**Mathematical Programming manuscript No.**
(will be inserted by the editor)

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# Valid Inequalities for Mixed Integer Linear Programs

December 2005, revised July 2006

**Abstract.** This tutorial presents a theory of valid inequalities for mixed integer linear sets. It introduces the necessary tools from polyhedral theory and gives a geometric understanding of several classical families of valid inequalities such as lift-and-project cuts, Gomory mixed integer cuts, mixed integer rounding cuts, split cuts and intersection cuts, and it reveals the relationships between these families. The tutorial also discusses computational aspects of generating the cuts and their strength.

**Key words:** mixed integer linear program, lift-and-project, split cut, Gomory cut, mixed integer rounding, elementary closure, polyhedra, union of polyhedra

## 1. Introduction

In this tutorial we consider *mixed integer linear programs*. These are problems of the form

$$\begin{aligned} \max \quad & cx + hy \\ & Ax + Gy \leq b \\ & x \geq 0 \ \text{ integral} \\ & y \geq 0, \end{aligned}$$

where the data are the row vectors $c \in \mathbb{Q}^n$, $h \in \mathbb{Q}^p$, the matrices $A \in \mathbb{Q}^{m \times n}$, $G \in \mathbb{Q}^{m \times p}$ and the column vector $b \in \mathbb{Q}^m$; and the variables are the column vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. The set $S$ of feasible solutions to this mixed integer linear program is called a *mixed integer linear set*. That is, if $P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$, then $S := P \cap (\mathbb{Z}^n_+ \times \mathbb{R}^p_+)$. It can be shown that $\text{conv}(S)$ is a polyhedron (see for example Nemhauser and Wolsey [46]). In order to solve a mixed integer linear program, it is therefore equivalent to solve the linear program

$$\begin{aligned} \max \ & cx + hy \\ & (x, y) \in \text{conv}(S) \end{aligned}$$

since the extreme points of $\text{conv}(S)$ are in $S$. This is the *polyhedral approach*. The difficulty, of course, is to construct inequalities defining $\text{conv}(S)$ given those

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⋆ Supported by NSF grant DMI-0352885 and ONR grant N00014-03-1-0188.

defining $P$. This tutorial presents approaches for constructing improved approximations of $\mathrm{conv}(S)$ recursively. An inequality is said to be *valid* for a set if it is satisfied by every point in this set. A *cut* with respect to a point $(\overline{x}, \overline{y}) \notin \mathrm{conv}(S)$ is a valid inequality for $\mathrm{conv}(S)$ that is violated by $(\overline{x}, \overline{y})$. The polyhedral approach to solving mixed integer linear programs leads naturally to the *cutting plane approach*: Solve the *linear programming relaxation* obtained by ignoring the integrality requirements on $x$; if this relaxation is unbounded or infeasible, then stop: the mixed integer linear progam is infeasible or unbounded; otherwise let $(\overline{x}, \overline{y})$ be an optimal extreme point solution; if $(\overline{x}, \overline{y}) \in S$, then stop: $(\overline{x}, \overline{y})$ is an optimal solution of the mixed integer linear program; otherwise generate a cut with respect to $(\overline{x}, \overline{y})$, add it to the previous linear programming relaxation, solve this improved relaxation, and repeat. In this tutorial, we give a geometric understanding of several classical families of cuts such as split cuts [25], Gomory mixed integer cuts [34], intersection cuts [4] and, in the mixed 0,1 case, lift-and-project cuts [54], [43], [7]. We study the relationships between these various families and discuss computational aspects of these cuts. Cuts that exploit specific structures of the constraints are also the object of a vast literature but they are not addressed in this tutorial. We start with some useful results in polyhedral theory.

## 2. Polyhedra

A *polyhedron* in $\mathbb{R}^n$ is a set of the form $P := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ where $A$ is a real matrix and $b$ a real vector. If $A$ and $b$ have rational entries, $P$ is a *rational polyhedron*. The polyhedron $\{x \in \mathbb{R}^n : Ax \leq 0\}$ is called a *polyhedral cone*. Note that a polyhedral cone is always nonempty since it contains the null vector 0.

For $S \subseteq \mathbb{R}^n$, the *convex hull* of $S$ is the set $\mathrm{conv}(S) := \{x \in \mathbb{R}^n : \; x = \sum_{i=1}^k \lambda_i x^i$ where $k \geq 1, \; \lambda \in \mathbb{R}^k_+, \sum_{i=1}^k \lambda_i = 1$ and $x^1, \ldots, x^k \in S\}$. This is the smallest convex set that contains $S$. It will sometimes be useful to work with $\overline{\mathrm{conv}}(S)$, the closure of $\mathrm{conv}(S)$, which is the smallest closed convex set that contains $S$. The *conic hull* of a nonempty set $S \subseteq \mathbb{R}^n$ is $\mathrm{cone}(S) := \{x \in \mathbb{R}^n : x = \sum_{i=1}^k \lambda_i x^i$ where $k \geq 1, \; \lambda \in \mathbb{R}^k_+$ and $x^1, \ldots, x^k \in S\}$.

The convex hull of a finite set of points in $\mathbb{R}^n$ is called a *polytope*.

An important theorem, due to Minkowski-Weyl, states that every polyhedron $P$ can be written as the sum of a polytope $Q$ and a polyhedral cone $C$. Here $Q + C := \{x \in \mathbb{R}^n : \; x = y + z$ for some $y \in Q$ and $z \in C\}$. Note that $P = \emptyset$ if and only if $Q = \emptyset$. If $P := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ is nonempty, then $C = \{x \in \mathbb{R}^n : \; Ax \leq 0\}$ is unique. The cone $\{x \in \mathbb{R}^n : \; Ax \leq 0\}$ is called the *recession cone* of $P$.

Another useful result is Farkas's lemma: For a nonempty polyhedron $P := \{x : \; Ax \leq b\}$, the inequality $\alpha x \leq \beta$ is valid for $P$ if and only if there exists a vector $v \geq 0$ such that $vA = \alpha$ and $vb \leq \beta$. [This is also a consequence of LP duality: $\beta \geq \max\{\alpha x \; : \; Ax \leq b\} \; = \; \min\{vb \; : \; vA = \alpha, \; v \geq 0\}$.] Another form of Farkas's lemma is: $Ax \leq b$ has a solution if and only if $vb \geq 0$ for all

$v \geq 0$ such that $vA = 0$. [Equivalently, by LP duality, $\max\{0 : \ Ax \leq b\} = \min\{vb : \ vA = 0, \ v \geq 0\}$.]

The proof of these results can be found in textbooks such as Schrijver [53] or Ziegler [58].

### *2.1. Some properties of polyhedra and convex sets*

This section contains two lemmas that will be used later.

**Lemma 1.** *Let $P := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ be a polyhedron and let $\Pi := P \cap \{x : \pi x \leq \pi_0\}$. If $\Pi \neq \emptyset$ and $\alpha x \leq \beta$ is a valid inequality for $\Pi$, then there exists a scalar $\lambda \in \mathbb{R}_+$ such that*

$$\alpha x - \lambda(\pi x - \pi_0) \leq \beta$$

*is valid for $P$.*

*Proof.* By Farkas's lemma, since $\Pi \neq \emptyset$, there exist $u \geq 0$, $\lambda \geq 0$ such that

$$\alpha = uA + \lambda\pi$$
$$\text{and} \ \ \beta \geq ub + \lambda\pi_0.$$

Since $uAx \leq ub$ is valid for $P$, so is $uAx \leq \beta - \lambda\pi_0$. Since $uAx = \alpha x - \lambda\pi x$, the inequality $\alpha x - \lambda(\pi x - \pi_0) \leq \beta$ is valid for $P$. □

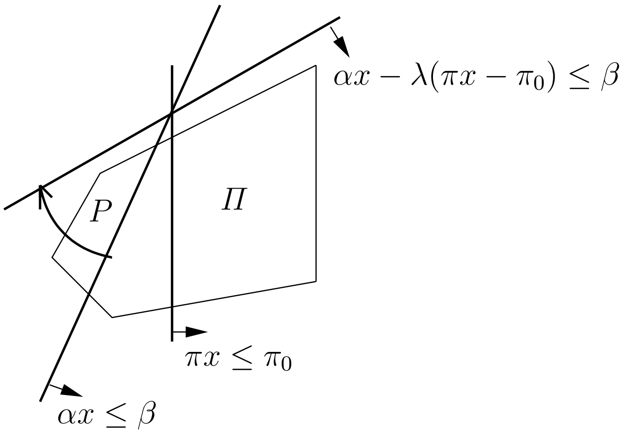


**Fig. 1.** Illustration of Lemma 1

*Remark 1.* Lemma 1 has two possible outcomes: When the hyperplane $\pi x = \pi_0$ is parallel to $\alpha x = \beta$, then by choosing $\lambda$ such that $\alpha = \lambda\pi$, we get the trivial inequality $\lambda\pi_0 \leq \beta$, satisfied by all points in $\mathbb{R}^n$ (for instance, the lemma holds when $P = \mathbb{R}^n$); in the more interesting case when $\pi x = \pi_0$ is not parallel to $\alpha x = \beta$, the outcome of the lemma is a closed half-space containing $P$ (see Figure 1).

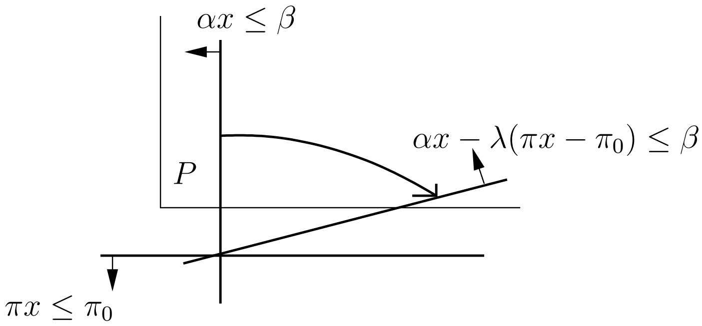


**Fig. 2.** Lemma 1 does not hold when $\Pi = \emptyset$

*Remark 2.* The assumption $\Pi \neq \emptyset$ is necessary in Lemma 1, as shown by the following example (see Figure 2): $P := \{x \in \mathbb{R}^2 : \ x_1 \geq 0, \ x_2 \geq 0\}$ and $\Pi := P \cap \{x : \ x_2 \leq -1\}$. Thus $\Pi$ is empty. The inequality $x_1 \leq 1$ is valid for $\Pi$ but there is no scalar $\lambda$ such that $x_1 - \lambda(x_2 + 1) \leq 1$ is valid for $P$.

Let $H \subseteq \mathbb{R}^n$ be a hyperplane and $S \subseteq \mathbb{R}^n$. In general, $\text{conv}(S) \cap H \neq \text{conv}(S \cap H)$ as shown by the following example: $S$ consists of two points not in $H$ but the line segment connecting them intersects $H$. The following lemma shows that equality holds when $S$ lies entirely in one of the closed half spaces defined by the hyperplane $H$ (see Figure 3).

**Lemma 2.** *Let* $H := \{x \in \mathbb{R}^n : \ ax = b\}$ *be a hyperplane and* $S \subseteq \{x : \ ax \leq b\}$. *Then* $conv(S) \cap H = conv(S \cap H)$.

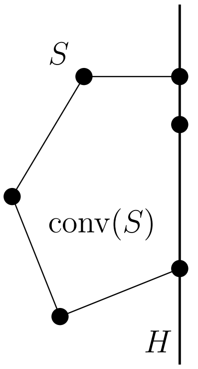


**Fig. 3.** Illustration of Lemma 2

*Proof.* We prove first $\text{conv}(S \cap H) \subseteq \text{conv}(S) \cap H$ and then $\text{conv}(S) \cap H \subseteq \text{conv}(S \cap H)$.

Clearly $\mathrm{conv}(S \cap H) \subseteq \mathrm{conv}(S)$ and $\mathrm{conv}(S \cap H) \subseteq H$ so the first inclusion is obvious.

To prove the other inclusion, let $x \in \mathrm{conv}(S) \cap H$. This means $ax = b$ and $x = \sum_{i=1}^{k} \lambda_i x^i$ where $x^1, \ldots, x^k \in S$, $\lambda \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.

$$b = ax = \sum_{i=1}^{k} \lambda_i a x^i \leq \sum_{i=1}^{k} \lambda_i b = b \tag{1}$$

where the inequality follows from $ax^i \leq b$ and $\lambda_i \geq 0$. Relation (1) implies that these inequalities are in fact equations, i.e. $ax^i = b$ for $i = 1, \ldots, k$. Therefore $x^i \in S \cap H$. This implies $x \in \mathrm{conv}(S \cap H)$. □

### *2.2. Facets*

Let $P := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ be a polyhedron. A *face* of $P$ is a set of the form

$$F := P \cap \{x \in \mathbb{R}^n : \ \alpha x = \beta\}$$

where $\alpha x \leq \beta$ is a valid inequality for $P$ (the inequality is said to *define* the face $F$). A face is itself a polyhedron since it is the intersection of the polyhedron $P$ with another polyhedron (the hyperplane $\alpha x = \beta$). The *dimension* of a set is the dimension of the smallest affine space that contains it. The faces of dimension 0 are called *vertices* of $P$, those of dimension 1 are called *edges* and those of dimension $\dim(P) - 1$ are called *facets*. The proof of the following theorem can be found in Schrijver [53].

**Theorem 1.** *Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron.*

*(i) For each facet $F$ of $P$, at least one of the inequalities defining $F$ is necessary in any description $Ax \leq b$ of $P$.*

*(ii) Inequalities defining faces of dimension less than $\dim(P) - 1$ are not needed in the description of $P$ and can be removed.*

This result states that, if a polyhedron in $\mathbb{R}^n$ has $m$ facets, any representation by a system of linear inequalities in $\mathbb{R}^n$ contains at least $m$ inequalities. In integer linear programming, we often consider polyhedra that are given implicitly as $\mathrm{conv}(S)$ (see the Introduction). It is not unusual for such polyhedra to have a number of facets that is exponential in the size of the input. Thus their representation by linear inequalities in $\mathbb{R}^n$ is large. In some cases, there is a way to get around this difficulty: a polyhedron with a large number of facets can sometimes be obtained as the projection of a polyhedron with a small number of facets. For this reason, projections turn out to be an important topic in this tutorial.

*2.3. Projections*

Let $P \subseteq \mathbb{R}^{n+p}$ where $(x, y) \in P$ will be interpreted as meaning $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. The *projection* of $P$ onto the $x$-space $\mathbb{R}^n$ is

$$\mathrm{proj}_x(P) := \{x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^p \text{ with } (x, y) \in P\}.$$

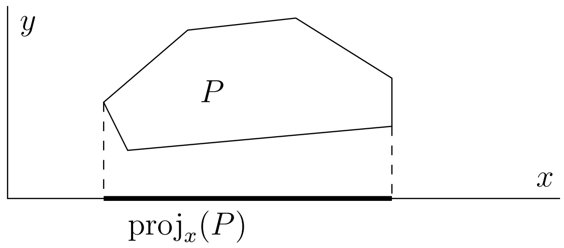


**Fig. 4.** Projection

**Theorem 2.** *Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$. Then $proj_x(P) = \{x \in \mathbb{R}^n : v^t(b - Ax) \geq 0$ for all $t \in T\}$ where $\{v^t\}_{t \in T}$ is the set of extreme rays of $Q := \{v \in \mathbb{R}^m : vG = 0, v \geq 0\}$.*

*Proof.* Let $x \in \mathbb{R}^n$. By Farkas's Lemma, $Gy \leq b - Ax$ has a solution $y$ if and only if $v^t(b - Ax) \geq 0$ for all extreme rays $v^t$ of $vG = 0, v \geq 0$. □

*Remark 3.* Variants of Theorem 2 can be proved similarly:
If $y \geq 0$ in $P$, then the relevant cone $Q$ is $\{v : vG \geq 0,\ v \geq 0\}$.
If $y \geq 0$ and $Ax + Gy = b$ in $P$, the relevant cone is $\{v : vG \geq 0\}$ with $v$ unrestricted in sign.

Enumerating the extreme rays of $Q$ may not be an easy task in applications. Another way of obtaining the projection of $P$ is to eliminate the variables $y_i$ one at a time (Fourier-Motzkin elimination procedure):

Consider a polyhedron $P \subseteq \mathbb{R}^{n+1}$ defined by the system of inequalities:

$$\sum_{j=1}^{n} a_{ij}x_j + g_i z \leq b_i \quad \text{for } i \in I. \tag{2}$$

Let $I^0 = \{i \in I : g_i = 0\}$, $I^+ = \{i \in I : g_i > 0\}$, $I^- = \{i \in I : g_i < 0\}$. The Fourier-Motzkin procedure eliminates the variable $z$ as follows: It keeps the inequalities of (2) in $I^0$, and it combines each pair of inequalities $i \in I^+$ and $l \in I^-$ to eliminate $z$.

**Theorem 3.** *The system of $|I^0| + |I^+||I^-|$ inequalities obtained by the Fourier-Motzkin procedure is the projection $proj_x(P)$ of $P$ in the $x$-space $\mathbb{R}^n$.*

*Proof.* (2) implies

$$z \le \frac{1}{g_i}(b_i - \sum_{j=1}^{n} a_{ij}x_j) \ \ \forall i \in I^+$$

$$z \ge \frac{1}{g_l}(b_l - \sum_{j=1}^{n} a_{lj}x_j) \ \ \forall l \in I^-.$$

Therefore, every $(x, z)$ in (2) satisfies

$$\frac{1}{g_l}(b_l - \sum_{j=1}^{n} a_{lj}x_j) \le \frac{1}{g_i}(b_i - \sum_{j=1}^{n} a_{ij}x_j). \tag{3}$$

Conversely, if $x$ satisfies the inequalities (2) for $i \in I^0$ and the system (3) for every $i \in I^+, l \in I^-$, then

$$\max_{l \in I^-} \frac{1}{g_l}(b_l - \sum_{j=1}^{n} a_{lj}x_j) \le \min_{i \in I^+} \frac{1}{g_i}(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

Let $z$ be any value in this interval. Then $(x, z)$ satisfies (2). □

### *2.4. Union of polyhedra*

In this section, we prove a result of Balas [5], [6] about the union of polyhedra. Consider $k$ polyhedra $P_i := \{x \in \mathbb{R}^n : \ A_i x \le b^i\}$. We will show that the smallest closed convex set that contains $\cup_{i=1}^k P_i$ is a polyhedron. This set is denoted by $\overline{\text{conv}}(\cup_{i=1}^k P_i)$.

The closure is needed as shown by the following example: $P_1$ consists of a single point, and $P_2$ is a line that does not contain the point $P_1$ (see Figure 5). Let $P_3$ denote the line going through $P_1$ that is parallel to $P_2$. It is easy to verify that $\overline{\text{conv}}(P_1 \cup P_2) = \text{conv}(P_2 \cup P_3)$ and that $\text{conv}(P_1 \cup P_2) = \text{conv}(P_2 \cup P_3) \setminus (P_3 \setminus P_1)$ (indeed, a point $x^*$ in $P_3 \setminus P_1$ is not in $\text{conv}(P_1 \cup P_2)$, but there is a sequence of points $x^k \in \text{conv}(P_1 \cup P_2)$ that converges to $x^*$). Here $\text{conv}(P_1 \cup P_2)$ is not a closed set, and therefore it is not a polyhedron.

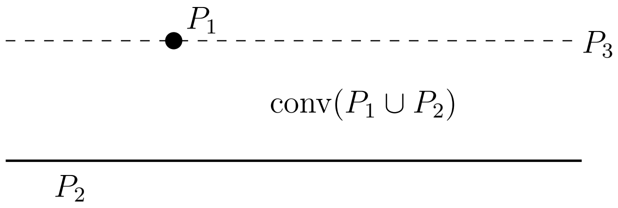


**Fig. 5.** $\overline{\text{conv}}(P_1 \cup P_2) \ne \text{conv}(P_1 \cup P_2)$

The following example shows that $\overline{\text{conv}}(\cup_{i=1}^k P_i)$ can have an exponential number of facets in the size of the input $P_i, i = 1, \ldots, k$.

Let $e^i$ denote the $i$-th unit vector in $\mathbb{R}^n$. Let $P_i$ be the single point $e^i$ for $i = 1, \ldots, n$, and $P_{n+i}$ the single point $-e^i$ for $i = 1, \ldots, n$. Then $\overline{\text{conv}}(\cup_{i=1}^{2n} P_i)$ has $2^n$ facets, namely

$$\overline{\text{conv}}(\bigcup_{i=1}^{2n} P_i) = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} \epsilon_i x_i \leq 1 \text{ for all } \epsilon \in \{-1, +1\}^n\}.$$

This polyhedron is called the *octahedron*.

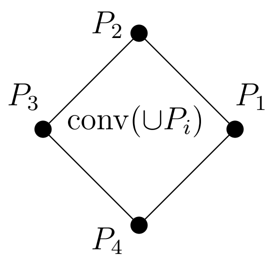


**Fig. 6.** Octahedron

Although $\overline{\text{conv}}(\cup_{i=1}^k P_i)$ may have exponentially many facets, Balas [5], [6] proved that it is the projection of a higher dimensional polyhedron $Y$ with a polynomial size representation:

$$Y := \begin{cases} A_i x^i \leq b^i y_i \\ \sum x^i = x \\ \sum y_i = 1 \\ \quad y_i \geq 0 \ \text{ for } i = 1, \ldots, k. \end{cases} \tag{4}$$

In this formulation, $x^i$ is a vector in $\mathbb{R}^n$ and $y_i$ is a scalar, for $i = 1, \ldots, k$. The vector $x \in \mathbb{R}^n$ corresponds to the original space onto which $Y$ is projected. Thus, the polyhedron $Y$ is defined in $\mathbb{R}^{kn+n+k}$. A formulation with a polynomial number of variables and constraints is said to be *compact*. The gist of Balas's result is that $\overline{\text{conv}}(\cup_{i=1}^k P_i)$ has a compact representation. The proof is given below.

Let $P_i := \{x \in \mathbb{R}^n : \ A_i x \leq b^i\}$ be a polyhedron and $C_i := \{x \in \mathbb{R}^n : \ A_i x \leq 0\}$ its recession cone. By the Minkowski-Weyl theorem, there exists a polytope $Q_i$ such that $P_i = Q_i + C_i$.

**Theorem 4.** *Consider $k$ polyhedra $P_i := \{x \in \mathbb{R}^n : \ A_i x \leq b^i\}$ and let $C_i := \{x \in \mathbb{R}^n : \ A_i x \leq 0\}$. Let $P_i =: Q_i + C_i$ where $Q_i$ is a polytope. If $\cup P_i \neq \emptyset$, assume that $C_j \subseteq \text{cone} \cup_{i: P_i \neq \emptyset} C_i$ for all $j$ such that $P_j = \emptyset$. Then the following sets are identical.*

*(i)* $\overline{conv}(\cup_{i=1}^k P_i)$
*(ii)* $conv(\cup_{i=1}^k Q_i) + cone(\cup_{i=1}^k C_i)$
*(iii)* $proj_x Y$*, where* $Y$ *is defined in (4).*

*Proof.* Define $Q := \text{conv}(\cup_{i=1}^k Q_i)$ and $C := \text{cone}(\cup_{i=1}^k C_i)$. We will show that

$$\text{conv}(\cup_{i=1}^k P_i) \subseteq \text{proj}_x Y \subseteq Q + C \subseteq \overline{\text{conv}}(\cup_{i=1}^k P_i).$$

Since $\text{proj}_x Y$ is a closed set, the first inclusion implies $\overline{\text{conv}}(\cup_{i=1}^k P_i) \subseteq \text{proj}_x Y$ and the theorem follows.

(a) $\text{conv}(\cup_{i=1}^k P_i) \subseteq \text{proj}_x Y$

The result holds when $\cup_{i=1}^k P_i = \emptyset$, so we assume $\cup_{i=1}^k P_i \neq \emptyset$. Without loss of generality, $P_1, \ldots, P_h$ are nonempty and $P_{h+1}, \ldots, P_k$ are empty, where $1 \leq h \leq k$.

Let $x \in \text{conv}(\cup_{i=1}^k P_i)$. Then $x$ is a convex combination of a finite number of points in $\cup_{i=1}^h P_i$. Since each $P_i$ is convex, we can write $x$ as a convex combination of points $z^i \in P_i$, say $x = \sum_{i=1}^h y_i z^i$ where $y_i \geq 0$ for $i = 1, \ldots, h$ and $\sum_{i=1}^h y_i = 1$. Let $x^i = y_i z^i$ for $i = 1, \ldots, h$ and $y_i = 0$, $x^i = 0$ for $i = h+1, \ldots, k$. Then $A_i x^i \leq b^i y_i$ for $i = 1, \ldots k$ and $x = \sum_{i=1}^k x^i$. This shows that $x \in \text{proj}_x Y$ and therefore (a) holds.

(b) $\text{proj}_x Y \subseteq Q + C$

The result holds if $Y = \emptyset$, so we assume $Y \neq \emptyset$.

Let $x \in \text{proj}_x Y$. By the definition of projection, there exist $x^1, \ldots, x^k, y$ such that $x = \sum_{i=1}^k x^i$ where $Ax^i \leq b^i y_i$, $\sum y_i = 1$, $y \geq 0$. Let $I := \{i : \ y_i > 0\}$.

For $i \in I$, let $z^i := \frac{x^i}{y_i}$. Then $z^i \in P_i$. Since $P_i = Q_i + C_i$, we can write $z^i = w^i + x^i$ where $w^i \in Q_i$ and $x^i \in C_i$.

For $i \notin I$, we have $A_i x^i \leq 0$. That is $x^i \in C_i$.

$$\begin{aligned} x &= \sum_{i \in I} y_i z^i + \sum_{i \notin I} x^i \\ &= \underbrace{\sum_{i \in I} y_i w^i}_{\in \text{conv}(\cup Q_i)} + \underbrace{\sum_{i=1}^k \lambda_i x^i}_{\in \text{cone}(\cup C_i)} \qquad \text{where } \lambda_i \geq 0 \\ &\in Q + C. \end{aligned}$$

(c) $Q + C \subseteq \overline{\text{conv}}(\cup_{i=1}^k P_i)$

The result holds when $Q = \emptyset$, so we assume $Q \neq \emptyset$. Without loss of generality $Q_1, \ldots, Q_h$ are nonempty and $Q_{h+1}, \ldots, Q_k$ are empty, where $1 \leq h \leq k$.

Let $x \in Q + C$. Then $x = \sum_{i=1}^h y_i w^i + \sum_{i=1}^k x^i$ where $w^i \in Q_i$, $y_i \geq 0$ for $i = 1, \ldots, h$, $\sum_{i=1}^h y_i = 1$ and $x^i \in C_i$ for $i = 1, \ldots k$. By the assumption in the statement of the theorem we have, for $j = h+1, \ldots, k$, $x^j = \sum_{i=1}^h \mu_i^j x^{ij}$, where $\mu_i^j \geq 0$ and $x^{ij} \in C_i$ for $i = 1, \ldots, h$.

Based on the above vector $y \in \mathbb{R}^h_+$, define $I := \{i : \; y_i > 0\}$ and consider the point

$$x^\epsilon := \sum_{i \in I} (y_i - \frac{h}{|I|}\epsilon) w^i + \sum_{i=1}^{h} \epsilon (w^i + \frac{1}{\epsilon}(x^i + \sum_{j=h+1}^{k} \mu_i^j x^{ij}))$$

for $\epsilon > 0$ small enough so that $y_i - \frac{h}{|I|}\epsilon \geq 0$ for all $i \in I$.

$x^\epsilon \in \text{conv}(\cup_{i=1}^k P_i)$ since $\sum_{i \in I}(y_i - \frac{h}{|I|}\epsilon) + \sum_{i=1}^h \epsilon = 1$ and $w^i + \frac{1}{\epsilon}(x^i + \sum_{j=h+1}^k \mu_i^j x^{ij}) \in P_i$.

Furthermore $x^\epsilon \to x$ as $\epsilon \to 0$.

Therefore $x \in \overline{\text{conv}}(\cup_{i=1}^k P_i)$. □

*Remark 4.* The assumption in Theorem 4 is necessary as shown by the following example (see Figure 7): $P_1 := \{x \in \mathbb{R}^2 : \; 0 \leq x \leq 1\}$ and $P_2 := \{x \in \mathbb{R}^2 : \; x_1 \leq 0, x_1 \geq 1\}$. Note that $P_2 = \emptyset$ and $C_2 = \{x \in \mathbb{R}^2 : \; x_1 = 0\}$. The theorem does not hold in this case since $\text{proj}_x Y = P_1 + C_2 = \{x \in \mathbb{R}^2 : \; 0 \leq x_1 \leq 1\}$, which is different from $\overline{\text{conv}}(P_1 \cup P_2) = P_1$.

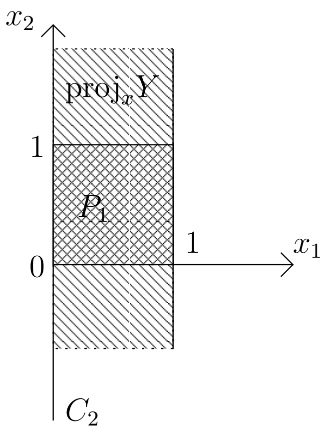


**Fig. 7.** $\overline{\text{conv}}(P_1 \cup P_2) \neq \text{proj}_x Y$

*Remark 5.* The assumption in Theorem 4 is automatically satisfied if

(i) $C_i = \{0\}$ whenever $P_i = \emptyset$, or

(ii) all the polyhedra $P_i$ have the same recession cone.

For example (i) holds when all the $P_i$s are nonempty, or when $C_i = \{0\}$ for all $i$.

*Remark 6.* If all the polyhedra $P_i$ have the same recession cone $C = C_i$ for $i = 1, \ldots, k$, then $\text{conv}(\cup_{i=1}^k P_i) = \overline{\text{conv}}(\cup_{i=1}^k P_i)$. Indeed, in this case, part (c) of the above proof simplifies and shows that $Q + C \subseteq \text{conv}(\cup_{i=1}^k P_i)$. Furthermore $\cup P_i$ is the $x$-projection of

$$Y_I := \begin{cases} A_i x^i \leq b^i y_i \\ \sum x^i = x \\ \sum y_i = 1 \\ \quad y_i \in \{0,1\} \ \text{ for } i = 1,\ldots,k. \end{cases}$$

To prove the last claim, observe that if $x \in \cup P_i$, then $x \in P_j$ for some $j = 1,\ldots,k$. Setting $x^j = x$, $y_j = 1$ and $x^i = 0$, $y_i = 0$ for $i \neq j$, we have a solution $(x, x^1, \ldots, x^k, y) \in Y_I$. Conversely, given $(x, x^1, \ldots, x^k, y) \in Y_I$, let $j$ be the index for which $y_j = 1$. We have $A_j x^j \leq b^j$ and $A_i x^i \leq 0$ for $i \neq j$, namely $x^j \in P_j$ and $x^i$ is in the recession cone of $P_i$ for $i \neq j$. Since the recession cone of $P_i$ is also the recession cone of $P_j$, the point $x = \sum_{i=1}^k x^i$ belongs to $P_j$.

*Remark 7.* When the $P_i$s have different recession cones, the $x$-projection of $Y_I$ is usually different from $\cup P_i$ as shown by the following example (see Figure 5).

Let $P_1 \subseteq \mathbb{R}^2$ consist of the single point $(0,1)$, and let $P_2 \subseteq \mathbb{R}^2$ be the line $x_2 = 0$. Then we claim that the $x$-projection of $Y_I$ is $P_2 \cup P_3$ where $P_3$ denotes the line $x_2 = 1$. Indeed, when $y_1 = 1$ in $Y_I$, $x = x^1 + x^2$ with $x^1 \in P_1$ and $x^2$ in the recession cone of $P_2$, which is $P_2$ itself in this example. So the $x$-projection of $Y_I$ restricted to $y_1 = 1$ is the line $P_3$. The $x$-projection of $Y_I$ restricted to $y_2 = 1$ is the line $P_2$, which proves the claim.

In fact, under the assumption of Theorem 4, we have

$$\text{proj}_x Y_I = \cup_{i=1}^k Q_i + \text{cone}(\cup_{i=1}^k C_i).$$

**Application:** Theorem 4 has been applied to several basic models that appear repeatedly in mixed integer programming [37], [45], [56], [3], [23]. As an example, we consider a mixed integer linear set studied by Miller and Wolsey [45] and Van Vyve [56] under the name of continuous mixing set. We present it here in a form that arises in the context of robust mixed 0,1 programming, following Atamtürk [3]:

$$S := \{(x,y,z) \in \{0,1\}^n \times \mathbb{R}^n_+ \times \mathbb{R}_+ : \ y_i + z \geq d_i x_i \ \text{ for } i = 1,\ldots,n\}$$

where $d \in \mathbb{R}^n_+$ is a given vector. The polyhedron $\text{conv}(S)$ has an exponential number of facets but has a compact formulation in a higher dimensional space. To derive this formulation, observe that the variable $z$ takes either the value 0 or $d_i$ for $i = 1,\ldots,n$ in every vertex of $\text{conv}(S)$. Set $d_0 := 0$. Let $\delta$ denote any of these $n+1$ values, and define

$$S(\delta) := \{(x,y,z) \in \{0,1\}^n \times \mathbb{R}^n_+ \times \delta : \ y_i \geq d_i x_i - \delta \ \text{ for } i = 1,\ldots,n\}.$$

Since the constraints of $S(\delta)$ are decoupled, one can strengthen each separately to obtain the convex hull of $S(\delta)$:

$$\text{conv}(S(\delta)) := \{(x,y,z) \in [0,1]^n \times \mathbb{R}^n \times \delta : \ y_i \geq (d_i - \delta)^+ x_i \ \text{ for } i = 1,\ldots,n\}.$$

Since $z \in \{d_0, d_1, \ldots, d_n\}$ for every vertex of conv($S$), we have

$$\text{conv}(S) = \text{conv}(\cup_{i=0}^{n} \text{conv}(S(d_i))) + C$$

where $C := \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \; x = 0, \; y \geq 0, \; z \geq 0\}$ is the recession cone of conv($S$). Defining $P_i := \text{conv}(S(d_i)) + C$ (all $P_i$s have the same recession cone $C$), it is straightforward to obtain the desired compact formulation $Y$ as defined in (4).

## 3. Lift-and-Project

In this section, we consider mixed 0,1 linear programs. These are mixed integer linear programs where the integer variables are only allowed to take the values 0 or 1. It will be convenient to write mixed 0,1 linear programs in the form

$$\begin{aligned} \min \;\; & cx \\ & Ax \geq b \\ & x_j \in \{0,1\} \text{ for } j = 1, \ldots, n \\ & x_j \geq 0 \qquad \text{for } j = n+1, \ldots, n+p, \end{aligned}$$

where the matrix $A \in \mathbb{Q}^{m \times (n+p)}$, the row vector $c \in \mathbb{Q}^{n+p}$ and the column vector $b \in \mathbb{Q}^m$ are data, and $x \in \mathbb{R}^{n+p}$ is a column vector of variables.

Consider the polyhedron $P := \{x \in \mathbb{R}_+^{n+p} : \;\; Ax \geq b\}$ and the mixed 0,1 linear set $S := \{x \in \{0,1\}^n \times \mathbb{R}_+^p : \; Ax \geq b\}$. The set conv($S$) is a polyhedron and, ideally, we would like to have its linear description in the form conv($S$) $=$ $Dx \geq d$. Constructing the description $Dx \geq d$ would reduce the solution of a mixed 0,1 linear program to that of a linear program. But this goal is too ambitious in general as the number of inequalities needed in the description $Dx \geq d$ is typically enormous. This is not surprising considering that mixed 0,1 linear programming is NP-hard! A more reasonable goal is to obtain an intermediate set between $P$ and conv($S$) over which one can optimize a linear function in polynomial time, and then to use recursion to get tighter approximations of conv($S$).

Sherali and Adams [54], Lovász and Schrijver [43] and Balas, Ceria and Cornuéjols [7] propose an approach for doing this which generates intermediate sets $Q$ between $P$ and conv($S$) as projections of higher dimensional sets that have a polynomial description. The polyhedron $P \subseteq \mathbb{R}^{n+p}$ is first lifted into a higher dimensional space $\mathbb{R}^{n+p+q}$ where the formulation is strengthened. This strengthened formulation is then projected back onto the original space $\mathbb{R}^{n+p}$, thus defining $Q$. In this process the constraints of the higher dimensional formulation are defined explicitly whereas those of $Q$ are only known implicitly through projection, thus allowing $Q$ to have a nonpolynomial number of constraints. This approach is known under the name of *lift-and-project*.

### *3.1. The lift-and-project relaxation*

Consider a polyhedron $P := \{x \in \mathbb{R}_+^{n+p} : \;\; Ax \geq b\}$ and the mixed 0,1 set $S := \{x \in \{0,1\}^n \times \mathbb{R}_+^p : \;\; Ax \geq b\}$. Without loss of generality, we assume that the constraints $Ax \geq b$ include $x_j \geq 0$ for $j = 1, \dots, n+p$, and $x_j \leq 1$ for $j = 1, \dots, n$. Balas, Ceria and Cornuéjols [7] study the following "lift-and-project" procedure:

**Step 0:** Select $j \in \{1, \dots, n\}$.
**Step 1:** Generate the nonlinear system $x_j(Ax-b) \geq 0,\ (1-x_j)(Ax-b) \geq 0$.
**Step 2:** Linearize the system by substituting $y_i$ for $x_i x_j$, $i \neq j$, and $x_j$ for $x_j^2$. Call this polyhedron $M_j$.
**Step 3:** Project $M_j$ onto the $x$-space. Let $P_j$ be the resulting polyhedron.

$S \subseteq P_j$ follows from the fact that, for any $x \in S$, we have $(x, y) \in M_j$ by choosing $y_i = x_i x_j$ for $i \neq j$ since $x_j^2 = x_j$ (this holds because $x_j$ is a 0,1 variable). We also have $P_j \subseteq P$ since $Ax \geq b$ is obtained by adding the constraints defining $M_j$. How tight is the relaxation $P_j$ of $S$ compared to the initial relaxation $P$? The next theorem shows that it is tightest possible among the relaxations that ignore the integrality of all the variables $x_i$ for $i \neq j$.

**Theorem 5.** $P_j = conv\{(Ax \geq b, x_j = 0) \cup (Ax \geq b, x_j = 1)\}$

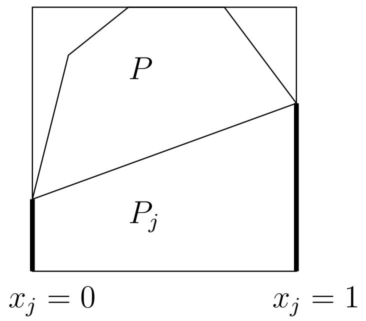


**Fig. 8.** Illustration of Theorem 5

*Proof.* Call $P^*$ the set $\text{conv}\{(Ax \geq b, x_j = 0) \cup (Ax \geq b, x_j = 1)\}$.

First we show $P_j \subseteq P^*$.

We assume $P \neq \emptyset$ since otherwise the result is trivial.

If $P \cap \{x_j = 0\} = \emptyset$, then $P^* = P \cap \{x_j = 1\}$. We already know that $P_j \subseteq P$. Thus, to show that $P_j \subseteq P^*$, it suffices to show that $P_j \subseteq \{x_j = 1\}$, i.e. that $x_j \geq 1$ is valid for $P_j$. Let $\epsilon := \min\{x_j :\ x \in P\}$. Since $P \cap \{x_j = 0\} = \emptyset$, we have $\epsilon > 0$. The inequality $x_j \geq \epsilon$ is valid for $P$ and, by Farkas's lemma, it is implied by a nonnegative combination of rows of $Ax \geq b$. Therefore $(1-x_j)x_j \geq (1-x_j)\epsilon$ is valid for the nonlinear system of Step 1. Step 2 replaces $x_j^2$ by $x_j$. This gives that $x_j \geq 1$ is valid for $P_j$.

Similarly if $P \cap \{x_j = 1\} = \emptyset$, we get $P_j \subseteq P^*$.

Now assume $P \cap \{x_j = 0\} \neq \emptyset$ and $P \cap \{x_j = 1\} \neq \emptyset$. Take $\alpha x \geq \beta$ valid for $P^*$. Since it is valid for $\Pi := P \cap \{x :\ x_j \leq 0\}$ we can find a $\lambda$ such that

$\alpha x + \lambda x_j \geq \beta$ is valid for $P$ (Lemma 1). Similarly, we can find $\mu$ such that $\alpha x + \mu(1 - x_j) \geq \beta$ is valid for $P$.

Therefore $(1 - x_j)(\alpha x + \lambda x_j - \beta) \geq 0$ and $x_j(\alpha x + \mu(1 - x_j) - \beta) \geq 0$ are valid for the nonlinear system of Step 1, and their sum is too:

$$\alpha x + (\lambda + \mu)(x_j - x_j^2) - \beta \geq 0.$$

Step 2 replaces $x_j^2$ by $x_j$. This gives $\alpha x \geq \beta$ valid for $M_j$, and thus for $P_j$. This completes the proof that $P_j \subseteq P^*$.

Now we show $P^* \subseteq P_j$. We assume $P^* \neq \emptyset$ since otherwise the result is trivial. Let $\overline{x}$ be a point in $P \cap \{x_j = 0\}$ or in $P \cap \{x_j = 1\}$. Define $\overline{y}_i = \overline{x}_i \overline{x}_j$ for $i \neq j$. Then $(\overline{x}, \overline{y}) \in M_j$ since $\overline{x}_j^2 = \overline{x}_j$. So, $\overline{x} \in P_j$. By convexity of $P_j$ it follows that $P^* \subseteq P_j$. $\square$

The set $\cap_{j=1}^n P_j$ is called the *lift-and-project closure*. It is a better approximation of $\text{conv}(S)$ than $P$:

$$\text{conv}(S) \subseteq \cap_{j=1}^n P_j \subseteq P.$$

How much better is it in practice? Bonami and Minoux [18] performed computational experiments (see also Bonami's dissertation [15]). On 35 mixed 0,1 linear programs from the MIPLIB library [13], they found that the lift-and-project closure reduces the integrality gap by 37 % on average (the *integrality gap* is the difference between the objective value optimized over $\text{conv}(S)$ and over $P$).

Sherali and Adams [54] define a stronger relaxation by skipping Step 0 and considering the nonlinear constraints $x_j(Ax - b) \geq 0$ and $(1 - x_j)(Ax - b) \geq 0$ for all $j = 1, \ldots, n$ in Step 1. Then, in Step 2, variables $y_{ij}$ are introduced for all $i = 1, \ldots, n+p$ and $j = 1, \ldots, n$ with $i > j$. Note that the size of the linear system generated in Step 2 is much larger than in the previous lift-and-project procedure ( $\frac{n(n-1)}{2} + np$ new variables and $2nm$ constraints, instead of just $n + p - 1$ new variables and $2m$ constraints before). Let $N$ denote the resulting polyhedron obtained by projection in Step 3. Clearly, the Sherali-Adams relaxation $N$ is at least as strong as the lift-and-project closure defined above, and it can be strictly stronger since the Sherali-Adams procedure takes advantage of the fact that $y_{ij} = y_{ji}$ whereas this is not the case for the lift-and-project closure $\cap_{j=1}^n P_j$. How much better is the Sherali-Adams relaxation in practice? We turn again to Bonami and Minoux [18]. For the 19 MIPLIB intances for which they could compute the Sherali-Adams bound within an hour, the improvement over the lift-and-project bound was 10 % on average. Specifically, on these 19 instances, the average integrality gap closed went from 29 % for the lift-and-project closure to 39 % for the Sherali-Adams relaxation. An even stronger relaxation can be obtained as follows:

*3.2. The Lovász-Schrijver relaxation*

**Step 1:** Generate the nonlinear system $x_j(Ax-b) \geq 0$ and $(1-x_j)(Ax-b) \geq 0$ for all $j = 1, \ldots, n$.

**Step 2:** Linearize the system by substituting $y_{ij}$ for $x_i x_j$, for all $i = 1, \ldots, n+p$, $j = 1, \ldots, n$ such that $j < i$, and $x_j$ for $x_j^2$ for all $j = 1, \ldots, n$. Denote by $Y$ the symmetric $(n+1) \times (n+1)$ matrix with the vector $(1, x_1, \ldots, x_n)$ in row 0, in column 0 and in the diagonal, and entry $y_{ij}$ in row $i$ and column $j$ for $i, j = 1, \ldots, n$ and $j < i$. Call $M_+$ the convex set in $\mathbb{R}_+^{\frac{n(n-1)}{2}+np}$ of all $(x, y)$ that satisfy the above linear inequalities and such that $Y$ is a positive semidefinite matrix.

**Step 3:** Project $M_+$ onto the $x$-space. Call $N_+$ the resulting convex set.

In this procedure, the fact that we get a relaxation $N_+$ instead of the set $S$ itself is because we replace the constraint $Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 & x^T \end{pmatrix}$ by "$Y$ positive semidefinite" in Step 2.

We have $\text{conv}(S) \subseteq N_+ \subseteq N \subseteq \cap_{j=1}^n P_j \subseteq P$.

The convex set $N_+$ is not a polyhedron in general. Nevertheless, a major interest in the above Lovász-Schrijver procedure [43] is due to the fact that a linear function can be optimized over $N_+$ in polynomial time. Indeed, optimizing a linear function over the constraint set defined in Step 2 is a *semidefinite program*: linear objective, linear constraints plus a symmetric positive semidefinite constraint on the matrix $Y$ of variables. Semidefinite programs can be solved in polynomial time by interior point algorithms [48].

How tight is the Lovász-Schrijver relaxation $N_+$? From a practical perspective the size of the semidefinite program creates a tremendous challenge: the number of variables has been multiplied by $n$ and the number of constraints as well! Burer and Vandenbussche [19] solve it using an augmented Lagrangian method and they report computational results on three classes of combinatorial problems, namely the maximum stable set problem, the quadratic assignment problem and the following problem of Erdös and Turan: Calculate the maximum size of a subet of numbers in $\{1, \ldots, n\}$ such that no three numbers are in arithmetic progression. In all three cases, the Lovász-Schrijver bound given by $N_+$ is substantially tighter than the Sherali-Adams bound given by $N$. To illustrate this, we give below the results obtained in [19] for the size of a maximum stable set (graphs with more than 100 nodes):

| Name | nodes | edges | optimum | $N_+$ | $N$ |
|---|---|---|---|---|---|
| brock200-1 | 200 | 5066 | 21 | 27.9 | 66.6 |
| c-fat200-1 | 200 | 18366 | 12 | 14.9 | 66.6 |
| johnson16-2-4 | 120 | 1680 | 8 | 10.2 | 23.3 |
| keller4 | 171 | 5100 | 11 | 15.4 | 57.0 |
| rand-200-05 | 200 | 982 | 64 | 72.7 | 75.1 |
| rand-200-50 | 200 | 10071 | 11 | 17.1 | 66.6 |

From a theoretical point of view, Goemans and Williamson [33] proved a striking result. They showed that, for the max-cut problem, the semidefinite relaxation is never more than 14% above the optimum value.

Is it possible to improve upon the Lovász-Schrijver relaxation in polynomial time? The answer is yes. See for example Lasserre [40], Laurent [41], and Bienstock and Zuckerberg [12]. These relaxations are even more computationally demanding than $N_+$. We do not discuss them here. Instead we return to relaxations that can yield cutting planes in a reasonably short amount of computing time.

### *3.3. Lift-and-project cuts*

In this section we return to the lift-and-project relaxation $P_j$. Optimizing a linear function over $P_j$ amounts to solving a linear program. So does the problem of generating a valid inequality for $P_j$ that cuts off a given point $\overline{x}$, or showing that none exists, as we explain below. Step 2 of the lift-and-project procedure in Section 3.1 constructs the following polyhedron $M_j$.

$$M_j := \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p} : \; Ay - bx_j \geq 0, \quad Ax + bx_j - Ay \geq b, \quad y_j = x_j\}.$$

The first two constraints come from linearizing the inequalities of Step 1. Actually, the variable $y_j$ is not introduced explicitly in Step 2. So, in fact, $M_j$ is the polyhedron in $\mathbb{R}_+^{n+p} \times \mathbb{R}_+^{n+p-\mathbf{1}}$ obtained by identifying the variables $y_j = x_j$ in the above set. Let $A_j$ be the $m \times (n+p-1)$ matrix obtained from $A$ by removing its $j$-th column $a^j$. We have

$$M_j := \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p-\mathbf{1}} : \; A_j y + (a^j - b)x_j \geq 0, \;\; Ax + (b - a^j)x_j - A_j y \geq b\}.$$

By renaming the coefficient matrices of $x$, we get

$$M_j = \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p-\mathbf{1}} : \; \tilde{B}_j x + A_j y \geq 0, \quad \tilde{A}_j x - A_j y \geq b\}$$

We want to project out the $y$ variables. This is done using Theorem 2. The appropriate cone is

$$Q := \{(u, v) : \; uA_j - vA_j = 0, \quad u \geq 0, v \geq 0\}.$$

Namely the set $P_j$ can be written as:

$$P_j = \{x \in \mathbb{R}_+^{n+p} : \; (u\tilde{B}_j + v\tilde{A}_j)x \geq vb \;\text{ for all } (u, v) \in Q\}.$$

Given a fractional solution $\overline{x}$, we can now express that an inequality $\alpha x \geq \beta$ is valid for $P_j$ and cuts off $\overline{x}$. Indeed $\alpha = u\tilde{B}_j + v\tilde{A}_j$ and $\beta = vb$ for $(u, v) \in Q$ expresses the validity for $P_j$ and $\alpha\overline{x} < \beta$ expresses that it is a cut. To get a deepest cut, one can solve the following linear program, the so-called *cut generating LP*:

$$\begin{aligned}
\max\ & vb - (u\tilde{B}_j + v\tilde{A}_j)\overline{x} \\
& uA_j - vA_j = 0 \\
& u \geq 0, \; v \geq 0.
\end{aligned}$$

These constraints along with a normalization constraint to truncate the cone will do. For example, we could add the constraint $\sum u_i + \sum v_i = 1$. Therefore, the cut generating LP has $2m$ variables and $n + p$ constraints other than the nonnegativity conditions. This is a fairly large linear program to solve just to generate one cut. Various tricks have been used to speed up the solution, such as working in the subspace of variables where $\overline{x}_j > 0$ for $j = 1, \dots, n+p$ and $\overline{x}_j < 1$ for $j = 1, \dots, n$, see [7]. Is it possible to generate lift-and-project cuts without explicitly formulating and solving the cut generating LP? Balas and Perregaard [10] give a positive answer. We explain this below.

Instead of expressing $P_j$ as the projection of $M_j$, it is also possible to express $P_j$ using Theorems 5 and 4 and then projecting onto the $x$-space. This is equivalent of course. By Theorem 5, $P_j$ is the convex hull of the union of two polyhedra:

$$\begin{aligned} Ax &\ge b \\ x &\ge 0 \\ -x_j &\ge 0 \end{aligned} \quad \text{and} \quad \begin{aligned} Ax &\ge b \\ x &\ge 0 \\ x_j &\ge 1. \end{aligned}$$

In this part, we assume that the inequalities $Ax \ge b$ contain $-x_j \ge -1$ for $j = 1, \dots, n$, but not $x \ge 0$. By Theorem 4,

$$P_j = \mathrm{proj}_x \left\{ \begin{array}{r} Ax^0 \ge by_0 \\ -x_j^0 \ge 0 \\ Ax^1 \ge by_1 \\ x_j^1 \ge y_1 \\ x^0 + x^1 = x \\ y_0 + y_1 = 1 \\ x, x^0, x^1, y_0, y_1 \ge 0. \end{array} \right.$$

Let $e_j$ denote the $j$-th unit vector. Using the projection theorem (Theorem 2), we get that $P_j$ is defined by the inequalities $\alpha x \ge \beta$ such that

$$\begin{array}{cccccccl} \alpha & & -uA & +u_0 e_j & & & \ge 0 \\ \alpha & & & & -vA & -v_0 e_j & \ge 0 \\ & \beta & -ub & & & & \le 0 \\ & \beta & & & -vb & -v_0 & \le 0 \\ & & u, & u_0, & v, & v_0 & \ge 0. \end{array}$$

Adding a normalization constraint, we obtain the cut generating LP:

$$\begin{array}{lllllll} \min \alpha\overline{x} - \beta & & & & & & \\ \quad \alpha & & -uA & +u_0 e_j & & & \ge 0 \\ \quad \alpha & & & & -vA & -v_0 e_j & \ge 0 \\ & \beta & -ub & & & & \le 0 \\ & \beta & & & -vb & -v_0 & \le 0 \\ & & \sum_{i=1}^m u_i & +u_0 & +\sum_{i=1}^m v_i & +v_0 & = 1 \\ & & u, & u_0, & v, & v_0 & \ge 0. \end{array} \tag{5}$$

Balas and Perregaard [10] give a precise correspondence between the basic feasible solutions of (5) and the basic solutions (possibly infeasible) of the usual LP relaxation

$$\text{(R)} \quad \min\{cx : \; Ax \geq b, x \geq 0\}.$$

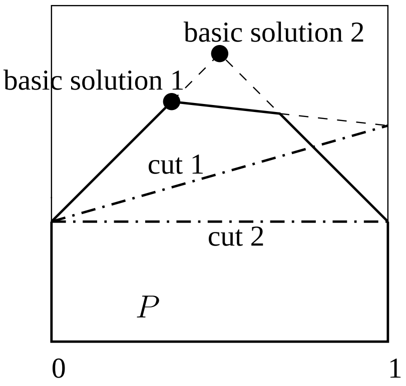


**Fig. 9.** Correspondence between basic solutions and lift-and-project cuts

A geometric view of this correspondence may be helpful: The $n+p$ extreme rays emanating from a basic solution of (R) intersect the hyperplanes $x_j = 0$ and $x_j = 1$ in $n+p$ points (some of these points may be at infinity). These points uniquely determine a hyperplane $\alpha x = \beta$ where $(\alpha, \beta)$ are associated with a basic feasible solution of the cut generating LP (5). For example, in Figure 9, cut 1 corresponds to the basic solution 1 of (R) and cut 2 corresponds to the basic (infeasible) solution 2 of (R).

Using the correspondence, Balas and Perregaard show how simplex pivots in the cut generating LP (5) can be mimicked by pivots in (R). The major practical consequence is that the cut generating LP (5) need not be formulated and solved explicitly. A sequence of increasingly deep lift-and-project cuts can be computed by pivoting directly in (R). We elaborate on these pivoting rules in Section 4.3.

### *3.4. Strengthened lift-and-project cuts*

Again we consider the mixed 0,1 linear set $S := \{x \in \{0,1\}^n \times \mathbb{R}^p_+ : \; Ax \geq b\}$. We assume that the constraints $Ax \geq b$ contain $-x_j \geq -1$ for $j = 1, \ldots, n$, but not $x \geq 0$. The cut generating LP (5) produces a lift-and-project inequality $\alpha x \geq \beta$ that is valid for $P_j$. The derivation only uses the integrality of variable $x_j$, not of the variables $x_k$ for $k = 1, \ldots, n$ and $k \neq j$. Balas and Jeroslow [9] found a simple way to use the integrality of the other variables to strengthen the lift-and-project cut. This strengthening has the nice property that it is straightforward to implement once the cut generating LP (5) has been solved.

Note that, given $u, u_0, v, v_0$, the optimal values of $\alpha_k$ and $\beta$ in (5) are:

$$\alpha_k = \begin{cases} \max(ua^k, va^k) & \text{for } k \neq j \\ \max(ua^j - u_0, va^j + v_0) & \text{for } k = j, \end{cases} \tag{6}$$

where $a^k$ denotes the $k$-th column of $A$, and

$$\beta = \min(ub, vb + u_0).$$

To strengthen the inequality $\alpha x \geq \beta$, one can try to decrease the coefficients $\alpha_k$. Balas and Jeroslow [9] found a way to do just that by using the integrality of the variables $x_k$ for $k = 1, \dots, n$.

**Theorem 6.** (Balas and Jeroslow [9]) *Let $\bar{x}$ satisfy $Ax \geq b, x \geq 0$. Given an optimal solution $u, u_0, v, v_0$ of the cut generating LP (5), define $m_k = \frac{va^k - ua^k}{u_0 + v_0}$,*

$$\alpha_k = \begin{cases} \min(ua^k + u_0\lceil m_k \rceil, va^k - v_0 \lfloor m_k \rfloor) & \text{for } k = 1, \dots, n \\ \max(ua^k, va^k) & \text{for } k = n+1, \dots, n+p \end{cases}$$

*and $\beta = \min(ub, vb + u_0)$. Then the inequality $\alpha x \geq \beta$ is valid for conv$(S)$.*

*Proof.* For $\pi \in \mathbb{Z}^n$, the following disjunction is valid for conv$(S)$:

$$\text{either } \sum_{k=1}^{n} \pi_k x_k \geq 0 \quad \text{or} \quad -\sum_{k=1}^{n} \pi_k x_k \geq 1.$$

Let us repeat the derivation of (5) with this disjunction in place of $-x_j \geq 0$ or $x_j \geq 1$ as before. We consider the union of

$$\begin{array}{rcr} Ax \geq b & & Ax \geq b \\ x \geq 0 & \text{and} & x \geq 0 \\ \sum_{k=1}^{n} \pi_k x_k \geq 0 & & -\sum_{k=1}^{n} \pi_k x_k \geq 1. \end{array}$$

Using Theorem 4 and the projection theorem (Theorem 2), we get that any inequality $\alpha x \geq \beta$ that satisfies

$$\begin{array}{lllllr} \alpha & -uA & -u_0(\sum_{k=1}^{n} \pi_k e_k) & & & \geq 0 \\ \alpha & & & -vA & +v_0(\sum_{k=1}^{n} \pi_k e_k) & \geq 0 \\ \beta & -ub & & & & \leq 0 \\ \beta & & & -vb & -v_0 & \leq 0 \\ & u, & u_0, & v, & v_0 & \geq 0 \end{array}$$

is valid for conv$(S)$. We can choose $u, u_0, v, v_0$ to be an optimal solution of the original cut generating LP (5). This implies that, for $k = 1, \dots, n$, we can choose $\alpha_k = \max(ua^k + u_0\pi_k, va^k - v_0\pi_k)$. Smaller coefficients $\alpha_k$ produce stronger inequalities since the variables are nonnegative. What is the best choice of $\pi_k \in \mathbb{Z}$ to get a small $\alpha_k$? It is obtained by equating $ua^k + u_0\pi_k$ and $va^k - v_0\pi_k$, which yields the value $m_k$ in the statement of the theorem (both $u_0$ and $v_0$ are strictly positive since otherwise $\alpha x \geq \beta$ is valid for $P$, contradicting that it is a cut for $\bar{x}$), and then rounding this value $m_k$ either up or down since $\pi_k$ must be integer. The best choice is the minimum stated in the theorem. $\square$

Bonami and Minoux [18] found that applying the Balas-Jeroslow strengthening step improves the average gap closed by an additional 8 %, as compared to the lift-and-project closure, on the 35 MIPLIB instances in their experiment. Specifically, the integrality gap closed goes from 37 % to 45 %. The time to perform the strengthening step is negligible.

### *3.5. Sequential convexification*

**Theorem 7.** (Balas [5]) $P_n(P_{n-1}(\ldots P_2(P_1)\ldots)) = conv(S)$.

*Proof.* By induction. Let $S_t := \{x \in \{0,1\}^t \times \mathbb{R}_+^{n-t+p} : \; Ax \geq b\}$. We want to show $P_t(P_{t-1}(\ldots P_2(P_1)\ldots)) = \mathrm{conv}(S_t)$. This is true for $t = 1$ by Theorem 5 applied to $j = 1$, so consider $t \geq 2$. Suppose that this is true for $t-1$. By the induction hypothesis we have

$$P_t(P_{t-1}(\ldots P_2(P_1)\ldots)) = P_t(\mathrm{conv}(S_{t-1}))$$

and by Theorem 5 applied to $j = t$, this is

$$= \mathrm{conv}(\mathrm{conv}(S_{t-1}) \cap \{x_t = 0\}) \cup (\mathrm{conv}(S_{t-1}) \cap \{x_t = 1\}).$$

For any set $S$ that lies entirely on one side of a hyperplane $H$, Lemma 2 states that

$$\mathrm{conv}(S) \cap H = \mathrm{conv}(S \cap H).$$

Therefore $\mathrm{conv}(S_{t-1}) \cap \{x_t = 0\} = \mathrm{conv}(S_{t-1} \cap \{x_t = 0\})$ and $\mathrm{conv}(S_{t-1}) \cap \{x_t = 1\} = \mathrm{conv}(S_{t-1} \cap \{x_t = 1\})$. Thus

$$P_t(P_{t-1}(\ldots P_2(P_1)\ldots)) = \mathrm{conv}(\mathrm{conv}(S_{t-1} \cap \{x_t = 0\})) \cup (\mathrm{conv}(S_{t-1} \cap \{x_t = 1\}))$$

$$= \mathrm{conv}((S_{t-1} \cap \{x_t = 0\}) \cup (S_{t-1} \cap \{x_t = 1\})) = \mathrm{conv}(S_t). \qquad \square$$

### *3.6. Rank*

Let $P^1$ denote the lift-and-project closure relative to $P$, i.e.

$$P^1 := \bigcap_{j=1}^{n} P_j.$$

For $k \geq 2$, let $P^k$ denote the lift-and-project closure relative to $P^{k-1}$. The set $P^k$ is a polyhedron. It is called the *k-th lift-and-project closure* relative to $P$. A valid inequality for $P^k$ but not $P^{k-1}$ is said to be *of rank k*. The lift-and-project *rank* of $P$ is the smallest integer such that $P^k = \mathrm{conv}(S)$. It follows from Theorem 7 that the lift-and-project rank of $P$ is at most $n$.

Similarly, $N_+$ is the Lovász-Schrijver closure relative to $P$, and the *k-th Lovász-Schrijver closure* $N_+^k$ is defined iteratively. The *Lovász-Schrijver rank* relative to $P$ is the smallest $k$ such that $N_+^k = \mathrm{conv}(S)$. Since $N_+ \subseteq P^1$, the

Lovász-Schrijver rank is at most $n$. Cook and Dash [24] and Goemans and Tunçel [32] give examples showing that the Lovász-Schrijver rank can be equal to $n$.

Let $P := \{x \in [0,1]^n : \sum_{j=1}^n x_j \geq \frac{1}{2}\}$ and $S := P \cap \mathbb{Z}^n$. Here $\text{conv}(S) = \{x \in [0,1]^n : \sum_{j=1}^n x_j \geq 1\}$. Cook and Dash show that the point $(\frac{1}{2n-k}, \ldots, \frac{1}{2n-k})$ belongs to $N_+^k$ for $k \leq n$. Note that this point does not belong to $\text{conv}(S)$ for $k < n$, showing that $N_+^k \neq \text{conv}(S)$ for $k < n$.

Another example is the following: $P := \{x \in \mathbb{R}^n : \sum_{j \in J} x_j + \sum_{j \notin J}(1 - x_j) \geq 1 \quad \text{for all } J \subseteq \{1, \ldots, n\}\}$ and $S := P \cap \{0,1\}^n$. Observe that $S = \emptyset$. However, Goemans and Tunçel and independently Cook and Dash show that $(\frac{1}{2}, \ldots, \frac{1}{2}) \in N_+^{n-1}$.

## 4. Mixed Integer Inequalities

In this section, we return to mixed integer linear sets $S := \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$ where the variables $x_j$ can take general nonnegative integer values, instead of just 0 or 1 in the previous section.

We present four families of valid inequalities for $\text{conv}(S)$: Gomory mixed integer inequalities, mixed integer rounding inequalities, split inequalities, and intersection inequalities. We compare the resulting elementary closures and address computational issues.

We do not address the general theory of superadditive functions [36] in this tutorial. The interested reader is referred to [46] pages 247-253 for an introduction to superadditivity applied to mixed integer linear sets.

### *4.1. Gomory's derivation*

Let us consider a mixed integer linear set comprising a single equality constraint:

$$S := \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \sum_{j=1}^n a_j x_j + \sum_{j=1}^p g_j y_j = b\}.$$

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$.

Let $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$. Then

$$\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j=1}^p g_j y_j \ = \ k + f_0$$

where $k$ is some integer. Since $k \leq -1$ or $k \geq 0$, any $x \in S$ satisfies the disjunction

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j=1}^p \frac{g_j}{f_0} y_j \geq 1 \tag{7}$$

OR

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j=1}^p \frac{g_j}{1 - f_0} y_j \geq 1. \tag{8}$$

This is of the form $a^1 z \geq 1$ or $a^2 z \geq 1$ which implies $\sum_j \max(a^1_j, a^2_j) z_j \geq 1$ for any $z \geq 0$. For each variable, what is the maximum coefficient in (7) and (8)? The answer is easy since one coefficient is positive and the other is negative. We get

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1-f_0} y_j \geq 1. \tag{9}$$

We have just proved that inequality (9) is valid for $S$. This is the *Gomory mixed integer inequality* (GMI inequality) [34].

*Remark 8.* In the pure integer programming case, the GMI inequality reduces to

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1-f_j}{1-f_0} x_j \geq 1$$

Since $\frac{1-f_j}{1-f_0} < \frac{f_j}{f_0}$ when $f_j > f_0$, it follows that the GMI inequality dominates

$$\sum_{j=1}^{n} f_j x_j \geq f_0,$$

which is known as the *fractional cut*. The fractional cut can also be derived using Chvátal's procedure [21].

Consider now mixed integer linear sets with $m$ inequality constraints, instead of one equality constraint above, $S := \{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ :\ Ax + Gy \leq b\}$. Let $P := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p :\ Ax + Gy \leq b,\ x \geq 0, y \geq 0\}$ denote the underlying polyhedron. If $P = \emptyset$ then $S = \emptyset$ and we are done. Therefore we may assume that $P \neq \emptyset$. Let $\alpha x + \gamma y \leq \beta$ be any valid inequality for $P$ (Recall that, by Farkas's lemma, the valid inequalities for $P$ are of the form $uAx + uGy - vx - wy \leq ub + t$ where $u \in \mathbb{R}^m_+$, $v \in \mathbb{R}^n_+$, $w \in \mathbb{R}^p_+$, $t \in \mathbb{R}_+$). Add a nonnegative slack variable $\alpha x + \gamma y + s = \beta$, derive a GMI inequality (9) from this equation, and eliminate $s = b - \alpha x - \gamma y$ from this inequality. The result is a valid inequality for $S$, in the space $\mathbb{R}^n \times \mathbb{R}^p$ of the variables $x, y$. Let us call these inequalities *GMI inequalities* for $S$. The *Gomory mixed integer closure* is obtained from $P$ by adding all the GMI inequalities for $S$.

In the previous section, we saw that one can optimize in polynomial time over the lift-and-project closure, the Sherali-Adams closure and the Lovász-Schrijver closure. A natural question is whether the same is true of the Gomory mixed integer closure. It turns out that the situation is totally different: It is NP-hard to optimize a linear function over the Gomory mixed integer closure relative to a polyhedron $P$ (Caprara and Letchford [20], Cornuéjols and Li [28]). Equivalently, given a point $(\overline{x}, \overline{y}) \in P \setminus S$, it is NP-hard to find a GMI cut with respect to $(\overline{x}, \overline{y})$ or show that none exists. A similar NP-hardness result was proved earlier by Eisenbrand [30] for the Chvátal closure [21].

Note that this is in contrast with the problem of finding a GMI cut that cuts off a *basic* solution $(\overline{x}, \overline{y}) \in P \setminus S$. Indeed, any row of the simplex tableau where $\overline{x}_j$ is fractional for some $j = 1, \ldots, n$ can be used to generate a GMI cut. We illustrate this in the next section.

*4.2. Gomory mixed integer cuts from the simplex tableau*

We illustrate the application of GMI cuts on a small example.

$$\begin{array}{lrl} \max & x & +2y \\ & -x & +y \leq 2 \\ & x & +y \leq 5 \\ & 2x & -y \leq 4 \\ & \multicolumn{2}{l}{x \in \mathbb{Z}_+,\ y \in \mathbb{R}_+.} \end{array}$$

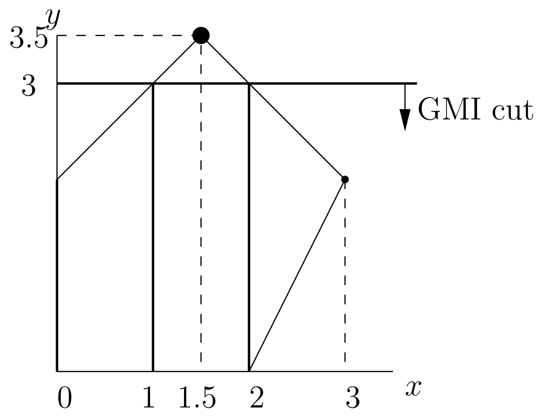


**Fig. 10.** Gomory mixed integer cut from the simplex tableau

Add nonnegative slacks $s_1, s_2, s_3$ and solve the LP relaxation $\max z$ where

$$\begin{array}{rrrrrrl} z & -x & -2y & & & & = 0 \\ & -x & +y & +s_1 & & & = 2 \\ & x & +y & & +s_2 & & = 5 \\ & 2x & -y & & & +s_3 & = 4. \end{array}$$

The optimal tableau is

$$\begin{array}{rrrrrrl} z & & & +0.5s_1 & +1.5s_2 & & = 8.5 \\ & & y & +0.5s_1 & +0.5s_2 & & = 3.5 \\ & x & & -0.5s_1 & +0.5s_2 & & = 1.5 \\ & & & 0.5s_1 & -0.5s_2 & +s_3 & = 4.5 \end{array}$$

and the corresponding solution is $\overline{x} = 1.5$, $\overline{y} = 3.5$. Since $x$ is not integer, we generate a cut from the row in which $x$ is basic, namely

$$x - 0.5s_1 + 0.5s_2 = 1.5$$

Here $f_0 = 0.5$ and applying the GMI formula, we get

$$s_1 + s_2 \geq 1.$$

Since $s_1 + s_2 = 7 - 2y$, we can express the cut in the $(x, y)$-space:

$$y \leq 3.$$

Adding this cut and solving the strengthened LP relaxation gives the solution $\overline{x} = 2$, $\overline{y} = 3$. Since $\bar{x}$ is integer in this solution, it is the optimal solution of the mixed integer program.

Gomory [34] introduced GMI cuts in the early 60s but, for decades, they were not used in commercial integer programming software. Most textbooks from the 80s considered them impractical for various reasons. They were revived in the 90s when Balas, Ceria, Cornuéjols and Natraj [8] demonstrated that a branch-and-cut algorithm based on GMI cuts was practical and superior to state-of-the-art algorithms of the time. GMI cuts turn out to be surprisingly good in practice (Bixby, Gu, Rothberg, Wunderling [14]). On 41 MIPLIB instances, adding the GMI cuts generated from the optimal simplex tableau reduces the integrality gap by 24 % on average [17]. Greater improvements can be obtained by performing several rounds of cutting. A *round* consists of solving the current linear programming relaxation, generating a GMI cut for each basic variable $x_i$ with a fractional value $\bar{x}_i$ in the current solution, and updating the linear program by adding these cuts. Since GMI cuts from the simplex tableau are extremely easy to generate and effective in reducing the integrality gap, they are widely used in commercial codes today. Numerical issues need to be addressed, of course: in order to avoid numerical instability (and possibly cutting off the optimal solution), one usually discards cuts that have too large a ratio between the largest and smallest coefficient; in order to avoid fill in of the basis inverse, one also discards cuts that are too dense.

### *4.3. Improving mixed integer Gomory cuts by lift-and-project*

In this section we return to mixed 0-1 programming and the correspondence between basic feasible solutions of the cut generating LP (5) and basic solutions (possibly infeasible) of the usual LP relaxation (R) introduced in Section 3.3. Balas and Perregaard [10] showed how this correspondence can be used to improve Gomory mixed integer cuts by mimicking simplex pivots in (5) through pivots in (R). The simplex tableaux of (5) and (R) will be referred to as *large* and *small* respectively.

Let

$$x_j = a_{j0} - \sum_{h \in J} a_{jh} x_h \tag{10}$$

be a row of the small optimal simplex tableau such that $0 < a_{j0} < 1$. The GMI cut from this row is equivalent to the strengthened lift-and-project cut from some basic feasible solution of (5), where index $j$ in (5) is the same as in (10). To identify this solution, partition $J$ into subsets $M_1$ and $M_2$, such that $h \in M_1$ if $a_{jh} < 0$, and $h \in M_2$ if $a_{jh} > 0$ ($h \in J$ such that $a_{jh} = 0$ can go into either subset). Then eliminating $\alpha$, $\beta$ from (5), the $n$ columns indexed by $M_1 \cup M_2$

together with the two columns indexed by $u_0$ and $v_0$ define a feasible basis of the resulting system of $n+2$ equations. The strengthened lift-and-project cut associated with this basic feasible solution to (5) is equivalent to the GMI cut from (10).

To evaluate the GMI cut generated from the small simplex tableau (10) as a lift-and-project cut, we calculate the reduced costs in the large tableau of the nonbasic variables of the above solution to (5). Each row $x_i$ of the small tableau corresponds to a pair of columns of the large tableau, associated with variables $u_i$ and $v_i$. The reduced costs $r(u_i)$, $r(v_i)$ of these variables in the large tableau are known simple functions of the entries $a_{ih}$ and $a_{jh}$, for $h \in J$, of rows $j$ and $i$ of the small tableau. If they are all nonnegative, the current large tableau is optimal, hence the GMI cut from (10) cannot be improved. Otherwise, the cut can be improved by executing a pivot in a row $i$ of the small tableau, such that $r(u_i) < 0$ or $r(v_i) < 0$.

To identify the nonbasic variable $x_k$ to replace $x_i$ in the basis of the small tableau, we calculate for each $h \in J$ the objective function value $f(a_{ih})$ of (5) resulting from the corresponding exchange in the large tableau. This value is a known simple function of the ratio $a_{jh}/a_{ih}$ and of the coefficients of rows $j$ and $i$ of the small tableau. Any column $h$ for which $f(a_{ih}) < 0$ is a candidate for an improving pivot, and the most negative value indicates the best column $k$.

Executing the pivot in the small tableau that exchanges $x_i$ for $x_k$ yields a new simplex tableau (whose solution is typically infeasible), whose $j$-th row (the same $j$ as before!) is of the form

$$x_j = a_{j0} + t a_{i0} - \sum_{h \in J \cup i \setminus k} (a_{jh} + t a_{ih}) x_h, \tag{11}$$

with $t := a_{jk}/a_{ik}$. The GMI cut from (11) is then stronger than the one from (10), in the sense that it cuts off the LP optimum of (R) by a larger amount.

These steps can then be repeated with (11) replacing (10) for as long as improvements are possible.

Practical experience shows that in about three quarters of the cases GMI cuts from the optimal simplex tableau can be improved by the pivoting procedure described above. On the other hand, improvements beyond 10 pivots are not frequent, and beyond 20 pivots they are very rare.

This procedure was extensively tested and has been incorporated into the mixed integer module of XPRESS, with computational results reported in [50].

### *4.4. K-cuts and reduce-and-split cuts*

What happens if, instead of generating a GMI inequality from an equation $\sum_{j=1}^n a_j x_j + \sum_{j=1}^p g_j y_j = b$, we first multiply the equation by $k$ before generating the GMI inequality?

Let us try it on an example.

$$\begin{aligned} &\max\ 10x_1 + 13x_2 \\ &\qquad 10x_1 + 14x_2 \le 43 \\ &\qquad x_1, x_2 \in \mathbb{Z}_+. \end{aligned}$$

Let $z = 10x_1 + 13x_2$ and introduce a nonnegative integer slack $x_3$. The optimal tableau is

$$\begin{array}{llll} z & +x_2 & +x_3 & = 43 \\ \quad x_1 & +1.4x_2 & +0.1x_3 & = 4.3 \end{array}$$

The GMI cut is $\frac{1-0.4}{1-0.3}x_2 + \frac{0.1}{0.3}x_3 \ge 1$, i.e. $18x_2 + 7x_3 \ge 21$.

If we multiply the equation $x_1 + 1.4x_2 + 0.1x_3 = 4.3$ by $k = 2$, we get $2x_1 + 2.8x_2 + 0.2x_3 = 8.6$. The resulting GMI cut is $3x_1 + 2x_2 \ge 6$.

If we multiply by $k = 3$, we get the GMI cut $2x_2 + 3x_3 \ge 9$.

If we multiply by $k = 4$, we get $2x_2 + 3x_3 \ge 4$.

If we multiply by $k = 5$, we get $x_3 \ge 1$.

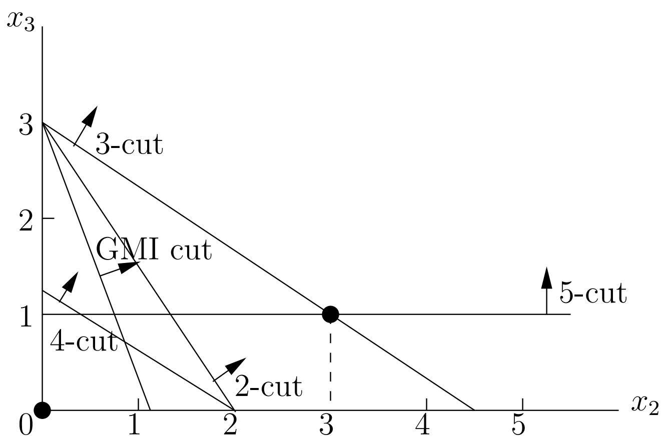


**Fig. 11.** K-cuts for the 2-variable example

Clearly, these simple operations on the equation $x_1 + 1.4x_2 + 0.1x_3 = 4.3$ produce very different GMI cuts. See Figure 11.

More generally, if we have an optimal simplex tableau, what sort of operations should we perform on the rows in order to generate useful GMI cuts?

Observe that, if we multiply $\sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b$ by a factor $k > 1$, the coefficients of the $x_j$ variables in the GMI inequality remain between 0 and 1, but the coefficients of the $y_j$ variables are multiplied by the factor $k$, which makes the inequality weaker.

This suggests the following idea (Andersen, Cornuéjols and Li [2]): Take linear combinations of the constraints in order to produce equations

$$\sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b$$

where the norm of the vector $g$ is small, and generate GMI inequalities from these "reduced" equations. Specifically, consider the rows of the optimal simplex tableau corresponding to the basic variables $x_i$ (but not the basic $y_i$s):

$$x_i = \overline{x}_i - \sum_{j \in J} \overline{a}_{ij} x_j - \sum_{h \in H} \overline{g}_{ih} y_h \quad \text{for } i \in B.$$

This gives a set of vectors $\{\overline{g}_i\}_{i \in B}$. Iteratively reduce the norm of at least one vector $\overline{g}_i$ in this set by *integral* combinations of the others (Integrality is useful to guarantee that the basic variables end up with integral coefficients when combining the rows). The procedure for reducing the vectors $\{\overline{g}_i\}_{i \in B}$ is similar to Lovász's basis reduction algorithm [42]. It terminates with a set of vectors $\{g'_i\}_{i \in B}$ with reduced norms. The corresponding combinations of the tableau rows are

$$\pi^i x_B = x'_i - \sum_{j \in J} a'_{ij} x_j - \sum_{h \in H} g'_{ih} y_h \quad \text{for } i \in B.$$

The GMI inequalities generated from the equations where $x'_i \notin \mathbb{Z}$ cut off $(\overline{x}, \overline{y})$ since only the nonbasic variables end up with nonzero coefficients in these inequalities (indeed, the vectors $\pi^i$ are integral). These cuts are called *reduce-and-split* cuts (the name comes from the equivalence between GMI inequalities and split inequalities, which we prove in Theorem 10). Computational experiments reported in Andersen, Cornuéjols and Li [2] show that the reduce-and-split cuts are usually quite different from the GMI cuts generated from the rows of the optimal tableau and that, in some cases, they can be significantly stronger. To illustrate this, we present a few instances of the MIPLIB where the improvement was particularly striking. The gap closed is reported after 20 rounds of cutting. The last two columns give the number of nodes in a branch-and-bound algorithm using these cuts in the formulation.

| Name | GMI gap | R&S gap | GMI nodes | R&S nodes |
|---|---|---|---|---|
| flugpl | 14% | 100% | 184 | 0 |
| gesa2 | 46% | 97% | 743 | 116 |
| gesa2o | 92% | 98% | 9145 | 75 |
| mod008 | 47% | 88% | 1409 | 82 |
| pp08a | 83% | 92% | 7467 | 745 |
| rgn | 15% | 100% | 874 | 0 |
| vpm1 | 44% | 98% | 7132 | 1 |
| vpm2 | 41% | 61% | 38946 | 4254 |

In other instances the reduce-and-split cuts did not improve on the GMI cuts from the optimal tableau. Therefore it seems that a hybrid approach that uses both types of cuts is a reasonable strategy.

The idea of using basis reduction to generate strong GMI cuts has been suggested independently by Köppe and Weismantel [39].

### *4.5. Mixed integer rounding inequalities*

Nemhauser and Wolsey [46], [47] and Wolsey [57] introduced two definitions of mixed integer rounding (MIR) inequalities. We follow Wolsey [57].

**Lemma 3.** *Consider the 2-variable mixed integer set* $S := \{(x,y) \in \mathbb{Z} \times \mathbb{R}_+ : x - y \le b\}$. *Let* $f_0 := b - \lfloor b \rfloor$. *Then the inequality*

$$x - \frac{1}{1-f_0} y \le \lfloor b \rfloor \tag{12}$$

*is a valid inequality for conv*$(S)$.

*Proof.* We show the validity of (12) in two different ways, for $x \le \lfloor b \rfloor$ and $x \ge \lfloor b \rfloor + 1$.

If $x \le \lfloor b \rfloor$, then adding this inequality to $\frac{1}{1-f_0}$ times $-y \le 0$ yields (12).

If $x \ge \lfloor b \rfloor + 1$, then adding $\frac{f_0}{1-f_0}$ times $-x \le -\lfloor b \rfloor - 1$ to $\frac{1}{1-f_0}$ times $x - y \le b$ yields (12). □

Note that the assumption $y \ge 0$ is critical in the above derivation, whereas we can have indifferently $x \in \mathbb{Z}$ or $x \in \mathbb{Z}_+$.

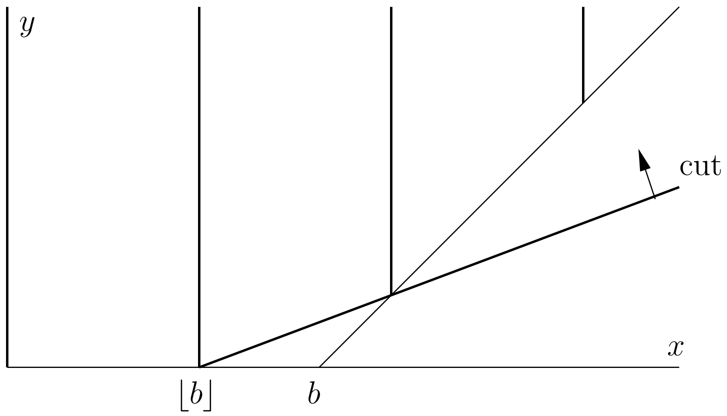


**Fig. 12.** Illustration of Lemma 3

**Theorem 8.** *Consider a mixed integer set defined by a single inequality:* $S := \{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ :\ ax + gy \le b\}$. *Let* $f_0 := b - \lfloor b \rfloor$ *and* $f_j := a_j - \lfloor a_j \rfloor$. *Then the inequality*

$$\sum_{j=1}^{n} (\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1-f_0}) x_j + \frac{1}{1-f_0} \sum_{j: g_j < 0} g_j y_j \le \lfloor b \rfloor \tag{13}$$

*is a valid inequality for conv*$(S)$.

*Proof.* Relax $ax + gy \le b$ to

$$\sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} a_j x_j + \sum_{j:g_j<0} g_j y_j \le b.$$

The validity of the relaxation follows from $x \ge 0$ and $y \ge 0$. Let

$$w := \sum_{j:f_j \le f_0} \lfloor a_j \rfloor x_j + \sum_{j:f_j > f_0} \lceil a_j \rceil x_j \quad \text{and}$$

$$z := -\sum_{j:g_j<0} g_j y_j + \sum_{j:f_j>f_0} (1 - f_j) x_j.$$

We have $w - z \le b$ and since $w \in \mathbb{Z}$ and $z \in \mathbb{R}_+$, we can apply Lemma 3. Thus

$$w - \frac{1}{1 - f_0} z \le \lfloor b \rfloor.$$

Subtituting $w$ and $z$ yields (13). □

Wolsey [57] calls the inequality (13) *MIR inequality.*

**Lemma 4.** *Consider a mixed integer set defined by a single inequality:* $S := \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \ ax + gy \le b\}$. *The MIR inequality (13) is identical to the GMI inequality.*

*Proof.* The Gomory mixed integer inequality is obtained by adding a slack variable $ax + gy + s = b$, generating (9) in the $(x, y, s)$-space:

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j>0} \frac{g_j}{f_0} y_j - \sum_{g_j<0} \frac{g_j}{1 - f_0} y_j + \frac{1}{f_0} s \ge 1$$

and substituting $s = b - ax - gy$ in this inequality to get the GMI inequality in the $(x, y)$-space. It is straightforward to check that the resulting inequality is identical to (13). □

Next we return to the general mixed integer linear set $S := \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \ Ax + Gy \le b\}$. The proof of the previous lemma can be slightly modified to show the next result.

**Lemma 5.** *Consider a mixed integer set with* $m$ *constraints* $S := \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \ Ax + Gy \le b\}$. *Let* $s := b - Ax - Gy$ *be a nonnegative vector of slack variables. For any* $\lambda \in \mathbb{R}^m$, *let* $a := \lambda A$, $g := \lambda G$, $\beta := \lambda b$, $f_j := a_j - \lfloor a_j \rfloor$ *and* $f_0 := \beta - \lfloor \beta \rfloor$. *The GMI inequality generated from* $\lambda A x + \lambda G y + \lambda s = \lambda b$ *is*

$$\sum_{j=1}^{n} (\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0}) x_j + \frac{1}{1 - f_0} \sum_{j:g_j<0} g_j y_j + \frac{1}{1 - f_0} \sum_{i:\lambda_i<0} \lambda_i s_i \le \lfloor \beta \rfloor \quad (14)$$

Any valid inequality for $P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : \ Ax + Gy \le b, \ x \ge 0, y \ge 0\}$ can be used to generate a MIR inequality (13) valid for $S$. Define the *MIR closure* relative to $P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : \ Ax + Gy \le b\}$ as the set obtained from $P$ by adding all these MIR inequalities. Lemma 4 implies that the MIR closure contains the GMI closure. This inclusion can be strict as observed by Bonami and Cornuéjols [16], and Dash, Günlük and Lodi [29]: For the integer set $S := \{(x_1, x_2) \in \mathbb{Z}^2_+ : 2x_1 + x_2 \le 2, \ -2x_1 + x_2 \le 0\}$, the inequality $x_2 \le 0$ is a GMI inequality (to see this, use $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = -\frac{1}{4}$) but one can show that it is not a MIR inequality. Thus, in general, inequalities (14) can be stronger than (13). This is because the vector $\lambda$ in Lemma 5 can have both positive and negative components, whereas (13) corresponds to the special case of (14) where only nonnegative multipliers $\lambda_i$ are used. With the other definition of MIR inequalities (Nemhauser and Wolsey [47]), one can show that the MIR closure is identical to the GMI closure [47], [26].

Marchand and Wolsey [44] showed that several classical inequalities in integer programming, such as flow cover inequalities, are MIR inequalities. They implemented an aggregation heuristic to generate MIR cuts with excellent computational results in the resolution of mixed integer linear programs. For example, on 41 MIPLIB instances, the Marchand-Wolsey aggregation heuristic (as available in the COIN-OR repository) reduces the integrality gap by 23% on average [17].

### *4.6. Split inequalities*

Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : \ Ax + Gy \le b\}$ where $A$, $G$, $b$ have rational entries, and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. For $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, define

$$\Pi_1 := P \cap \{(x, y) : \ \pi x \le \pi_0\}$$

$$\Pi_2 := P \cap \{(x, y) : \ \pi x \ge \pi_0 + 1\}$$

Clearly $S \subseteq \Pi_1 \cup \Pi_2$. Therefore any inequality $cx + hy \le c_0$ that is valid for $\Pi_1 \cup \Pi_2$ is also valid for $S$. An inequality $cx + hy \le c_0$ is called a *split inequality* [25] if there exists $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $cx + hy \le c_0$ is valid for $\Pi_1 \cup \Pi_2$ (see Figure 13). The disjunction $\pi x \le \pi_0$ or $\pi x \ge \pi_0 + 1$ is called a *split disjunction*.

Many of the inequalities that we have studied in this tutorial are split inequalities. Lift-and-project inequalities (i.e. the valid inequalities for $\cap_{j=1}^n P_j$) are split inequalities as a consequence of Theorem 5 (the split disjunction is $x_j \le 0$ or $x_j \ge 1$). Strengthened lift-and-project cuts are also split inequalities (recall the split disjunction $\sum_{k=1}^n \pi_k x_k \ge 0$ or $-\sum_{k=1}^n \pi_k x_k \ge 1$ used in the proof of Theorem 6). GMI inequalities are split inequalities (indeed the disjunction (7) or (8) is a split disjunction). $k$-cuts and reduce-and-split cuts are split inequalities since they are special types of GMI cuts. MIR inequalities are also split inequalities (remember the derivation in the proof of Lemma 3).

The intersection of all split inequalities, denoted by $P^1$, is called the *split closure* relative to $P$.

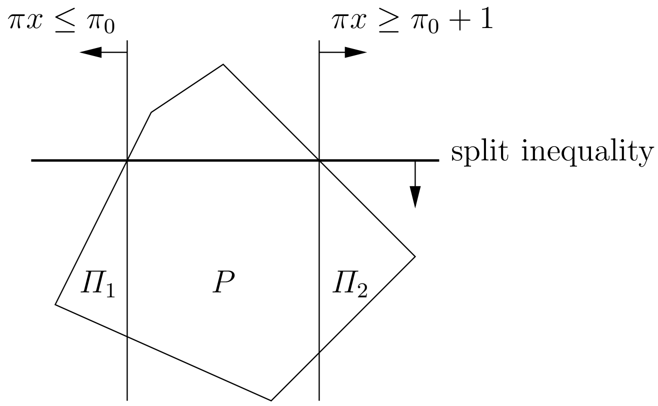


**Fig. 13.** A split inequality

**Theorem 9.** (Cook, Kannan and Schrijver [25]) *If $P$ is a rational polyhedron, the split closure relative to $P$ is a rational polyhedron.*

For $k \geq 2$, $P^k$ denotes the split closure relative to $P^{k-1}$ and it is called the $k$-th split closure relative to $P$. It follows from the above theorem that $P^k$ is a polyhedron. Unlike for the pure integer case [21], [52], there is in general no finite $r$ such that $P^r = \text{conv}(S)$ in the mixed integer case, as shown by the following example [25].

*Example 1.* Let $S := \{(x, y) \in \mathbb{Z}_+^2 \times \mathbb{R}_+ : \ x_1 \geq y, \ x_2 \geq y, \ x_1 + x_2 + 2y \leq 2\}$. Starting from $P := \{(x_1, x_2, y) \in \mathbb{R}_+^3 : \ x_1 \geq y, \ x_2 \geq y, \ x_1 + x_2 + 2y \leq 2\}$, we claim that there is no finite $r$ such that $P^r = \text{conv}(S)$.

To see this, note that $P$ is a simplex with vertices $O = (0, 0, 0)$, $A = (2, 0, 0)$, $B = (0, 2, 0)$ and $C = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (see Figure14). $S$ is contained in the plane $y = 0$. More generally, consider a simplex $P$ with vertices $O, A, B$ and $C = (\frac{1}{2}, \frac{1}{2}, t)$ with $t > 0$. Let $C_1 = C$, let $C_2$ be the point on the edge from $C$ to $A$ with coordinate $x_1 = 1$ and $C_3$ the point on the edge from $C$ to $B$ with coordinate $x_2 = 1$. Observe that no split disjunction removes all three points $C_1$, $C_2$, $C_3$. Let $Q_i$ be the intersection of all split inequalities that do not cut off $C_i$. All split inequalities belong to at least one of these three sets, thus $P^1 = Q_1 \cap Q_2 \cap Q_3$. Let $S_i$ be the simplex with vertices $O, A, B, C_i$. Clearly, $S_i \subseteq Q_i$. Thus $S_1 \cap S_2 \cap S_3 \subseteq P^1$. It is easy to verify that $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in S_i$ for $i = 1, 2$ and 3. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in P^1$. By induction, $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3^k}) \in P^k$.

*Remark 9.* For mixed 0,1 programs, Theorem 7 implies that $P^n = \text{conv}(S)$ (Indeed, the lift-and-project polytope $P_1$ contains the split closure relative to $P$ by Theorem 5. Similiarly, $P_2(P_1))$ contains the $2^{nd}$ split closure, etc).

*Example 2.* Cornuéjols and Li [27] observed that the $n$-th split closure is needed for 0,1 programs, i.e. there are examples where $P^k \neq \text{conv}(S)$ for all $k < n$. They

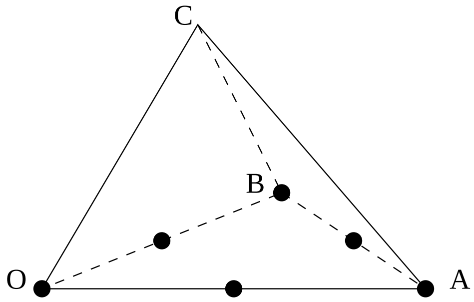


**Fig. 14.** Example showing that the split rank can be unbounded

use the following well-known polytope studied by Chvátal, Cook, and Hartmann [22]:

$$P_{CCH} := \{x \in [0,1]^n : \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \frac{1}{2}, \text{ for all } J \subseteq \{1, 2, \cdots, n\}\}$$

Let $F_j$ be the set of all vectors $x \in \mathbb{R}^n$ such that $j$ components of $x$ are $\frac{1}{2}$ and each of the remaining $n - j$ components are equal to 0 or 1. The polytope $P_{CCH}$ is the convex hull of $F_1$.

**Lemma 6.** *If a polyhedron $P \subseteq \mathbb{R}^n$ contains $F_j$, then its split closure $P^1$ contains $F_{j+1}$.*

*Proof.* It suffices to show that, for every $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the polyhedron $\Pi = \text{conv}((P \cap \{x : \pi x \leq \pi_0\}) \cup (P \cap \{x : \pi x \geq \pi_0 + 1\}))$ contains $F_{j+1}$. Let $v \in F_{j+1}$ and assume w.l.o.g. that the first $j + 1$ elements of $v$ are equal to $\frac{1}{2}$. If $\pi v \in \mathbb{Z}$, then clearly $v \in \Pi$. If $\pi v \notin \mathbb{Z}$, then at least one of the first $j + 1$ components of $\pi$ is nonzero. Assume w.l.o.g. that $\pi_1 > 0$. Let $v_1, v_2 \in F_j$ be equal to $v$ except for the first component which is 0 and 1 respectively. Notice that $v = \frac{v_1 + v_2}{2}$. Clearly, each of the intervals $[\pi v_1, \pi v]$ and $[\pi v, \pi v_2]$ contains an integer. Since $\pi x$ is a continuous function, there are points $\tilde{v}_1$ on the line segment $\text{conv}(v, v_1)$ and $\tilde{v}_2$ on the line segment $\text{conv}(v, v_2)$ with $\pi \tilde{v}_1 \in \mathbb{Z}$ and $\pi \tilde{v}_2 \in \mathbb{Z}$. This means that $\tilde{v}_1$ and $\tilde{v}_2$ are in $\Pi$. Since $v \in \text{conv}(\tilde{v}_1, \tilde{v}_2)$, this implies $v \in \Pi$. $\square$

Starting from $P = P_{CCH}$ and applying the lemma recursively, it follows that the $(n - 1)$-st split closure relative to $P_{CCH}$ contains $F_n$, which is nonempty. Since $\text{conv}(P_{CCH} \cap \{0, 1\}^n)$ is empty, the $n$-th split closure is needed to obtain $\text{conv}(P_{CCH} \cap \{0, 1\}^n)$. End of Example 2.

*Remark 10.* In view of Example 1 showing that no bound may exist on the split rank when the integer variables are general, and Remark 9 showing that the rank is always bounded when they are 0,1 valued, one is tempted to convert general integer variables into 0,1 variables. For a bounded integer variable $0 \leq x \leq u$, there are several natural tranformations:

(i) a binary expansion of $x$ (see Owen and Mehrotra [49]);

(ii) $x = \sum_{i=1}^{u} iz_i$, $\sum z_i \le 1$, $z_i \in \{0,1\}$ (see Sherali and Adams [55] and Köppe, Louveaux and Weismantel [38]);

(iii) $x = \sum_{i=1}^{u} z_i$, $z_i \le z_{i-1}$, $z_i \in \{0,1\}$ (see Roy [51]).

Roy [51] also shows that practical benefits can be gained from such a transformation.

### *4.7. Chvátal inequalities*

Let $P := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ :\ Ax + Gy \le b\}$ be a rational polyhedron and let $S := \{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ :\ Ax + Gy \le b\}$. Define a *Chvátal inequality* to be an inequality $\pi x \le \pi_0$ with $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, where

$$P \cap \{(x,y) :\ \pi x \ge \pi_0 + 1\} = \emptyset.$$

It follows from the definition that Chvátal inequalities are split inequalities where one of the sets $\Pi_1$ or $\Pi_2$ is empty.

**Lemma 7.** *Chvátal inequalities are GMI inequalities.*

*Proof.* The lemma holds when $P = \emptyset$. Assume now that $P \ne \emptyset$. Let

$$\beta = \max_{(x,y) \in P} \pi x$$

Then $\pi x \le \beta$ is a valid inequality for $P$. Since $P \cap \{(x,y) :\ \pi x \ge \pi_0 + 1\} = \emptyset$, it follows that $\beta < \pi_0 + 1$. Applying the MIR formula (13) to $\pi x \le \beta$, it follows that the inequality $\pi x \le \pi_0$ is a MIR inequality. By Lemma 4, this inequality is a GMI inequality. □

Define the *Chvátal closure* relative to $P$ as the intersection of $P$ with all the Chvátal inequalities. How does the Chvátal closure compare to the lift-and-project closure? The answer is that neither is included in the other in general, as shown by the following 2-variable examples: For $P := \{x \in \mathbb{R}^2_+ :\ x_2 \le 2x_1,\ 2x_1 + x_2 \le 2\}$ and $S := P \cap \{0,1\}^2$, the inequality $x_2 \le 0$ is a lift-and-project inequality but not a Chvátal inequality. On the other hand, for $P := \{x \in \mathbb{R}^2_+ :\ x_1 + x_2 \le 1.5,\ x_1 \le 1,\ x_2 \le 1\}$ and $S := P \cap \{0,1\}^2$, the inequality $x_1 + x_2 \le 1$ is a Chvátal inequality but not a lift-and-project inequality.

Although it is NP-hard to optimize over the Chvátal closure, there are empirical results on its strength. Bonami, Cornuéjols, Dash, Fischetti, Lodi [17] found that the Chvátal closure closes at least 29 % of the integrality gap on average on 41 MIPLIB instances (all the MIPLIB 3 instances that have at least one continuous variable and nonzero integrality gap). For the remaining 24 instances (pure integer programs in MIPLIB 3 with nonzero integrality gap), Fischetti and Lodi [31] found that the Chvátal closure closes at least 63 % of the integrality gap on average.

*4.8. Equivalence between split inequalities and GMI inequalities*

Nemhauser and Wolsey [46], [47] studied the relation between split closure and GMI closure. The equivalence proof given here is based on [27]. It uses Lemma 1.

**Theorem 10.** *Let* $P := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : \; Ax + Gy \leq b\}$ *be a rational polyhedron and let* $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$*. The split closure relative to* $P$ *is identical to the Gomory mixed integer closure relative to* $P$*.*

*Proof.* We may assume that the constraints $x \geq 0$ and $y \geq 0$ are part of $Ax + Gy \leq b$ in the description of $P$.

Consider first a GMI inequality. Its derivation in Section 4.1 was obtained by arguing that $k = \lfloor b \rfloor - \sum_{f_j \leq f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0} \lceil a_j \rceil x_j$ is an integer, and either $k \leq -1$ or $k \geq 0$. This is a disjunction of the form $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ with $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Thus the derivation of the GMI inequality implies that it is a split inequality.

Conversely, let $cx + hy \leq c_0$ be a split inequality. Let $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ denote the split disjunction used in deriving this inequality, where $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and let $\Pi_1$, $\Pi_2$ be the corresponding intersections with $P$.

First assume that $\Pi_2 = \emptyset$. Then the inequality $cx + hy \leq c_0$ is valid for $\Pi_1$. Since all the inequalities that define $\Pi_1$ are valid for $P$ except possibly for $\pi x \leq \pi_0$, it suffices to show that $\pi x \leq \pi_0$ is a GMI inequality. This follows from the fact that $\pi x \leq \pi_0$ is a Chvátal inequality (since $\Pi_2 := P \cap \{(x,y) : \; \pi x \geq \pi_0 + 1\} = \emptyset$) and from Lemma 7.

A similar argument holds when $\Pi_1 = \emptyset$, so we now assume that $\Pi_1 \neq \emptyset$ and $\Pi_2 \neq \emptyset$.

By Lemma 1, there exist $\alpha, \beta \in \mathbb{R}_+$ such that

$$cx + hy - \alpha(\pi x - \pi_0) \leq c_0 \text{ and} \tag{15}$$

$$cx + hy + \beta(\pi x - (\pi_0 + 1)) \leq c_0 \tag{16}$$

are both valid for $P$. We can assume $\alpha > 0$ and $\beta > 0$ since, otherwise, $cx + hy \leq c_0$ is valid for $P$ and therefore also for its Gomory mixed integer closure. We now apply the Gomory mixed integer procedure to (15) and (16). Introduce slack variables $s_1$ and $s_2$ in (15) and (16) respectively and subtract (15) from (16).

$$(\alpha + \beta)\pi x + s_2 - s_1 = (\alpha + \beta)\pi_0 + \beta.$$

Dividing by $\alpha + \beta$ we get

$$\pi x + \frac{s_2}{\alpha + \beta} - \frac{s_1}{\alpha + \beta} = \pi_0 + \frac{\beta}{\alpha + \beta}.$$

From this equation, we can derive a GMI inequality. Note that $f_0 = \frac{\beta}{\alpha+\beta}$ and that the continuous variable $s_2$ has a positive coefficient while $s_1$ has a negative coefficient. So the GMI inequality is

$$\frac{\frac{1}{\alpha+\beta}}{\frac{\beta}{\alpha+\beta}} s_2 + \frac{\frac{1}{\alpha+\beta}}{1 - \frac{\beta}{\alpha+\beta}} s_1 \geq 1$$

which simplifies to

$$\frac{1}{\alpha} s_1 + \frac{1}{\beta} s_2 \geq 1.$$

We now replace $s_1$ and $s_2$ as defined by the equations (15) and (16) to get the GMI inequality in the space of the $x, y$ variables. The resulting inequality is

$$cx + hy \leq c_0.$$

Therefore $cx + hy \leq c_0$ is a GMI inequality. □

The split closure is identical to the GMI closure. How tight is it in practice? Balas and Saxena [11] addressed this question by formulating the separation problem for the split closure as a parametric mixed integer linear program with a single parameter in the objective function and the right hand side. They found that the split closure closes 82 % of the integrality gap on average for 33 MIPLIB instances of mixed integer programs, and 71 % on average for 24 MIPLIB instances of pure integer programs. This experiment shows that the split closure is surprisingly strong. It is interesting to compare the above 82 % figure with the 48 % obtained by adding GMI cuts from the optimal tableau + MIR cuts + lift-and-project cuts [17]. This is a clear indication that there are useful split cuts that are currently not exploited in integer programming codes. Finding deep split cuts efficiently remains a challenging practical issue.

### *4.9. Intersection cuts*

Intersection cuts were introduced by Balas [4]. They are obtained from a basis of the linear programming relaxation and a convex set that contains the corresponding basic solution but no integer feasible solution in its interior. In this section, we focus on convex sets derived from split disjunctions. For convenience, we assume that the constraints are in equality form. Let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$ with

$$P := \{x \in \mathbb{R}_+^{n+p} : \ Ax = b\}$$

where $A \in \mathbb{R}^{m\times(n+p)}$ and $b \in \mathbb{R}^m$. Wlog assume that $A$ is of full row rank. Let $B$ index $m$ linearly independent columns of $A$ ($B$ is a *basis*) and let $J$ index the remaining columns of $A$ (the nonbasic variables). Let $P(B)$ be the relaxation of $P$ obtained by deleting the nonnegativity constraints on the basic variables:

$$P(B) := \{x \in \mathbb{R}^{n+p} : Ax = b \text{ and } x_j \geq 0 \text{ for } j \in J\}. \tag{17}$$

The convex hull of $P(B) \cap (\mathbb{Z}^n \times \mathbb{R}^p)$ was studied by Gomory [35] under the name of *corner polyhedron*. Clearly, $S$ is contained in the corner polyhedron and, therefore, valid inequalities for the corner polyhedron are also valid for $S$.

We may write $P(B) = \bar{x} + C$, where $C$ is the polyhedral cone $C := \{x \in \mathbb{R}^{n+p} : Ax = 0 \text{ and } x_j \geq 0 \text{ for } j \in J\}$, and $\bar{x}$ is the basic solution that solves the system $Ax = b$ and $x_j = 0$ for $j \in J$:

$$x_i = \bar{x}_i - \sum_{j \in J} \bar{a}_{ij} x_j, \qquad i \in B. \tag{18}$$

The extreme rays of $C$ are the following vectors $r^j$ for $j \in J$:

$$r^j_k := \begin{cases} -\bar{a}_{kj} & \text{if } k \in B, \\ 1 & \text{if } k = j, \\ 0 & \text{if } k \in J \setminus \{j\}. \end{cases} \tag{19}$$

Thus $P(B) = \bar{x} + \text{cone}(\{r^j\}_{j \in J})$, where $\text{cone}(\{r^j\}_{j \in J})$ denotes the polyhedral cone generated by the vectors $\{r^j\}_{j \in J}$.

We now derive the intersection cut. Consider an arbitrary split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$, where $\pi \in \mathbb{Z}^n \times \{0\}^p$ and $\pi_0 \in \mathbb{Z}$. Assume $\bar{x}$ violates the split disjunction, and define $\epsilon := \pi\bar{x} - \pi_0$ to be the amount by which $\bar{x}$ violates the first term of the disjunction. Since $\pi_0 < \pi\bar{x} < \pi_0 + 1$, we have $0 < \epsilon < 1$. Also, for $j \in J$, define scalars:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi r^j} & \text{if } \pi r^j < 0, \\ \frac{1-\epsilon}{\pi r^j} & \text{if } \pi r^j > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{20}$$

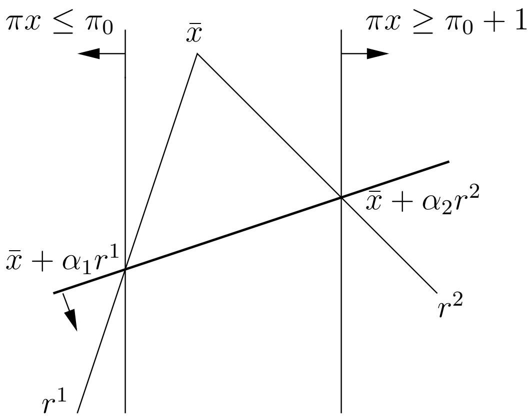


**Fig. 15.** Intersection cut

The interpretation of $\alpha_j$ is the following. Consider the half-line $\bar{x} + \alpha r^j$, where $\alpha \in \mathbb{R}_+$, starting from $\bar{x}$ in the direction $r^j$. The value $\alpha_j$ is the smallest

$\alpha \in \mathbb{R}_+$ such that $\bar{x}+\alpha r^j$ satisfies the split disjunction. In other words, the point $\bar{x}+\alpha_j r^j$ is the intersection of the half-line starting in $\bar{x}$ in direction $r^j$ with the hyperplane $\pi x = \pi_0$ or the hyperplane $\pi x = \pi_0 + 1$ (see Figure 15). Note that $\alpha_j = +\infty$ when the direction $r^j$ is parallel to the hyperplane $\pi x = \pi_0$. Given the numbers $\alpha_j$ for $j \in J$, the intersection cut associated with $B$ and the split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$, is given by:

$$\sum_{j \in J} \frac{x_j}{\alpha_j} \geq 1. \tag{21}$$

This inequality is valid for the corner polyhedron $\mathrm{conv}\big(P(B) \cap (\mathbb{Z}^n \times \mathbb{R}^p)\big)$ since it is a split inequality. In fact, the intersection cut gives a complete description of the set of points in $P(B)$ that satisfy the split disjunction:

$$\mathrm{conv}\big(P(B) \cap (\{x:\ \pi x \leq \pi_0\} \cup \{x:\ \pi x \geq \pi_0 + 1\})\big) = P(B) \cap \{x : \sum_{j \in J} \frac{x_j}{\alpha_j} \geq 1\}.$$

Andersen, Cornuéjols and Li [1] showed that intersection cuts are sufficient to describe the split closure $P^1$ relative to $P$. Let $\mathcal{B}$ denote the set of all bases of $A$. We have:

**Theorem 11.**

$$P^1 = \bigcap_{B \in \mathcal{B}} \ \bigcap_{\pi \in \mathbb{Z}^n \times \{0\}^p,\ \pi_0 \in \mathbb{Z}} conv\big(P(B) \cap (\{x:\ \pi x \leq \pi_0\} \cup \{x:\ \pi x \geq \pi_0 + 1\})\big).$$

The following lemma shows that the GMI cuts derived from rows of the simplex tableau are intersection cuts:

**Lemma 8.** *Let $B$ be a basis of $A$, and let $\bar{x}$ be the corresponding basic solution. Also, let $x_i$ be a basic integer constrained variable, and suppose $\bar{x}_i$ is fractional. The GMI cut obtained from the row of the simplex tableau in which $x_i$ is basic is given by the inequality $\sum_{j \in J} \frac{x_j}{\alpha_j} \geq 1$, where $\alpha_j$ is computed using formula (20) with $\pi_0 := \lfloor \bar{x}_i \rfloor$, and for $j = 1, \ldots, n$:*

$$\pi_j := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in J \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in J \text{ and } f_j > f_0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

*Proof.* Let us compute $\alpha_j$ for the above disjunction using formula (20), where $j \in J$. We have:

$$\epsilon = \pi\bar{x} - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_0.$$

Using (19) and (22), we get

$$\pi r^j = \pi_i r_i^j - \pi_j r_j^j = \begin{cases} -f_j & \text{if } 1 \leq j \leq n \text{ and } f_j \leq f_0, \\ 1 - f_j & \text{if } 1 \leq j \leq n \text{ and } f_j > f_0, \\ -\bar{a}_{ij} & \text{if } n+1 \leq j \leq n+p. \end{cases} \tag{23}$$

Now $\alpha_j$ follows from formula (20). This yields the GMI cut as claimed. □

## 5. Conclusion

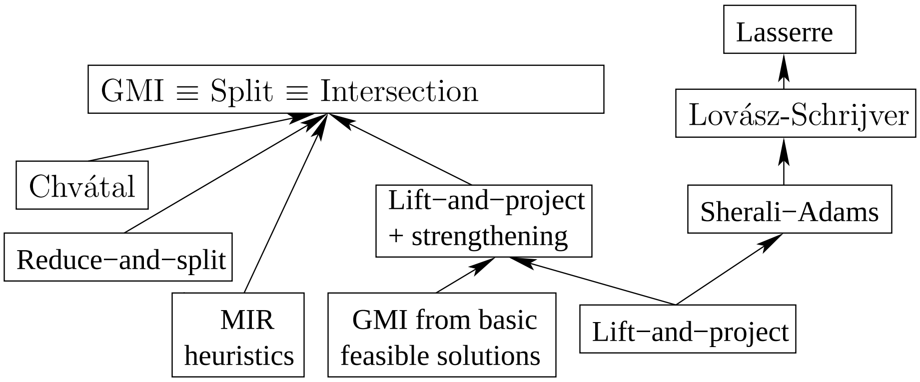


**Fig. 16.** Relations between families of cuts

This tutorial considered mixed integer linear sets $S := P \cap (\mathbb{Z}^n_+ \times \mathbb{R}^p_+)$ where $P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$ is a polyhedron. It presented several families of valid inequalities for $S$ and compared the corresponding elementary closures. See Figure 16. The Gomory mixed integer (GMI) closure is identical to the split closure. This set can also be obtained using intersection cuts generated from all the basic solutions (both feasible and infeasible). Computational experiments show that the split closure is surprisingly tight (it closes over 75 % of the integrality gap on average on MIPLIB 3 instances). Unfortunately separating or optimizing over the split closure is NP-hard. This justifies our interest in families of split cuts that can be generated efficiently: GMI cuts from the optimal basis, MIR cuts obtained heuristically, reduce-and-split cuts and lift-and-project cuts. They appear in the bottom of the figure. On the right hand side of Figure 16 appear elementary closures that are defined for mixed 0,1 programs only, from weakest at the bottom to tightest at the top.

Integer programming solvers went from using no general cutting planes twenty years ago to using many rounds of such cuts nowadays. Much larger instances can be solved, but this aggressive cutting may sometimes result in the optimum solution being cut off due to numerical difficulties with cuts from later rounds. Have we reached the limits of what can be acheived with general cuts? I do not think so. The split closure is surprisingly tight and integer programming solvers should probably do a better job of approximating it before generating cuts of higher rank.

We conclude with three possible research directions. The previous paragraph suggests that one should look for new classes of deep split cuts that can be separated rapidly, or just use the currently known families more efficiently. Variable transformations are intriguing (see Remark 10) and deserve more attention in the context of cutting planes. Obtaining compact higher dimensional formulations by applying Theorem 4 is another promising direction for research.

**Aknowledgements:** This tutorial benefited from PhD students' feedback in short courses in Lausanne (2004), Lisbon (2005) and Zinal (2006) and from discussions with colleagues. I am most grateful to Egon Balas for his help with Theorem 4 and Section 4.3. Thanks also to Claude Lemarechal, Laurence Wolsey and the referees for their useful comments on an earlier draft.

## References

1. K. Andersen, G. Cornuéjols and Y. Li, Split closure and intersection cuts, *Mathematical Programming A 102* (2005) 457–493.
2. K. Andersen, G. Cornuéjols and Y. Li, Reduce-and-split cuts: Improving the performance of mixed integer Gomory cuts, *Management Science 51* (2005) 1720–1732.
3. A. Atamtürk, Strong formulations of robust mixed 0–1 programming, to appear in *Mathematical Programming* (2006).
4. E. Balas, Intersection cuts – A new type of cutting planes for integer programming, *Operations Research 19* (1971) 19–39.
5. E. Balas, Disjunctive programming: properties of the convex hull of feasible points, GSIA Management Science Research Report MSRR 348, Carnegie Mellon University (1974), published as invited paper in *Discrete Applied Mathematics 89* (1998) 1–44.
6. E. Balas, Disjunctive programming and a hierarchy of relaxations for discrete optimization problems, *SIAM Journal on Algebraic and Discrete Methods 6* (1985) 466–486.
7. E. Balas, S. Ceria and G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed 0-1 programs, *Mathematical Programming 58* (1993) 295–324.
8. E. Balas, S. Ceria, G. Cornuéjols and R.N. Natraj, Gomory cuts revisited, *Operations Research Letters 19* (1996) 1–9.
9. E. Balas and R. Jeroslow, Strengthening cuts for mixed integer programs, *European Journal of Operations Research 4* (1980) 224–234.
10. E. Balas and M. Perregaard, A Precise correspondence between lift-and-project cuts, simple disjunctive cuts and mixed integer Gomory cuts for 0-1 programming, *Mathematical Programming B 94* (2003) 221–245.
11. E. Balas and A. Saxena, Optimizing over the split closure, Tepper School of Business Management Science Research Report MSRR 674, Carnegie Mellon University (2006).
12. D. Bienstock and M. Zuckerberg, Subset algebra lift operators for 0-1 integer programming, *SIAM Journal on Optimization 15* (2004) 63–95.
13. R.E. Bixby, S. Ceria, C.M. McZeal and M.W.P. Savelsbergh, An updated mixed integer programming library: MIPLIB 3.0, *Optima 58* (1998) 12–15.
14. R.E. Bixby, Z. Gu, E. Rothberg and R. Wunderling, Mixed integer programming: A progress report, M. Grötschel ed., The sharpest cut: The impact of Manfred Padberg and his work, *MPS/SIAM Series in Optimization* (2004) 309–326.
15. P. Bonami, Etude et mise en oeuvre d'approches polyédriques pour la résolution de programmes en nombres entiers ou mixtes généraux, PhD Thesis, Université de Paris 6 (2003).
16. P. Bonami and G. Cornuéjols, On MIR inequalities (2006).
17. P. Bonami, G. Cornuéjols, S. Dash, M. Fischetti and A. Lodi, Projected Chvátal-Gomory cuts for mixed integer linear programs, technical report (2005).
18. P. Bonami and M. Minoux, Using rank-1 lift-and-project closures to generate cuts for 0–1 MIPs, a computational investigation, *Discrete Optimization 2* (2005) 288–307.
19. S. Burer and D. Vandenbussche, Solving lift-and-project relaxations of binary integer programs, *SIAM Journal on Optimization 16* (2006) 726–750.
20. A. Caprara and A.N. Letchford, On the separation of split cuts and related inequalities, *Mathematical Programming B 94* (2003) 279–294.
21. V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial optimization, *Discrete Mathematics 4* (1973) 305–337.
22. V. Chvátal, W. Cook, and M. Hartmann, On cutting-plane proofs in combinatorial optimization, *Linear Algebra and its Applications 114/115* (1989) 455–499.
23. M. Conforti and L. Wolsey, Compact formulations as a union of polyhedra, CORE discussion paper, Louvain-la-Neuve, Belgium (2005).
24. W. Cook and S. Dash, On the matrix-cut rank of polyhedra, *Mathematics of Operations Research 26* (2001) 19–30.

25. W. Cook, R. Kannan and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming 47* (1990) 155–174.
26. G. Cornuéjols and Y. Li, Elementary closures for integer programs, *Operations Research Letters 28* (2001) 1–8.
27. G. Cornuéjols and Y. Li, On the rank of mixed 0,1 polyhedra, *Mathematical Programming A 91* (2002) 391–397.
28. G. Cornuéjols and Y. Li, A connection between cutting plane theory and the geometry of numbers, *Mathematical Programming A 93* (2002) 123–127.
29. S. Dash, O. Günlük and A. Lodi, Separating from the MIR closure of polyhedra, presentation at the International Symposium on Mathematical Programming, Rio de Janeiro, Brazil, 2006, paper in preparation.
30. F. Eisenbrand, On the membership problem for the elementary closure of a polyhedron, *Combinatorica 19* (1999) 297–300.
31. M. Fischetti and A. Lodi, Optimizing over the first Chvátal closure, in M. Jünger and V. Kaibel eds., *IPCO 2005, LNCS 3509* (2005) 12–22.
32. M.X. Goemans and L. Tunçel, When does the positive semidefiniteness constraint help in lifting procedures, *Mathematics of Operations Research 26* (2001) 796–815.
33. M.X. Goemans and D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *Journal of the ACM 42* (1995) 1115–1145.
34. R.E. Gomory, An algorithm for integer solutions to linear programs, *Recent Advances in Mathematical Programming*, R.L. Graves and P. Wolfe eds., McGraw-Hill, New York (1963) 269–302.
35. R.G. Gomory, Some polyhedra related to combinatorial problems, *Linear Algebra and Applications 2* (1969) 451–558.
36. R.E. Gomory and E.L. Johnson, Some continuous functions related to corner polyhedra I, *Mathematical Programming 3* (1972) 23–85.
37. O. Günlük and Y. Pochet, Mixing mixed-integer inequalities, *Mathematical Programming 90* (2001) 429–458.
38. M. Köppe, Q. Louveaux and R. Weismantel, Intermediate integer programming representations using value disjunctions, technical report, Department of Mathematics, University of Magdeburg (2005).
39. M. Köppe and R. Weismantel, A mixed-integer Farkas lemma and some consequences, *Operations Research Letters 32* (2004) 207–211.
40. J.B. Lasserre, An explicit exact SDP relaxation for nonlinear 0-1 program, *Lecture Notes in Computer Science 2081* (2001) 293–303.
41. M. Laurent, A comparison of the Sherali-Adams, Lovász-Schrijver and Lasserre relaxation for 0-1 programming, *SIAM Journal on Optimization 28* (2003) 345–375.
42. A.K. Lenstra, H.W. Lenstra and L. Lovász, Factoring polynomials with rational coefficients, *Mathematische Annalen 261* (1982) 515–534.
43. L. Lovász and A. Schrijver, Cones of matrices and set-functions and 0-1 optimization, *SIAM Journal of Optimization 1* (1991) 166–190.
44. H. Marchand and L.A. Wolsey, Aggregation and mixed integer rounding to solve MIPs, *Operations Research 49* (2001) 363–371.
45. A. Miller and L.A. Wolsey, Tight formulations for some simple MIPs and convex objective IPs, *Mathematical Programming B 98* (2003) 73–88.
46. G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York (1988).
47. G.L. Nemhauser and L.A. Wolsey, A recursive procedure to generate all cuts for 0-1 mixed integer programs, *Mathematical Programming 46* (1990) 379–390.
48. Y.E. Nesterov and A.S. Nemirovski, *Interior Point Polynomial Algorithms in Convex Programming*, SIAM Publications, SIAM, Philadelphia (1994).
49. J.H. Owen and S. Mehrotra, On the value of binary expansions for general mixed-integer linear programs, *Operations Research 50* (2002) 810–819.
50. M. Perregaard, A practical implementation of lift-and-project cuts: A computational exploration of lift-and-project cuts with XPRESS-MP, 18th ISMP, Copenhagen (2003).
51. J.-S. Roy, "Binarize and project" to generate cuts for general mixed-integer programs, technical report, EDF R&D, Clamart, France (2006).
52. A. Schrijver, On cutting planes, *Annals of Discrete Mathematics 9* (1980) 291–296.
53. A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York (1986).

54. H. Sherali and W. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM Journal on Discrete Mathematics 3* (1990) 311–430.
55. H. Sherali and W. Adams, A reformulation-linearization technique for solving discrete and continuous nonconvex problems, Chapter 4, Kluwer Academic Publishers, Norwell, MA (1999).
56. M. Van Vyve, The continuous mixing polyhedron, *Mathematics of Operations Research 30* (2005) 441–452.
57. L.A. Wolsey, *Integer Programming* (1999).
58. G.M. Ziegler, *Lectures on Polytopes*, Springer, New York (1995).