. Furthermore, $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in T$.*

*Proof.* Let $S$, $T$ and $\bar{x}$ be as stated. The fact that $D_{LP}(\bar{x},S)$ is feasible and bounded follows from the facts $\bar{x} \in \bigcap_{i \in T} Conv(P(S \setminus \{i\}) \subseteq P(S)$ and $|T| \geq 2$.

Now, suppose $\bar{u}_{i'} = 0$ and $\bar{v}_{i'} = 0$ for some $i' \in T$. Then the problem $(D'_{LP}(\bar{x}, S \setminus \{i'\}))$, obtained from $D_{LP}(\bar{x},S)$ by eliminating $u_{i'}$, $v_{i'}$ and the normalization constraint:

$$\begin{aligned}
\min \quad & \beta^2(S \setminus \{i'\},v,v^0) - (\alpha^2(S \setminus \{i'\},v,v^0))^T\bar{x} + w_0 & \\
\text{s.t.} \quad & \alpha_j^1(S \setminus \{i'\},u,u^0) - \alpha_j^2(S \setminus \{i'\},v,v^0) = 0, \qquad \forall j \in N, & (x_j^1)\\
& \beta^2(S \setminus \{i'\},v,v^0) - \beta^1(S \setminus \{i'\},u,u^0) + w_0 - t_0 = 0, & (\lambda^1)\\
& u^0 \geq 0_{m_1},\ v^0 \geq 0_{m_2},\ w_0, t_0 \geq 0,\ u_i, v_i \geq 0,\ \forall i \in S \setminus \{i'\}, &
\end{aligned}$$

is unbounded (since $\bar{x} \notin Conv(P(S) \cap F_D)$). This means that the dual of $D'_{LP}(\bar{x}, S \setminus \{i'\})$, the problem $P'_{LP}(\bar{x}, S \setminus \{i'\})$, is infeasible:

$$
\begin{aligned}
\max \quad & 0_n^T x^1 + 0\lambda^1 \\
\text{s.t.} \quad & -\lambda^1 b_i + a_{i.}^T x^1 \le 0, \qquad \forall i \in S \setminus \{i'\}, && (u_i)\\
& \lambda^1 b_i - a_{i.}^T x^1 \le b_i - a_{i.}^T \bar{x}, \qquad \forall i \in S \setminus \{i'\}, && (v_i)\\
& \lambda^1 \le 1, && (w_0)\\
& -\lambda^1 d^1 + D^1 x^1 \le 0_{m_1}, && (u^0)\\
& \lambda^1 d^2 - D^2 x^1 \le d^2 - D^2 \bar{x}, && (v^0)\\
& \lambda^1 \ge 0. && (t_0)
\end{aligned}
$$

However, these constraints are the conditions that must be satisfied for $\bar{x}$ to be in $Conv(P(S \setminus \{i'\}) \cap F_D)$, which is a contradiction. □

With the above lemmas, we are ready to prove the converse of Theorem 4:

**Lemma 14** *Let $S \subseteq M$, and suppose that either $|S| \ge r(S) + 2$ or $r(S) = n$ and $|S| = n + 1$. Then*

$$Conv(P(S) \cap F_D) \supseteq \bigcap_{i \in S} Conv(P(S \setminus \{i\}) \cap F_D). \tag{33}$$

*Proof.* Let $\bar{x} \in \bigcap_{i \in S} Conv(P(S \setminus \{i\}) \cap F_D)$, and suppose $\bar{x} \notin Conv(P(S) \cap F_D)$. Define $B_u := \{i \in S : u_i \text{ basic }\}$ and $B_v := \{i \in S : v_i \text{ basic }\}$ to be the set of basic $u$'s and $v$'s respectively in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}(\bar{x}, S)$. From Lemma 13 and the fact that a variable with positive value, that does not have an upper bound, is basic, it follows that $(B_u \cup B_v) = S$.

The feasible set for the problem $D_{LP}(\bar{x}, S)$ is of the form $\{y \in \mathbb{R}^{n'} : Wy = w_0, y \ge 0_{n'}\}$, where $W$ and $w_0$ are of suitable dimensions. The column of $W$ corresponding to the variable $u_i$, $i \in S$, is given by $[a_{i.}^T, -b_i, 0]^T$. Similarly, the column of $W$ corresponding to the variable $v_i$, $i \in S$, is given by $[-a_{i.}^T, b_i, 0]^T$. Since for all $i \in S$, either $u_i$ or $v_i$ is basic, the vectors $[a_{i.}^T, -b_i]^T$, $i \in S$, are linearly independent. Clearly, there can be at most $r(S) + 1$ of these. Hence $|S| = r(S) + 1$. This excludes the case $|S| \ge r(S) + 2$, so we must have $r(S) = n$ and $|S| = n + 1$.

The number of basic variables in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}(\bar{x}, S)$ is at most $n + 2$, since the number of basic variables is bounded by the number of equality constraints in $D_{LP}(\bar{x}, S)$. The number of basic variables among the variables $u_i$ and $v_i$, $i \in S$, is $|S| = n + 1$. However, according to Lemma 12, at least two of the variables in $(\bar{u}^0, \bar{v}^0)$ are basic, which gives a total of $n + 3$ basic variables — a contradiction. □

Now, we strengthen Theorem 4 for the case where $|S| \ge r(S) + 2$. Let $\bar{I}(S)$ be the set of constraints $i \in S$ for which $r(S \setminus \{i\}) = r(S)$, i.e. $\bar{I}(S) := \{i \in S : r(S) = r(S \setminus \{i\})\}$. We have:

**Theorem 5** *Let $S \subseteq M$ satisfy $|S| \geq r(S) + 2$. Then*

$$Conv(P(S) \cap F_D) = \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D). \tag{34}$$

Like in Theorem 4, and with the same proof, one direction of Theorem 5 is easy. For the converse, observe that $|S| \geq r(S) + 2$ implies $|\bar{I}(S)| \geq 2$. It also implies $\bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D) \subseteq P(S)$. From Lemma 13, we have:

**Corollary 3** *Suppose $|S| \geq r(S) + 2$, $\bar{x} \in \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D)$ and $\bar{x} \notin Conv(P(S) \cap F_D)$. Then $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in \bar{I}(S)$.*

We can now prove the converse of Theorem 5:

**Lemma 15** *Let $S \subseteq M$ satisfy $|S| \geq r(S) + 2$. Then:*

$$Conv(P(S) \cap F_D) \supseteq \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D). \tag{35}$$

*Proof.* Let $\bar{x} \in \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D)$, and suppose $\bar{x} \notin Conv(P(S) \cap F_D)$. Observe that it suffices to prove that the vectors $[a_{i.}^T, -b_i]$, $i \in S$, are linearly independent, since that would contradict $|S| \geq r(S)+2$. Suppose they are linearly dependent, and let $\bar{i} \in S$ satisfy $[a_{\bar{i}.}^T, -b_{\bar{i}}] = \sum_{i \in S \setminus \{i\}} \mu_i [a_{i.}^T, -b_i]$, where $\mu_i$, for $i \in S \setminus \{i\}$, are scalars, not all zero. Suppose first that $\bar{i} \in S \setminus \bar{I}(S)$. Then we have $a_{\bar{i}.} \in span(\{a_{i.} : i \in S \setminus \{\bar{i}\})$. However, that implies $r(S) = r(S \setminus \{\bar{i}\})$, which is a contradiction. Hence, we must have $\bar{i} \in \bar{I}(S)$ and $\mu_i = 0$ for all $i \in S \setminus \bar{I}(S)$. However, according to Corollary 3, the vectors $[a_{i.}^T, -b_i]$, $i \in \bar{I}(S)$, are linearly independent. □

Applying Theorem 5 iteratively, and Theorem 4 for sets of size $n+1$, we get the following:

**Theorem 6** *Suppose $|M| = m \geq r + 1$. Also, define $\mathcal{C}_1^* := \{S \subseteq M : |S| = r+1$ and $r(S) = r\}$ and $\mathcal{C}_2^* := \{S \subseteq M : |S| = r$ and $(r(S) = r \vee r(S) = r-1)\}$. We have,*

$$Conv(P \cap F_D) = \bigcap_{T \in C_1^*} Conv(P(T) \cap F_D). \tag{36}$$

*Furthermore, if $r = n$,*

$$Conv(P \cap F_D) = \bigcap_{T \in C_2^*} Conv(P(T) \cap F_D). \tag{37}$$

The example in Figure 1 demonstrates that the assumption $|S| \geq r(S) + 2$ is necessary for (35) to hold. In this example, $P$ has 3 constraints $a_{i.}^T x \leq b_i$, $i = 1, 2, 3$, and $D$ is a two-term disjunction involving the 2 constraints $D^1 x \leq d^1$ and $D^2 x \leq d^2$. $C^1$ denotes the cone defined by $a_{1.}^T x \leq b_1$ and $a_{3.}^T x \leq b_3$,

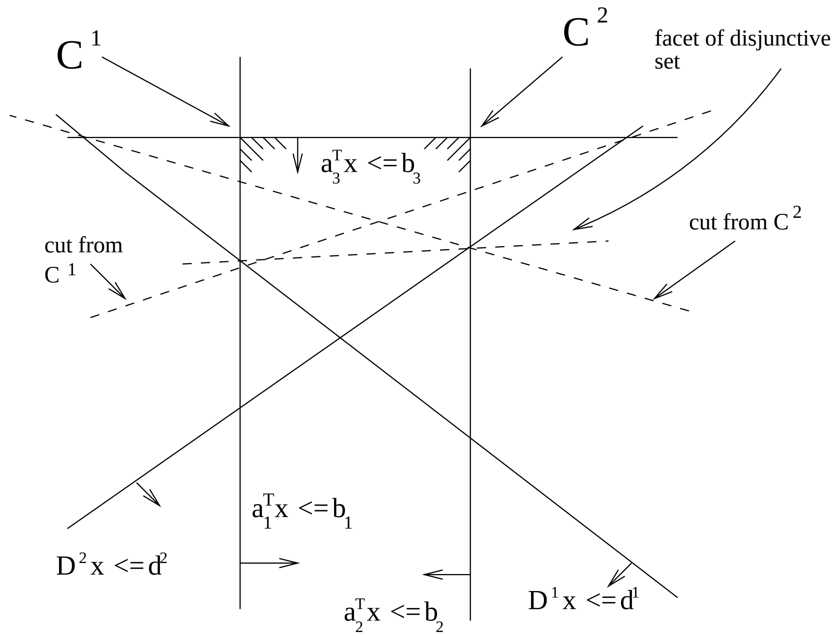


**Fig. 1.** Example of two-term disjunction

and $C^2$ denotes the cone defined by $a_{2.}^T x \leq b_2$ and $a_{3.}^T x \leq b_3$. We clearly see $Conv(P(M) \cap F_D) \neq \bigcap_{i \in \bar{I}(M)} Conv(P(M \setminus \{i\}) \cap F_D)$.

The example does not exclude, however, that (35) is true for sets $S$ satisfying $|S| = r(S) + 1$ for the special case of a split disjunction. In fact, the example suggests that it is also necessary for the disjunction to be a split disjunction. In the following, we will prove that (35) remains valid for $|S| = r(S) + 1$ for the special case of a split disjunction.

Let $D(\pi, \pi_0)$ be a split disjunction. The problem $P_{LP}(\bar{x}, S)$, for a split disjunction $D(\pi, \pi_0)$, which will be called $P_{LP}^S(\bar{x}, S)$ in the following, is obtained from the problem $P_{LP}(\bar{x}, S)$ by replacing (23) and (24) with:

$$-\lambda^1 \pi_0 + \pi^T x^1 - s \leq 0, \qquad (u^0) \qquad (38)$$

$$-\lambda^1(\pi_0 + 1) + \pi^T x^1 - s \leq -(\pi_0 + 1) + \pi^T \bar{x}. \qquad (v^0) \qquad (39)$$

The dual of $P_{LP}^S(\bar{x}, S)$ is the problem $D_{LP}^S(\bar{x}, S)$ defined as:

$$\min \quad \sum_{i\in S} v_i(b_i - a_{i.}^T\bar{x}) + w_0 + v^0(\pi^T\bar{x} - (\pi_0+1)) \tag{40}$$

$$\text{s.t.} \quad \sum_{i\in S} a_{i.}(u_i - v_i) + \pi(u_0+v_0) = 0_n, \qquad (x^1) \tag{41}$$

$$\sum_{i\in S} b_i(v_i-u_i) - \pi_0(u_0+v_0) - v^0 + w_0 - t_0 = 0, \quad (\lambda^1) \tag{42}$$

$$u^0 + v^0 + t_1 = 1, \qquad (\text{s}) \tag{43}$$

$$u^0, v^0, w_0, t_0, t_1 \geq 0, \tag{44}$$

$$u_i, v_i \geq 0, \quad \forall i \in S. \tag{45}$$

The solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$, for the case where $|S| = r(S)+1$, can be characterized as follows (see also Lemma 2 in [3]):

**Lemma 16** *Suppose $|S| = r(S)+1$, $\bar{x} \in \bigcap_{i\in \bar{I}(S)} Conv(P(S\setminus\{i\}) \cap F_{D(\pi,\pi_0)})$ and $\bar{x} \notin Conv(P(S) \cap F_{D(\pi,\pi_0)})$. Let $B_u := \{i \in S : u_i \text{ basic }\}$ and $B_v := \{i \in S : v_i \text{ basic }\}$ be the set of basic u's and v's respectively in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$. Then $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, and the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly independent.*

*Proof.* As mentioned earlier, the feasible set for $D_{LP}^S(\bar{x}, S)$ can be written as $\{y \in \mathbb{R}^{n'} : Wy = z_0, y \geq 0_{n'}\}$, where $W$ and $z_0$ are of suitable dimensions.

We first argue that the variables $w_0$, $t_0$ and $t_1$ are non-basic in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$. $t_1$ is clearly non-basic, since $s$ is basic in $P_{LP}^S(\bar{x}, S)$. From Lemma 12 it follows that both $u^0$ and $v^0$ are basic. The column corresponding to $u^0$ is $[\pi, -\pi_0, 1]$ and the column corresponding to $v^0$ is $[\pi, -(\pi_0+1), 1]$. Subtracting the column corresponding to $v_0$ from the column corresponding to $u_0$ gives $e_{n+1}$, i.e. the $(n+1)^{th}$ unit vector in $\mathbb{R}^{n+2}$. Since this is exactly the column corresponding to $w_0$ and $-t_0$, and since basic columns must be linearly independent, it follows that $w_0$ and $t_0$ are both non-basic.

As argued earlier, not both $v_i$ and $u_i$, $i \in S$, can be in the basis, since their corresponding columns in $W$ are multiples of each other. Hence $B_u \cap B_v = \emptyset$. Now, since $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ is a basic solution to $D_{LP}^S(\bar{x}, S)$, the solution to the system:

$$\sum_{i\in B_u} a_{i.}u_i - \sum_{i\in B_v} a_{i.}v_i + \pi \quad = 0_n \tag{46}$$

$$\sum_{i\in B_v} b_i v_i - \sum_{i\in B_u} b_i u_i - \pi_0 - v^0 \;= 0 \tag{47}$$

$$u_0 + v_0 \quad = 1 \tag{48}$$

is unique. The system (46)-(48) is of the form $W^B y = z_0^B$. The number of rows of $W^B$ (and $z_0^B$) is $n+2$, and the number of columns is $|B_u \cup B_v| + 2$. All columns of $W^B$ are linearly independent. If $|B_u \cup B_v| + 2 < n+2$, multiple solutions would exist. Hence, we must have $|B_u \cup B_v| = n$.

Now suppose the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly dependent. Then there exists a non-zero solution $(u^*, v^*)$ to the system $\sum_{i\in B_u} a_{i.}u_i^* - \sum_{i\in B_v} a_{i.}v_i^* =$

$0_n$. Define the scalars $u_i(\delta) := \bar{u}_i + \delta u_i^*$ for $i \in B_u$ and $v_i(\delta) := \bar{v}_i + \delta v_i^*$ for $i \in B_v$, where $\delta \in \mathbb{R}$. We have that $(u(\delta), v(\delta), u_0, v_0)$ satisfies (46)-(48) if and only if $u_0 + v_0 = 1$ and $\bar{v}_0 + \delta(\sum_{i \in B_v} b_i v_i^* - \sum_{i \in B_u} b_i u_i^*) - v_0 = 0$. Defining $v_0(\delta) := \bar{v}_0 + \delta(\sum_{i \in B_v} b_i v_i^* - \sum_{i \in B_u} b_i u_i^*)$ and $u_0(\delta) := 1 - v_0(\delta)$, we have that $(u(\delta), v(\delta), u_0(\delta), v_0(\delta))$ satisfies (46)-(48). Since $u_i^*$ for $i \in B_u$ and $v_i^*$ for $i \in B_v$ are not all zero, and none of the vectors $a_{i.}$, $i \in M$, are zero vectors, there must exist $\delta^* \in \mathbb{R}$ such that $(u(\delta^*), v(\delta^*), u_0(\delta^*), v_0(\delta^*))$ is a different solution to (46)-(48) than $(\bar{u}, \bar{v}, \bar{u}_0, \bar{v}_0)$, a contradiction. □

From the above lemma, we immediately have the desired extension of Theorem 5 for the split disjunction:

**Lemma 17** *Suppose $S \subseteq M$ satisfies $|S| = r(S) + 1$. Then:*

$$Conv(P(S) \cap F_{D(\pi,\pi_0)}) = \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_{D(\pi,\pi_0)}). \tag{49}$$

*Proof.* Suppose $\bar{x} \in \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_{D(\pi,\pi_0)})$ and $\bar{x} \notin Conv(P(S) \cap F_{D(\pi,\pi_0)})$. Let $B_u$ and $B_v$ and $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ be as in Lemma 6. We have $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, and the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly independent. Let $\{\bar{i}\} = S \setminus (B_u \cup B_v)$. We can not have $\bar{i} \in \bar{I}(S)$, since by Corollary 1, that would imply $\bar{u}_{\bar{i}} > 0$ or $\bar{v}_{\bar{i}} > 0$, which contradicts $\bar{i} \notin B_u \cup B_v$. Hence, we must have $\bar{i} \in S \setminus \bar{I}(S)$. But that means that $\bar{i}$ is in every basis of S, which contradicts $\bar{i} \notin B_u \cup B_v$. □

From Theorem 5 and Lemma 17, we get the following:

**Theorem 7**

$$Conv(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{T \in \mathcal{B}_r^*} Conv(P(T) \cap F_{D(\pi,\pi_0)}). \tag{50}$$

By intersection over all possible split disjunctions, and interchanging intersections, we get Theorem 1 of section 3.

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