Ideal Clutters

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July 1999, revised September 2001

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This work was supported in part by NSF grants DMI-0098427, DMI-9802773, DMS-9509581, DMS 96-32032 and ONR grant N00014-97-1-0196

Abstract

The Operations Research model known as the Set Covering Problem has a wide range of applications. See for example the survey by Ceria, Nobili and Sassano in *Annotated Bibliographies in Combinatorial Optimization* edited by Dell'Amico, Maffioli and Martello [16]. Sometimes, due to the special structure of the constraint matrix, the natural linear programming relaxation yields an optimal solution that is integer, thus solving the problem. Under which conditions do such integrality properties hold? This question is of both theoretical and practical interest. On the theoretical side, polyhedral combinatorics and graph theory come together in this rich area of discrete mathematics. In this tutorial, we present the state of the art and open problems on this question.

Keywords: Ideal clutter, ideal matrix, set covering, integer polyhedron, width-length inequality, Max Flow Min Cut property.

1 Introduction

A *clutter* $\mathcal{C}$ is a family $E(\mathcal{C})$ of subsets of a finite ground set $V(\mathcal{C})$ with the property that $A_1 \not\subseteq A_2$ for all distinct $A_1, A_2 \in E(\mathcal{C})$. $V(\mathcal{C})$ denotes the set of *vertices* and $E(\mathcal{C})$ the set of *edges* of $\mathcal{C}$. A clutter is *ideal* if $\{x \geq \mathbf{0} : \ x(A) \geq \mathbf{1} \ \text{ for all } A \in E(\mathcal{C})\}$ is an *integral polyhedron*, i.e. all its extreme points have 0,1 coordinates. Here $x(A)$ denotes $\sum_{i \in A} x_i$. This concept is also known under the name of *width-length property*, *weak Max Flow Min Cut property* or *$\mathcal{Q}_+$-MFMC property*. We prefer the term "ideal" because it stresses the parallel with "perfection".

A clutter is *trivial* if it has no edge or if it has the empty set as unique edge. Given a nontrivial clutter $\mathcal{C}$, we define $M(\mathcal{C})$ to be a 0,1 matrix whose columns are indexed by $V(\mathcal{C})$, whose rows are indexed by $E(\mathcal{C})$ and where $M_{ij} = 1$ if and only if the vertex corresponding to column j belongs to the edge corresponding to row i. In other words, the rows of $M(\mathcal{C})$ are the characteristic vectors of the sets in $E(\mathcal{C})$. Note that the definition of $M(\mathcal{C})$ is unique up to permutation of rows and permutation of columns. $M(\mathcal{C})$ contains no dominating row, since $\mathcal{C}$ is a clutter (A vector $r \in F$ is said to be *dominating* if there exists $v \in F$ distinct from r such that $r \geq v$). A 0,1 matrix M containing no dominating rows is called a *clutter matrix.* Given any 0,1 clutter matrix M, we denote by $\mathcal{C}(M)$ the unique clutter for which $M(\mathcal{C}(M)) = M$. The 0,1 matrix M is *ideal* if the clutter $\mathcal{C}(M)$ is ideal. Clearly, $\mathcal{C}(M)$ is ideal if and only if $\{x \geq \mathbf{0} : Mx \geq \mathbf{1}\}$ is an integral polyhedron. In this tutorial we present the state of the art and open problems on ideal clutters and matrices. Parts of the tutorial overlap with [10].

1.1 Blockers

A *transversal* of a clutter $\mathcal{C}$ is a set of vertices that intersects all the edges. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is the clutter with $V(\mathcal{C})$ as vertex set and the minimal transversals of $\mathcal{C}$ as edge set. That is, $E(b(\mathcal{C}))$ consists of the minimal members of $\{B \subseteq V(\mathcal{C}) : |B \cap A| \geq 1 \text{ for all } A \in E(\mathcal{C})\}$. In other words, the rows of $M(b(\mathcal{C}))$ are the minimal 0,1 vectors x^T such that x belongs to the polyhedron $P(\mathcal{C}) = \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$.

Example 1.1 *Let G be a graph and s, t be distinct nodes of G. If $\mathcal{C}$ is the clutter of st-paths, then $b(\mathcal{C})$ is the clutter of minimal st-cuts.*

Edmonds and Fulkerson [19] observed that $b(b(\mathcal{C})) = \mathcal{C}$. Before proving this property, we make the following remark.

Remark 1.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be two clutters defined on the same vertex set. If*
(i) every edge of $\mathcal{H}$ contains an edge of $\mathcal{K}$ and
(ii) every edge of $\mathcal{K}$ contains an edge of $\mathcal{H}$,
then $\mathcal{H} = \mathcal{K}$.

Theorem 1.3 *If $\mathcal{C}$ is a clutter, then $b(b(\mathcal{C})) = \mathcal{C}$.*

Proof: Let A be an edge of $\mathcal{C}$. The definition of $b(\mathcal{C})$ implies that $|A \cap B| \geq 1$, for every edge B of $b(\mathcal{C})$. So A is a transversal of $b(\mathcal{C})$, i.e. A contains an edge of $b(b(\mathcal{C}))$.

Now let A be an edge of $b(b(\mathcal{C}))$. We claim that A contains an edge of $\mathcal{C}$. Suppose otherwise. Then $V(\mathcal{C}) - A$ is a transversal of $\mathcal{C}$ and therefore it contains an edge B of $b(\mathcal{C})$. But then $A \cap B = \emptyset$ contradicts the fact that A is an edge of $b(b(\mathcal{C}))$. So the claim holds.

Now the theorem follows from Remark 1.2. □

Two 0,1 matrices of the form $M(\mathcal{C})$ and $M(b(\mathcal{C}))$ are said to form a *blocking pair*. The next theorem is an important result due to Lehman [35]. It states that, for a blocking pair A, B of 0,1 matrices, the polyhedron P defined by

$$Ax \geq \mathbf{1} \tag{1}$$
$$x \geq 0 \tag{2}$$

is integral if and only if the polyhedron Q defined by

$$Bx \geq \mathbf{1} \tag{3}$$
$$x \geq 0 \tag{4}$$

is integral. The proof of this result uses the following remark.

Remark 1.4

(i) The rows of B are exactly the 0,1 extreme points of P.

(ii) If an extreme point x of P satisfies $x^T \geq \lambda^T B$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then x is a 0,1 extreme point of P.

Proof: (i) follows from the fact that the rows of B are the minimal 0,1 vectors in P.

To prove (ii), note that x is an extreme point of $P_I = \{\chi : \chi^T \geq \lambda^T B$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1\}$ for otherwise x would be a convex combination of distinct $x^1, x^2 \in P_I$ and, since $P_I \subseteq P$, this would contradict the assumption that x is an extreme point of P. Now (ii) follows by observing that the extreme points of P_I are exactly the rows of B. $\square$

Theorem 1.5 (Lehman [35]) *A clutter is ideal if and only if its blocker is.*

Proof: By Theorem 1.3, it suffices to show that if P defined by (1)-(2) is integral, then Q defined by (3)-(4) is also integral.

Let a be an arbitrary extreme point of Q. By (3), $Ba \geq \mathbf{1}$, i.e. $a^T x \geq 1$ is satisfied by every x such that x^T is a row of B. Since P is an integral polyhedron, it follows from Remark 1.4(i) that $a^T x \geq 1$ is satisfied by all the extreme points of P. By (4), $a \geq 0$. Therefore $a^T x \geq 1$ is satisfied by all points in P. Furthermore, $a^T x = 1$ for some $x \in P$. Now, by linear programming duality, we have

$$1 \quad = \quad \min\{a^T x : \ x \in P\} \quad = \quad \max\{\lambda^T \mathbf{1} : \ \lambda^T A \leq a^T, \ \lambda \geq 0\}.$$

Therefore, by Remark 1.4(ii), a is a 0,1 extreme point of Q. $\square$

1.2 Related Concepts

Let $M \neq 0$ be a 0,1 clutter matrix and consider the following pair of dual linear programs.

$$min\{wx : x \geq \mathbf{0}, \ Mx \geq \mathbf{1}\} \tag{5}$$
$$= max\{y\mathbf{1} : y \geq \mathbf{0}, \ yM \leq w\} \tag{6}$$

The clutter $\mathcal{C}(M)$ is ideal if (5) has an optimal solution vector x that is integral for all $w \geq 0$. Next, we consider concepts that involve integrality in both the primal and the dual problems.

Definition 1.6 *The clutter $\mathcal{C}(M)$* packs *if both (5) and (6) have optimal solution vectors x and y that are integral when $w = \mathbf{1}$.*

Definition 1.7 *The clutter $\mathcal{C}(M)$ has the* packing property *if both (5) and (6) have optimal solution vectors x and y that are integral for all vectors w with components equal to $0, 1$ or $+\infty$.*

Definition 1.8 *The clutter $\mathcal{C}(M)$ has the* Max Flow Min Cut *property (or MFMC property) if both (5) and (6) have optimal solution vectors x and y that are integral for all nonnegative integral vectors w.*

Clearly, the MFMC property for a clutter implies the packing property which itself implies that the clutter packs. Conforti and Cornuéjols [6] conjectured that, in fact, the MFMC property and the packing property are identical. This conjecture is still open.

Conjecture 1.9 *A clutter has the MFMC property if and only if it has the packing property.*

Clearly, the MFMC property implies idealness. In fact, the packing property implies idealness.

Theorem 1.10 *If a clutter has the packing property, then it is ideal.*

This follows from a result of Lehman [36] that we will prove in Section 4.

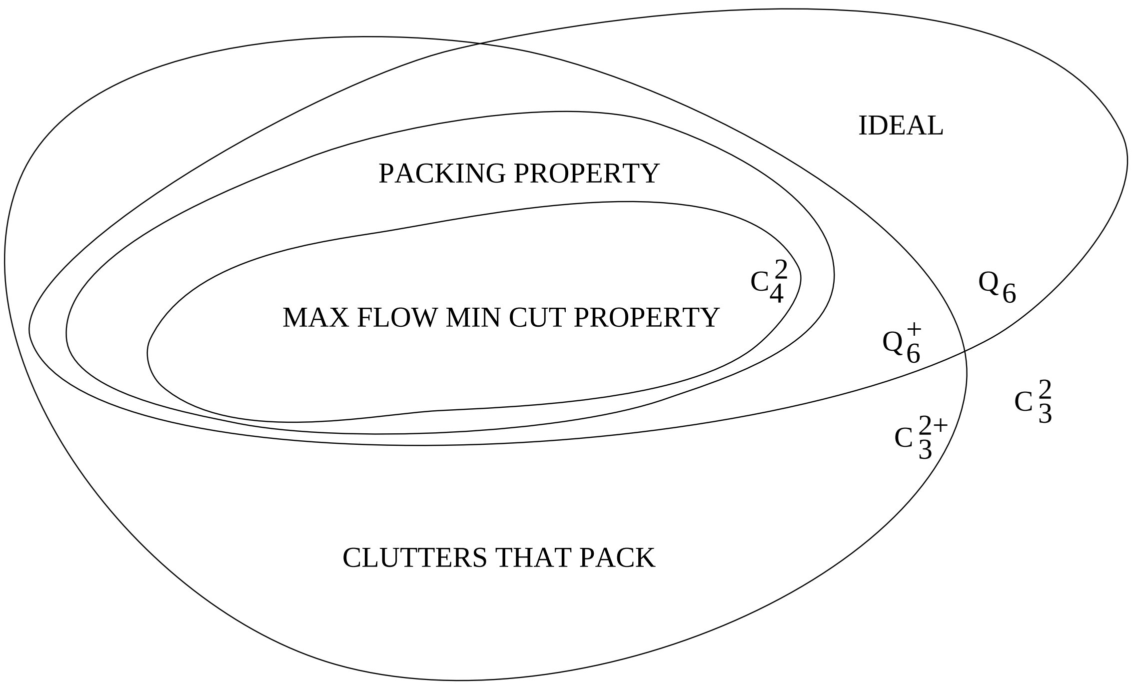

Figure 1: Classes of clutters.

A linear system $Ax \geq b$ is *Total Dual Integral* (TDI) if the linear program *min wx* subject to $Ax \geq b$ has an integral optimal dual solution y for every integral w for which the linear program has a finite optimum. Edmonds and Giles [20] proved that, if $Ax \geq b$ is TDI and b is integral, then $P = \{x : Ax \geq b\}$ is an integral polyhedron. The proof of the Edmonds-Giles theorem can be found in Schrijver [52], pages 310-311, or Nemhauser and Wolsey [40], pages 536-537. It follows that $\mathcal{C}(M)$ has the MFMC property if and only if (6) has an optimal integral solution y for all nonnegative integral vectors w.

Definition 1.11 *Let k be a positive integer. The clutter $\mathcal{C}(M)$ has the* $1/k$ - MFMC property *if it is ideal and, for all nonnegative integral vectors w, the linear program (6) has an optimal solution vector y such that ky is integral.*

When $k = 1$, this definition reduces to that of the MFMC property. If $\mathcal{C}(M)$ has the $1/k$-MFMC property, then it also has the $1/q$-MFMC property for every integer q that is a multiple of k.

Example 1.12 *Let $V(\mathcal{C})$ be the set of edges of K_4 and let $E(\mathcal{C})$ be the set of triangles of K_4. The reader can verify that $\mathcal{C}$ is ideal, does not have the MFMC property and, in fact, does not pack. Whereas $b(\mathcal{C})$ is ideal, packs and, in fact, has the MFMC property.*

1.3 Deletion, Contraction and Minor

Let $\mathcal{C}$ be a clutter. For $j \in V(\mathcal{C})$, the *contraction* $\mathcal{C}/j$ and *deletion* $\mathcal{C} \setminus j$ are clutters defined as follows: both have $V(\mathcal{C}) - \{j\}$ as vertex set, $E(\mathcal{C}/j)$ is the set of minimal members in $\{S - \{j\} : S \in E(\mathcal{C})\}$ and $E(\mathcal{C} \setminus j) = \{S : j \notin S \in E(\mathcal{C})\}$.

Contractions and deletions of distinct vertices can be performed sequentially, and it is easy to show that the result does not depend on the order.

Proposition 1.13 *For a clutter $\mathcal{C}$ and distinct vertices j_1, j_2,*

(i) $(\mathcal{C}\setminus j_1)\setminus j_2 = (\mathcal{C}\setminus j_2)\setminus j_1$

(ii) $(\mathcal{C}/j_1)/j_2 = (\mathcal{C}/j_2)/j_1$

(iii) $(\mathcal{C}\setminus j_1)/j_2 = (\mathcal{C}/j_2)\setminus j_1$

Proof: Use the definitions of contraction and deletion! □

Definition 1.14 *A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by a sequence of deletions and contractions is a* minor *of $\mathcal{C}$.*

If V_1 and V_2 are disjoint subsets of $V(\mathcal{C})$, we let $\mathcal{C}/V_1\setminus V_2$ be the minor obtained from $\mathcal{C}$ by contracting all vertices of V_1 and deleting all vertices of V_2. If $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$, the minor is *proper*.

Proposition 1.15 *For a clutter $\mathcal{C}$ and $U \subset V(\mathcal{C})$,* *(i)* $b(\mathcal{C}\backslash U) = b(\mathcal{C})/U$

(ii) $b(\mathcal{C}/U) = b(\mathcal{C})\backslash U$

Proof: Use the definitions of contraction, deletion and blocker! $\square$

We leave it as an exercise to prove the following result.

Proposition 1.16 *If a clutter is ideal, then so are all its minors.*

Contracting $j \in V(\mathcal{C})$ corresponds to setting $x_j = 0$ in the set covering constraints $Mx \geq \mathbf{1}$ of (5) since column j is removed from M as well as the resulting dominating rows. Deleting j corresponds to setting $x_j = 1$ since column j is removed from M as well as all the rows with a 1 in column j.

Corollary 1.17 *Let M be a 0,1 matrix. The following are equivalent.*

- *The polyhedron $\{x \geq \mathbf{0},\ Mx \geq \mathbf{1}\}$ is integral.*
- *The polytope $\{0 \leq x \leq \mathbf{1},\ Mx \geq \mathbf{1}\}$ is integral.*

2 st-cuts and st-paths

Consider a digraph (N, A) with $s, t \in N$. Let $\mathcal{C}$ be the clutter where $V(\mathcal{C}) = A$ and where $E(\mathcal{C})$ is the family of st-paths.

Theorem 2.1 (Ford and Fulkerson [22]) *The clutter $\mathcal{C}$ has the MFMC property.*

This theorem is a restatement of the famous Max Flow Min Cut theorem of Ford-Fulkerson: for any nonnegative integral arc capacities w, the minimum capacity of an st-cut equals the maximum number of st-paths such that every arc $a \in A$ belongs to at most w_a of the paths. Indeed, the Ford-Fulkerson theorem states that both (5) and (6) have optimal solutions that are integral.

Theorem 2.1 implies that $\mathcal{C}$ is ideal and therefore the polyhedron

$$\{x \in \Re^A_+ : x(P) \geq 1 \text{ for all } st\text{-paths } P\}$$

is integral. Its extreme points are the minimal st-cuts. In the remainder, it will be convenient to refer to minimal st-cuts simply as st-cuts.

As a consequence of Lehman's theorem (Theorem 1.5), the clutter of st-cuts is also ideal. So the polyhedron

$$\{x \in \Re^A_+ : x(C) \geq 1 \text{ for all } st\text{-cuts } C\}$$

is integral. In fact, it is easy to show that the clutter of st-cuts has the MFMC property.

2.1 The Width-Length Inequality

In a network, the product of the minimum number of edges in an st-path by the minimum number of edges in an st-cut is at most equal to the total number of edges in the network. This width-length inequality can be generalized to any nonnegative edge lengths ℓ_e and widths w_e: the minimum length of an st-path times the minimum width of an st-cut is at most equal to the scalar product $\ell^T w$. This width-length inequality was observed by Moore and Shannon [39] and Duffin [18]. A length and a width can be defined for any clutter and its blocker. Interestingly, Lehman [35] showed that the width-length inequality can be used as a characterization of idealness.

Theorem 2.2 (Width-length inequality, Lehman [35]) *For a clutter $\mathcal{C}$ and its blocker $b(\mathcal{C})$, the following statements are equivalent.*

- *$\mathcal{C}$ and $b(\mathcal{C})$ are ideal;*
- $\min\{w(C) :\ C \in E(\mathcal{C})\} \times \min\{\ell(D) :\ D \in E(b(\mathcal{C}))\} \ \leq\ w^T\ell$ *for all* $\ell, w \in \Re^n_+$.

Proof: Let $A = M(\mathcal{C})$ and $B = M(b(\mathcal{C}))$ be the blocking pair of 0,1 matrices associated with $\mathcal{C}$ and $b(\mathcal{C})$ respectively.

First we show that if $\mathcal{C}$ and $b(\mathcal{C})$ are ideal then, for all $\ell, w \in R^n_+$, $\alpha\beta \leq w^T\ell$ where $\alpha := \min\{w(C) :\ C \in E(\mathcal{C})\}$ and $\beta := \min\{\ell(D) :\ D \in E(b(\mathcal{C}))\}$.

If $\alpha = 0$ or $\beta = 0$, then this clearly holds.

If $\alpha > 0$ and $\beta > 0$, we can assume w.l.o.g. that $\alpha = \beta = 1$ by scaling ℓ and w. So $Aw \geq \mathbf{1}$, i.e. w belongs to the polyhedron $P := \{x \geq 0,\ Ax \geq \mathbf{1}\}$. Therefore w is greater than or equal to a convex combination of the extreme points of P, which are the rows of B by Remark 1.4(i) since P is an integral polyhedron. It follows that $w^T \geq \lambda^T B$ where $\lambda \geq 0$ and $\sum_i \lambda_i = 1$. Similarly, one shows that $\ell^T \geq \mu^T A$ where $\mu \geq 0$ and $\sum_i \mu_i = 1$. Since $BA^T \geq J$, where J denotes the matrix of all 1's, it follows that

$$w^T\ell \geq \lambda^T BA^T\mu \geq \lambda^T J\mu = 1 = \alpha\beta$$

Now we prove the converse. Let $\mathcal{C}$ be a nontrivial clutter and let w be any extreme point of $P := \{x \geq 0 :\ Ax \geq \mathbf{1}\}$. Since $Aw \geq \mathbf{1}$, it follows that $\min\{w(C) :\ C \in E(\mathcal{C})\} \geq 1$. For any point z in $Q := \{z \geq 0 :\ Bz \geq \mathbf{1}\}$, we also have $\min\{z(D) :\ D \in E(b(\mathcal{C}))\} \geq 1$. Using the hypothesis, it follows that $w^T z \geq 1$ is satisfied by all points z in Q. Furthermore, equality holds for at least one $z \in Q$. Now, by linear programming duality,

$$1 \ = \ \min\{w^T z :\ z \in Q\} \ = \ \max\{\mu^T \mathbf{1} :\ \mu^T B \leq w^T,\ \mu \geq 0\}.$$

It follows from Remark 1.4(ii) that w is a 0,1 extreme point of P. Therefore, $\mathcal{C}$ is ideal. By Theorem 1.5, $b(\mathcal{C})$ is also ideal. □

2.2 Two-Commodity Flows

Let G be an undirected graph and let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be two pairs of nodes of G. A *two-commodity cut* is a set of edges separating each of the pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$. A *two-commodity path* is an s_1t_1-path or an s_2t_2-path.

For any edge capacities $w \in \Re_+^{E(G)}$, Hu [33] showed that a minimum capacity two-commodity cut can be obtained by solving the linear program (5) where M is the incidence matrix of two-commodity paths versus edges.

Theorem 2.3 (Hu [33]) *The clutter of two-commodity paths is ideal.*

Hence, the polyhedron

$$\begin{array}{rcll} x(P) & \geq & 1 & \text{for all two-commodity paths } P \\ x_e & \geq & 0 & \text{for all } e \in E(G) \end{array}$$

is integral.

Using Lehman's theorem (Theorem 1.5), the polyhedron

$$\begin{array}{rcll} x(C) & \geq & 1 & \text{for all two-commodity cuts } C \\ x_e & \geq & 0 & \text{for all } e \in E(G) \end{array}$$

is integral.

The clutters of 2-commodity paths and of 2-commodity cuts do not pack, but both have the 1/2-MFMC property (Hu [33] and Seymour [58], respectively).

The clutter of multicommodity paths is not always ideal for more than two commodities, but conditions on the graph G and the source-sink pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ have been obtained under which it is ideal. See Papernov [48], Okamura and Seymour [44], Lomonosov [37] and Frank [23] for examples.

3 T-cuts and T-joins

Consider a connected graph G with nonnegative edge weights w_e, for $e \in E(G)$. The Chinese Postman Problem consists in finding a minimum weight closed walk going through each edge at least once (the edges of the graph represent streets where mail must be delivered and w_e is the length of the street). Equivalently, the postman must find a minimum weight set of edges $J \subseteq E(G)$ such that $J \cup E(G)$ induces an Eulerian graph, i.e. J induces a graph the odd degree nodes of which coincide with the odd degree nodes of G. Since $w \geq 0$, we can assume w.l.o.g. that J is acyclic. Such an edge set J is called a *postman set*.

The problem is generalized as follows. Let G be a graph and T a node set of G of even cardinality. An edge set J of G is called a *T-join* if it induces an acyclic graph the odd degree nodes of which coincide with T. For disjoint node sets S_1, S_2, let (S_1, S_2)

denote the set of edges with one endnode in S_1 and the other in S_2. A T-*cut* is a minimal edge set of the form $(S, V(G) - S)$ where S is a set of nodes with $|T \cap S|$ odd. Clearly every T-cut intersects every T-join.

Edmonds and Johnson [21] considered the problem of finding a minimum weight T-join. One way to solve this problem is to reduce it to the perfect matching problem in a complete graph K_p, where $p = |T|$. Namely, compute the lengths of shortest paths in G between all pairs of nodes in T, use these values as edge weights in K_p and find a minimum weight perfect matching in K_p. The union of the corresponding paths in G is a minimum weight T-join. Edmonds and Johnson developed a direct primal-dual algorithm for the minimum weight T-join problem and, as a by-product, obtained that the clutter of T-cuts is ideal.

Theorem 3.1 (Edmonds and Johnson [21]) *The polyhedron*

$$\begin{aligned} x(C) &\geq 1 \quad \textit{for all} \ \ T-\textit{cuts} \ \ C && (7)\\ x_e &\geq 0 \quad \textit{for all} \ \ \ e \in E(G). && (8) \end{aligned}$$

is integral.

In the next section, we give a non-algorithmic proof of this theorem suggested by Pulleyblank [49].

The Edmonds-Johnson theorem together with the fact that the blocker of an ideal clutter is ideal (Theorem 1.3 of Lehman) implies that the clutter of T-joins is also ideal. That is the polyhedron

$$\begin{aligned} x(J) &\geq 1 \quad \text{for all} \ \ T-\text{joins} \ \ J\\ x_e &\geq 0 \quad \text{for all} \ \ \ e \in E(G). \end{aligned}$$

is integral.

The clutter of T-cuts does not pack, but it has the 1/2-MFMC property (Seymour [61]). The clutter of T-joins does not have the 1/2-MFMC property (there is an example requiring multiplication by 4 to get an integer dual), but it may have the 1/4-MFMC property (open problem). Another intriguing conjecture is the following. Recall that, in a graph G, a *postman set* is a T-join where T coincides with the nodes of G having odd degree.

Conjecture 3.2 (Conforti and Johnson [9]) *The clutter of postman sets packs in graphs noncontractible to the Petersen graph.*

If true, this implies the four color theorem! Indeed, the special case where G is cubic is Tutte's conjecture, recently proved by Robertson, Sanders, Seymour and Thomas [51].

3.1 Proof of the Edmonds-Johnson Theorem

First, we prove the following lemma. For $v \in V(G)$, let $\delta(v)$ denote the set of edges incident with v. A *star* is a tree where one node is adjacent to all the other nodes.

Lemma 3.3 *Let $\tilde{x}$ be an extreme point of the polyhedron*

$$\begin{aligned} x(\delta(v)) &\geq 1 \quad \text{for all} \quad v \in T && (9) \\ x_e &\geq 0 \quad \text{for all} \quad e \in E(G). && (10) \end{aligned}$$

The connected components of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ are either

(i) *odd cycles with nodes in T and edges $\tilde{x}_e = 1/2$, or*

(ii) *stars with nodes in T, except possibly the center, and edges $\tilde{x}_e = 1$.*

Proof: Every connected component C of $\tilde{G}$ is either a tree or contains a unique cycle, since the number of edges in C is at most the number of inequalities (9) that hold with equality.

Assume first that C contains a unique cycle. Then (9) holds with equality for all nodes of C, which are therefore in T. Now C is a cycle since, otherwise, C has a pendant edge e with $\tilde{x}_e = 1$ and therefore C is disconnected, a contradiction. If C is an even cycle, then by alternately increasing and decreasing $\tilde{x}$ around the cycle by a small ϵ ($-\epsilon$ respectively), $\tilde{x}$ can be written as a convex combination of two points satisfying (9) and (10). So (i) must hold.

Assume now that C is a tree. Then (9) holds with equality for at least $|V(C)| - 1$ nodes of C. In particular, it holds with equality for at least one node of degree one. Since C is connected, this implies that C is a star and (ii) holds. □

Proof of Theorem 3.1: In order to prove the theorem, it suffices to show that every extreme point $\tilde{x}$ of the polyhedron (7)–(8) is the incidence vector of a T-join. We proceed by induction on the number of nodes of G.

Suppose first that $\tilde{x}$ is an extreme point of the polyhedron (9)–(10). Consider a connected component of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ and let S be its node set. Since $\tilde{x}(S, V(G) - S) = 0$, it follows from (7) that S contains an even number of nodes of T. By Lemma 3.3, $\tilde{G}$ contains no odd cycle, showing that $\tilde{x}$ is an integral vector. Furthermore, $\tilde{x}$ is the incidence vector of a T-join since, by Lemma 3.3 again, the component of $\tilde{G}$ induced by S is a star and $|S \cap T|$ even implies that the center is in T if and only if the star has an odd number of edges.

Assume now that $\tilde{x}$ is not an extreme point of the polyhedron (9)–(10). Then there is some T-cut $C = (V_1, V_2)$ with $|V_1| \geq 2$ and $|V_2| \geq 2$ such that

$$\tilde{x}(C) = 1.$$

Let $G_1 = (V_1 \cup \{v_2\}, E_1)$ be the graph obtained from G by contracting V_2 to a single node v_2. Similarly, $G_2 = (V_2 \cup \{v_1\}, E_2)$ is the graph obtained from G by contracting V_1 to a single node v_1. The new nodes v_1, v_2 belong to T. For $i = 1, 2$, let $\tilde{x}^i$ be the restriction of $\tilde{x}$ to E_i. Since every T-cut of G_i is also a T-cut of G, it follows by induction that $\tilde{x}^i$ is greater than or equal to a convex combination of incidence vectors of T-joins of G_i. Let $\mathcal{T}_i$ be this set of T-joins. Each T-join in $\mathcal{T}_i$ has exactly one edge incident with v_i. Since $\tilde{x}^1$ and $\tilde{x}^2$ coincide on the edges of C, it follows that the T-joins of $\mathcal{T}_1$ can be combined with those of $\mathcal{T}_2$ to form T-joins of G and that $\tilde{x}$ is greater than or equal to a convex combination of incidence vectors of T-joins of G. Since $\tilde{x}$ is an extreme point, it is the incidence vector of a T-join. □

We have just proved that the clutter of T-cuts is ideal. It does not have the MFMC property in general graphs. However Seymour proved that it does in bipartite graphs. Seymour also showed that, in a general graph, if the edge weights w_e are integral and their sum is even in every cycle, then the dual variables can be chosen to be integral in an optimum solution.

3.2 *st*-*T*-Cuts

Goemans and Ramakrishnan [26] introduced a generalization of st-cuts, T-cuts and two-commodity cuts as follows. In a graph G, let s, t be two distinct nodes and let T be a node set of even cardinality. An *st-T-cut* is a T-cut $\delta(U) := \{uv \in E : u \in U, v \notin U\}$ where U contains exactly one of s or t. The st-cut clutter is obtained when $T = \{s, t\}$, the T-cut clutter is obtained when t is an isolated node and the two-commodity cut clutter is obtained when $T = \{s', t'\}$.

Recently, Guenin [30] characterized exactly when the clutter of st-T-cuts is ideal. This generalizes theorems of Hu (Theorem 2.3) and Edmonds-Johnson (Theorem 3.1).

4 Minimally Nonideal Matrices

Lehman (Theorem 1.5) showed that ideal 0,1 matrices always come in pairs (if M is ideal, so is its blocker $b(M)$) and that the width-length inequality is in fact a characterization of idealness (recall Theorem 2.2). Another important result of Lehman about ideal 0,1 matrices is the following.

Theorem 4.1 (Lehman [36]) *For a 0,1 matrix A, the following statements are equivalent:*

(i) the matrix A is ideal,

(ii) $\min\ \{cx\ :\ \ Ax \geq \mathbf{1},\ \ x \geq \mathbf{0}\}$ has an integral optimal solution x for all $c \in \{0, 1, +\infty\}^n$.

The fact that (i) implies (ii) is an immediate consequence of the definition of idealness. The difficult part of Lehman's theorem is that (ii) implies (i). The main purpose of this section is to prove this result. This is done by studying properties of minimally nonideal matrices.

4.1 Lehman's Characterization

A 0,1 matrix A is *minimally nonideal* (*mni*) if

(i) A contains no dominating row,

(ii) $Q(A) := \{x \geq \mathbf{0} : \; Ax \geq \mathbf{1}\}$ is not an integral polyhedron,

(iii) For every $i = 1, \dots, n$, both $Q(A) \cap \{x : \; x_i = 0\}$ and $Q(A) \cap \{x : \; x_i = 1\}$ are integral polyhedra.

If A is *mni*, the clutter $\mathcal{C}(A)$ is also called *mni*. Equivalently, a clutter $\mathcal{C}$ is *mni* if it is not ideal but all its proper minors are ideal.

For $t \geq 2$ integer, let $\mathcal{J}_t$ denote the clutter with $t+1$ vertices and edges corresponding, respectively, to the points and lines of the finite degenerate projective plane. Namely, $V(\mathcal{J}_t) := \{0, \dots, t\}$, and $E(\mathcal{J}_t) := \{\{1, \dots, t\}, \{0, 1\}, \{0, 2\}, \dots, \{0, t\}\}$.

A matrix A is *isomorphic* to a matrix B if B can be obtained from A by a permutation of rows and a permutation of columns.

Let J denote a square matrix all of whose entries are 1's, and let I be the identity matrix. Given a *mni* matrix A, let $\bar{x}$ be an extreme point of the polyhedron $Q(A) := \{x \geq 0 : \; Ax \geq \mathbf{1}\}$ with fractional components. The maximum row submatrix $\bar{A}$ of A such that $\bar{A}\bar{x} = \mathbf{1}$ is called a *core* of A. So A has a core for each fractional extreme point of $Q(A)$.

Theorem 4.2 (Lehman [36]) *Let A be a mni matrix and $B = b(A)$. Then*

(i) A has a unique core $\bar{A}$ and B has a unique core $\bar{B}$;

(ii) $\bar{A}$ and $\bar{B}$ are square matrices;

(iii) Either A is isomorphic to $M(\mathcal{J}_t)$, $t \geq 2$, or the rows of $\bar{A}$ and $\bar{B}$ can be permuted so that

$$\bar{A}\bar{B}^T = J + dI$$

for some positive integer d.

Lehman's proof of this theorem is rather terse. Seymour [63], Padberg [47] and Gasparyan, Preissmann and Sebö [24] give more accessible presentations of Lehman's proof. In the next section, we present a proof of Lehman's theorem following Padberg's polyhedral point of view.

Bridges and Ryser [2] studied square matrices Y, Z that satisfy the matrix equation $YZ = J + dI$.

Theorem 4.3 (Bridges and Ryser [2]) *Let Y and Z be $n \times n$ 0,1 matrices such that $YZ = J + dI$ for some positive integer d. Then*

(i) each row and column of Y has the same number r of ones, each row and column of Z has the same number s of ones with $rs = n + d$,

(ii) $YZ = ZY$,

Proof: It is straightforward to check that $(J + dI)^{-1} = \frac{1}{d}I - \frac{1}{d(n+d)}J$. Hence

$$YZ = J + dI \Rightarrow YZ(\frac{1}{d}I - \frac{1}{d(n+d)}J) = I \Rightarrow Z(\frac{1}{d}I - \frac{1}{d(n+d)}J)Y = I$$

$$\text{i.e.} \qquad ZY = \frac{1}{n+d}ZJY + dI = \frac{1}{n+d}\mathbf{s}\mathbf{r}^T + dI$$

where $\mathbf{s} := Z\mathbf{1}$ and $\mathbf{r} := Y^T\mathbf{1}$.

It follows that, for each i and j, $n+d$ divides $r_i s_j$. On the other hand, the trace of the matrix ZY is equal to the trace of YZ, which is $n(d+1)$. This implies $\frac{1}{n+d}(\sum_1^n s_i r_i) = n$ and, since $s_i > 0$ and $r_i > 0$, we have $r_i s_i = n + d$. Now consider distinct i, j. Since $r_i s_i = r_j s_j = n + d$ and $n + d$ divides $r_i s_j$ and $r_j s_i$, it follows that $r_i = r_j$ and $s_i = s_j$. Therefore, all columns of Z have the same sum s and all rows of Y have the same sum r. Furthermore, $ZY = J + dI$ and, by symmetry, all columns of Y have the same sum and all rows of Z have the same sum. $\square$

Theorems 4.2 and 4.3 have the following consequence.

Corollary 4.4 *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$. Then it has a nonsingular row submatrix $\bar{A}$ with exactly r ones in every row and column. Moreover, rows of A not in $\bar{A}$ have at least $r + 1$ ones.*

This implies the next result, which is a restatement of Theorem 4.1.

Corollary 4.5 *Let A be a 0,1 matrix. The polyhedron $Q(A) = \{x \in R^n_+ : Ax \geq \mathbf{1}\}$ is integral if and only if $min\{wx : x \in Q(A)\}$ has an integral optimal solution for all $w \in \{0, 1, \infty\}^n$.*

Note that Theorem 1.10 mentioned in the introduction follows from Corollary 4.5.

Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$ and let B be its blocker. Let $\bar{A}$ be the unique core of A and $\bar{B}$ be the unique core of B. Define, $\mathcal{A} := \mathcal{C}(A), \mathcal{B} := \mathcal{C}(B), core(\mathcal{A}) := \mathcal{C}(\bar{A}), core(\mathcal{B}) := \mathcal{C}(\bar{B})$. Corollary 4.4 implies that $core(\mathcal{A})$ (resp. $core(\mathcal{B})$) is the set of edges of $\mathcal{A}$ (resp. $\mathcal{B}$) of minimum cardinality. Let L be the edge of $core(\mathcal{A})$ which corresponds to the i^{th} row of $\bar{A}$ and let U be the edge of $core(\mathcal{B})$ which corresponds to the i^{th} row of $\bar{B}$. Theorem 4.2 states that $\bar{A}\bar{B}^T = J + dI$. It follows that L intersects every edge of $core(\mathcal{B})$ exactly once except for U which is intersected $d + 1$ times. We say that L and U are *mates*. It follows from Theorem 4.3(ii) that $\bar{A}\bar{B}^T = \bar{B}^T\bar{A} = J + dI$. In particular for every column j of $\bar{B}$, $col(\bar{B}, j)^T\bar{A} = \mathbf{1} + de_j$. We can restate this as follows.

Corollary 4.6 *Let A and B be mni matrices which are not isomorphic to $M(\mathcal{J}_t)$. Suppose $\bar{A}$ has r ones per row and $\bar{B}$ has s ones per row. Let j be the index of a column of $\bar{B}$. Let $L_1, \ldots, L_s$ be the edges of core($\mathcal{A}$) corresponding to the rows of $\bar{A}$ whose indices are given by the characteristic set of column j of $\bar{B}$. Then $L_1 - \{j\}, \ldots, L_s - \{j\}$ are pairwise disjoint, and exactly $d+1$ of these edges contain j.*

The previous corollary implies immediately,

Remark 4.7 *Let $\mathcal{A}$ be a mni clutter distinct from $\mathcal{J}_t$. Let C_1, C_2 be edges of core($\mathcal{A}$) and let U_1, U_2 be their mates. If $e \in U_1 \cap U_2$ then $L_1 \cap L_2 \subseteq \{e\}$ and if $e \in C_1 \cap C_2$ then $U_1 \cap U_2 \subseteq \{e\}$.*

4.1.1 Proof of Lehman's Theorem

Let A be an $m \times n$ *mni* matrix, $\bar{x}$ a fractional extreme point of $Q(A) := \{x \in R^n_+ : Ax \geq \mathbf{1}\}$ and $\bar{A}$ a core of A. That is, $\bar{A}$ is the maximal row submatrix of A such that $\bar{A}\bar{x} = \mathbf{1}$. For simplicity of notation, assume that $\bar{A}$ corresponds to the first p rows of A, i.e. the entries of $\bar{A}$ are a_{ij} for $i = 1, \ldots, p$ and $j = 1, \ldots, n$. Since A is *mni*, every component of $\bar{x}$ is nonzero. Therefore $p \geq n$ and $\bar{A}$ has no row or column containing only 0's or only 1's.

The following easy result will be applied to the bipartite representation G of the 0,1 matrix $J - \bar{A}$ where J denotes the $p \times n$ matrix of all 1's, namely ij is an edge of G if and only if $a_{ij} = 0$, for $1 \leq i \leq p$ and $1 \leq j \leq n$. Let $d(u)$ denote the degree of node u.

Lemma 4.8 (de Bruijn and Erdös [15]) *Let $(I \cup J, E)$ be a bipartite graph with no isolated node. If $|I| \geq |J|$ and $d(i) \geq d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$, then $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$.*

Proof: $|I| = \sum_{i \in I}(\sum_{j \in N(i)} \frac{1}{d(i)}) \leq \sum_{i \in I} \sum_{j \in N(i)} \frac{1}{d(j)} = \sum_{j \in J} \sum_{i \in N(j)} \frac{1}{d(j)} = |J|$. Now the hypothesis $|I| \geq |J|$ implies that equality holds throughout. So $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$. $\square$

The key to proving Lehman's theorem is the following lemma.

Lemma 4.9 *$p = n$ and, if $a_{ij} = 0$ for $1 \leq i, j \leq n$, then row i and column j of $\bar{A}$ have the same number of ones.*

Proof: Let x^j be defined by

$$x^j_k = \begin{cases} \bar{x}_k & \text{if} \quad k \neq j \\ 1 & \text{if} \quad k = j \end{cases}$$

and let F_j be the face of $Q(A) \cap \{x_j = 1\}$ of smallest dimension that contains x^j. Since A is *mni*, F_j is an integral polyhedron. The proof of the lemma will follow unexpectedly from computing the dimension of F_j.

The point x^j lies at the intersection of the hyperplanes in $\bar{A}x = \mathbf{1}$ such that $a_{kj} = 0$ (at least $n - \sum_{k=1}^{p} a_{kj}$ such hyperplanes are independent since $\bar{A}$ has rank n) and of the hyperplane $x_j = 1$ (independent of the previous hyperplanes). It follows that

$$dim(F_j) \leq n - (n - \sum_{k=1}^{p} a_{kj} + 1) = \sum_{k=1}^{p} a_{kj} - 1$$

Choose a row a^i of $\bar{A}$ such that $a_{ij} = 0$. Since $x^j \in F_j$, it is greater than or equal to a convex combination of extreme points b^ℓ of F_j, say $x^j \geq \sum_{\ell=1}^{t} \gamma_\ell b^\ell$, where $\gamma > 0$ and $\sum \gamma_\ell = 1$.

$$1 = a^i x^j \geq \sum_{\ell=1}^{t} \gamma_\ell a^i b^\ell \geq 1 \tag{11}$$

Therefore, equality must hold throughout. In particular $a^i b^\ell = 1$ for $\ell = 1, \ldots, t$. Since b^ℓ is a 0,1 vector, it has exactly one nonzero entry in the set of columns k where $a_{ik} = 1$. Another consequence of the fact that equality holds in (11) is that $x_k^j = \sum_{\ell=1}^{t} \gamma_\ell b_k^\ell$ for every k where $a_{ik} = 1$. Now, since $x_k^j > 0$ for all k, it follows that F_j contains at least $\sum_{k=1}^{n} a_{ik}$ linearly independent points b^ℓ, i.e.

$$dim(F_j) \geq \sum_{k=1}^{n} a_{ik} - 1.$$

Therefore, $\sum_{k=1}^{n} a_{ik} \leq \sum_{k=1}^{p} a_{kj}$ for all i, j such that $a_{ij} = 0$.

Now Lemma 4.8 applied to the bipartite representation of $J - \bar{A}$ implies that $p = n$ and

$$\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} \text{ for all } i, j \text{ such that } a_{ij} = 0.$$

□

Lemma 4.10 *$\bar{x}$ has exactly n adjacent extreme points in $Q(A)$, all with 0,1 coordinates.*

Proof: By Lemma 4.9, exactly n inequalities of $A\bar{x} \geq \mathbf{1}$ are tight, namely $\bar{A}\bar{x} = \mathbf{1}$. In the polyhedron $Q(A)$, an edge adjacent to $\bar{x}$ is defined by $n-1$ of the n equalities in $\bar{A}x = \mathbf{1}$. Moving along such an edge from $\bar{x}$, at least one of the coordinates decreases. Since $Q(A) \in R_+^n$, this implies that $\bar{x}$ has exactly n adjacent extreme points on $Q(A)$. Suppose $\bar{x}$ has a fractional adjacent extreme point $\bar{x}'$. Since A is *mni*, $0 < \bar{x}'_j < 1$ for all j. Let $\bar{A}'$ be the $n \times n$ nonsingular submatrix of A such that $\bar{A}'\bar{x}' = \mathbf{1}$. Since $\bar{x}$ and $\bar{x}'$ are adjacent on $Q(A)$, $\bar{A}$ and $\bar{A}'$ differ in only one row. W.l.o.g. assume that $\bar{A}'$ corresponds to rows 2 to $n+1$. Since A contains no dominating row, there exists j such that $a_{1j} = 0$ and $a_{n+1,j} = 1$. Since $\bar{A}'$ cannot contain a column with only 1's, $a_{ij} = 0$ for some $2 \leq i \leq n$. But now, Lemma 4.8 is contradicted with row i and column j in either $\bar{A}$ or $\bar{A}'$. $\square$

Lemma 4.10 has the following implication. Let $\bar{B}$ denote the $n \times n$ 0,1 matrix whose rows are the extreme points of $Q(A)$ adjacent to $\bar{x}$. By Remark 1.4(i), $\bar{B}$ is a submatrix of B. By Lemma 4.10, $\bar{B}$ satisfies the matrix equation

$$\bar{A}\bar{B}^T = J + D$$

where J is the matrix of all 1's and D is a diagonal matrix with positive diagonal entries $d_1, \ldots, d_n$.

Lemma 4.11 *Either*

(i) $\bar{A} = \bar{B}$ *are isomorphic to* $M(\mathcal{J}_t)$, *for* $t \geq 2$, *or*

(ii) $D = dI$, *where* d *is a positive integer.*

Proof: Consider the bipartite representation G of the 0,1 matrix $J - \bar{A}$.

Case 1: G is connected.
Then it follows from Lemma 4.9 that

$$\sum_k a_{ik} = \sum_k a_{kj} \text{ for all } i, j. \tag{12}$$

Let α denote this common row and column sum.

$$(n + d_1, \ldots, n + d_n) = \mathbf{1}^T(J + D) = \mathbf{1}^T\bar{A}\bar{B}^T = (\mathbf{1}^T\bar{A})\bar{B}^T = \alpha\mathbf{1}^T\bar{B}^T$$

Since there is at most one d, $1 \leq d < \alpha$, such that $n + d$ is a multiple of α, all d_i must be equal to d, i.e. $D = dI$.

Case 2: G is disconnected.
Let $q \geq 2$ denote the number of connected components in G and let

$$\bar{A} = \begin{pmatrix} K_1 & & \mathbf{1} \\ & \ldots & \\ \mathbf{1} & & K_q \end{pmatrix}$$

where K_t are 0,1 matrices, for $t = 1, \ldots, q$. It follows from Lemma 4.9 that the matrices K_t are square and $\sum_k a_{ik} = \sum_k a_{kj} = \alpha_t$ in each K_t.

Suppose first that $\bar{A}$ has no row with $n - 1$ ones. Then every K_t has at least two rows and columns. We claim that, for every j, k, there exist i, l such that $a_{ij} = a_{ik} = a_{lj} = a_{lk} = 1$. The claim is true if $q \geq 3$ or if $q = 2$ and j, k are in the same component (simply take two rows i, l from a different component). So suppose $q = 2$, column j is in K_1 and column k is in K_2. Since no two rows are identical, we must have $\alpha_1 \geq 1$, i.e. $a_{ij} = 1$ for some row i of K_1. Similarly, $a_{lk} = 1$ for some row l of K_2. The claim follows.

For each row b of $\bar{B}$, the vector $\bar{A}b^T$ has an entry greater than or equal to 2, so there exist two columns j, k such that $b_j = b_k = 1$. By the claim, there exist rows a_i and a_l of $\bar{A}$ such that $a_i b^T \geq 2$ and $a_l b^T \geq 2$, contradicting the fact that $\bar{A}b^T$ has exactly one entry greater than 1.

Therefore $\bar{A}$ has a row with $n-1$ ones. Now it is routine to check that $\bar{A}$ is isomorphic to $M(\mathcal{J}_t)$, for $t \geq 2$. □

To complete the proof of Theorem 4.2, it only remains to show that the core $\bar{A}$ is unique and that $\bar{B}$ is a core of B and is unique.

If $\bar{A} = M(\mathcal{J}_t)$ for some $t \geq 2$, then the fact that A has no dominated rows implies that $A = \bar{A}$. Thus $B = \bar{B} = M(\mathcal{J}_t)$. So, the theorem holds in this case.

If $\bar{A}\bar{B}^T = J + dI$ for some positive integer d, then, by Theorem 4.3, all rows of $\bar{A}$ contain r ones. Therefore, $\bar{x}_j = \frac{1}{r}$, for $j = 1, \ldots, n$. The feasibility of $\bar{x}$ implies that all rows of A have at least r ones, and Lemma 4.9 implies that exactly n rows of A have r ones. Now $Q(A)$ cannot have a fractional extreme point $\bar{x}'$ distinct from $\bar{x}$, since the above argument applies to $\bar{x}'$ as well. Therefore A has a unique core $\bar{A}$. Since $\bar{x}$ has exactly n neighbors in $Q(A)$ and they all have s components equal to one, the inequality $\sum_1^n x_i \geq s$ is valid for the 0,1 points in $Q(A)$. This shows that every row of B has at least s ones and exactly n rows of B have s ones. Since B is *mni*, $\bar{B}$ is the unique core of B. □

4.2 Examples of *mni* Clutters

Let $Z_n = \{0, \ldots, n-1\}$. We define addition of elements in Z_n to be addition modulo n. Let $k \leq n-1$ be a positive integer. For each $i \in Z_n$, let C_i denote the subset $\{i, i+1, \ldots, i+k-1\}$ of Z_n. Define the *circulant* clutter $\mathcal{C}_n^k$ by $V(\mathcal{C}_n^k) := Z_n$ and $E(\mathcal{C}_n^k) := \{C_0, \ldots, C_{n-1}\}$.

Lehman [35] gave three infinite classes of minimally nonideal clutters: $\mathcal{C}_n^2$, $n \geq 3$ odd, their blockers, and the degenerate projective planes $\mathcal{J}_n$, $n \geq 2$.

Conjecture 4.12 (Cornuéjols and Novick [13]) *There exists n_0 such that, for $n \geq n_0$, all mni matrices have a core isomorphic to $\mathcal{C}_n^2$, $\mathcal{C}_n^{\frac{n+1}{2}}$ for $n \geq 3$ odd, or $\mathcal{J}_n$, for $n \geq 2$.*

However, there exist several known "small" *mni* matrices that do not belong to any of the above classes. For example, Lehman [35] noted that $\mathcal{F}_7$ is *mni*. $\mathcal{F}_7$ is the clutter with 7 vertices and 7 edges corresponding to points and lines of the Fano plane (finite projective geometry on 7 points):

$$M(\mathcal{F}_7) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let K_5 denote the complete graph on five nodes and let $\mathcal{O}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the odd cycles of K_5 (the triangles and the pentagons). Seymour [57] noted that $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, and $\mathcal{C}_9^2$ with the extra edge $\{3, 6, 9\}$ are *mni*.

Ding [17] found the following *mni* clutter: $V(\mathcal{D}_8) := \{1, \dots, 8\}$ and

$$E(\mathcal{D}_8) := \{\{1,2,6\}, \{2,3,5\}, \{3,4,8\}, \{4,5,7\}, \{2,5,6\}, \{1,6,7\}, \{4,7,8\}, \{1,3,8\}\}.$$

Cornuéjols and Novick [13] characterized the *mni* circulant clutters $\mathcal{C}_n^k$. They showed that the following ten clutters are the only *mni* $\mathcal{C}_n^k$ for $k \geq 3$:

$$\mathcal{C}_5^3,\ \mathcal{C}_8^3,\ \mathcal{C}_{11}^3,\ \mathcal{C}_{14}^3,\ \mathcal{C}_{17}^3,\ \mathcal{C}_7^4,\ \mathcal{C}_{11}^4,\ \mathcal{C}_9^5,\ \mathcal{C}_{11}^6,\ \mathcal{C}_{13}^7.$$

Independently, Qi [50] discovered $\mathcal{C}_9^5$ and $\mathcal{C}_{11}^6$ and Ding [17] discovered $\mathcal{C}_8^3$.

Let $\mathcal{T}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the triangles of K_5 (interestingly, $M(\mathcal{T}_{K_5})$ is also the node-node adjacency matrix of the Petersen graph). It can be shown that $\mathcal{T}_{K_5}$, $core(b(\mathcal{T}_{K_5}))$ and their blockers are *mni*. Often, when a *mni* clutter $\mathcal{H}$ has the property that $core(\mathcal{H})$ and $core(b(\mathcal{H}))$ are also *mni*, many more *mni* clutters can be constructed from $\mathcal{H}$ and from $b(\mathcal{H})$, see [13]. For example, Cornuéjols and Novick [13] have constructed more than one thousand mni clutters from $\mathcal{T}_{K_5}$. More results can be found in [42].

Lütolf and Margot [38] designed a computer program that enumerates possible cores of minimally nonideal matrices. It first enumerates the square 0,1 matrices Y, Z that satisfy the matrix equation $YZ = J + dI$, and then checks that the covering polyhedron has a unique fractional extreme point. Lütolf and Margot [38] enumerated all square *mni* matrices of dimension at most 12×12 and found 20 such matrices (previously, only 15 were known); they found 13 new square *mni* matrices of dimensions 14×14 and 17×17; and they found 38 new nonsquare *mni* matrices with 11, 14 and 17 columns with nonisomorphic cores. The overwhelming majority of these examples have $d = 1$: Only three cores with $d = 2$ are known (namely $\mathcal{F}_7, \mathcal{T}_{K_5}$ and the core of its blocker) and none with $d \geq 3$.

A clutter $\mathcal{C}$ is *minimally nonpacking* if it does not pack, but all its proper minors do. If $\mathcal{C}$ is minimally nonpacking, then $M(\mathcal{C})$ is also said to be minimally nonpacking.

Theorem 4.13 (Cornuéjols, Guenin and Margot [12]) *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$, $t \geq 2$. If A is minimally nonpacking, then $d = 1$.*

Conjecture 4.14 ([12]) *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$, $t \geq 2$. Then A is minimally nonpacking if and only if $d = 1$.*

Using a computer program, this conjecture was verified for all known minimally nonideal matrices with $n \leq 14$.

Proof of Theorem 4.13: We show that, if $\mathcal{C} \neq \mathcal{J}_t$ is a mni clutter with $d > 1$ then $\mathcal{C}$ is not minimally nonpacking. Let L be an edge of $core(\mathcal{C})$ and let U be its mate. Let $r := |L|$ and $s := |U|$. Let i be any vertex in $L \cap U$ and let $I := (L - U) \cup \{i\}$.

Claim 1: Every transversal of $\mathcal{C} \setminus I$ has cardinality at least $s - 1$.

Proof of claim: It suffices to show that every transversal of $core(\mathcal{C}) \setminus I$ has cardinality at least $s-1$. Suppose there exists a transversal T of $core(\mathcal{C}) \setminus I$ with $|T| \leq s-2$. Let j be any vertex in $U - \{i\}$. By Corollary 4.6, L is among the s edges of $core(\mathcal{C})$ that pairwise intersect at most in $\{j\}$. Since $I \subseteq L - \{j\}$, there are $s-1$ edges of $core(\mathcal{C}) \setminus I$ that pairwise intersect at most in $\{j\}$. Therefore, $|T| \leq s-2$ implies $j \in T$. By symmetry among the vertices of $U - \{i\}$, it follows that $U - \{i\} \subseteq T$. So in particular $|T| \geq s-1$, a contradiction. $\diamond$

Suppose $\mathcal{C} \setminus I$ packs. Then it follows from Claim 1 that $\mathcal{C} \setminus I$ contains $s-1$ disjoint edges $L_1, \ldots, L_{s-1}$.

Claim 2: None of $L_1, \ldots, L_{s-1}$ are edges of $core(\mathcal{C})$.

Proof of claim: Suppose that L_1 is an edge of $core(\mathcal{C})$ and let U_1 be its mate. Then $U_1 - (I \cup L_1)$ contains an edge T in $b(\mathcal{C})/(I \cup L_1)$. By assumption $|L_1 \cap U_1| = d+1 \geq 3$. Thus

$$|T| \quad \leq \quad |U_1 - L_1| = |U_1| - (d+1) = s - (d+1) \leq s - 3$$

By Proposition 1.15, T is a transversal of $\mathcal{C} \setminus (I \cup L_1)$. But $L_2, \ldots, L_{s-1}$ are disjoint edges of $\mathcal{C} \setminus (I \cup L_1)$, which implies that every tranversal of $\mathcal{C} \setminus (I \cup L_1)$ has cardinality at least $s-2$, a contradiction. $\diamond$

By Corollary 4.4, the edges $L_1, \ldots, L_{s-1}$ have cardinality at least $r+1$. Moreover they do not intersect I. Therefore we must have:

$$(r+1)(s-1) \leq n - |I| = (rs - d) - (r - d) = rs - r$$

Thus $r \leq 1$, a contradiction. $\square$

4.3 A Conjecture

As a parallel to Theorem 4.1, we can restate Conjecture 1.9 as follows.

Conjecture 4.15 (Conforti and Cornuéjols [6])
For a 0,1 matrix A, the following statements are equivalent:

(i) the matrix A has the MFMC property,

(ii) min $\{cx : Ax \geq \mathbf{1},\ x \geq \mathbf{0}\}$ has an integral optimal dual solution y for all $c \in \{0, 1, +\infty\}^n$.

4.4 Ideal Minimally Nonpacking Clutters

Minimally nonpacking clutters are either ideal or minimally nonideal. This follows from Theorem 1.10. Theorem 4.13 above discussed the minimally nonideal case. In this section, we discuss the ideal case. The clutter of triangles of K_4 is such an example: this clutter has 6 vertices (the 6 edges of K_4) and 4 edges (the 4 triangles of K_4 viewed as edge sets) and it is denoted by Q_6.

A clutter is *binary* if its edges have an odd intersection with its minimal transversals. Seymour [57] showed that Q_6 is the only ideal minimally nonpacking binary clutter. However, there are ideal minimally nonpacking clutters that are not binary, such as

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Note that, for this clutter, the minimum size of a transversal is 2. Other examples can be found in [12] but none is known with a minimum transversal of size greater than 2. Interestingly, all ideal minimally nonpacking clutters with a transversal of size 2 share strong structural properties with Q_6. A clutter $\mathcal{C}$ has the *Q_6-property* if $M(\mathcal{C})$ has 4 rows such that every column restricted to this set of rows contains two 0's and two 1's and each such 6 possible 0,1 vectors occurs at least once.

Theorem 4.16 (Cornuéjols, Guenin and Margot [12]) *Every ideal minimally nonpacking clutter with a transversal of size 2 has the Q_6-property.*

Conjecture 4.17 [12] *Every ideal minimally nonpacking clutter has a transversal of size 2.*

It is proved in [12] that this conjecture would imply Conjecture 1.9 or, equivalently, Conjecture 4.15.

5 Odd Cycles in Graphs

In this section, we consider the clutter of odd cycles in a graph. Seymour [57] characterized exactly the graphs for which $\mathcal{H}$ has the MFMC property and Guenin [28] characterized exactly when $\mathcal{H}$ is ideal.

For edge weights $w \in \Re_+^{E(G)}$, consider the minimization problem (5). Recall that an integral solution to (5) is the incidence vector of a transversal T of $\mathcal{H}$. Since T intersects all odd cycles, $E(G) - T$ induces a bipartite graph. Therefore, a minimal transversal T of $\mathcal{H}$ is the complement of a cut $\delta(U)$. In particular, when $\mathcal{H}$ is ideal, (5) finds a cut of maximum weight in G, i.e. (5) solves the famous *max cut* problem.

5.1 Planar Graphs

Orlova and Dorfman [45] showed that the clutter $\mathcal{H}$ of odd cycles is ideal when G is planar.

Theorem 5.1 (Orlova and Dorfman [45]) *In a planar graph, the clutter of odd cycles is ideal.*

Proof: Let G be a planar graph and D its dual. The bounded faces of G form a cycle basis. Thus any odd cycle of G is a symmetric difference of faces, an odd number of which are odd faces. Faces of G correspond to nodes of D. Let T be the set of odd degree nodes of D. An odd cycle of G corresponds to an edge set of D of the form $\delta(U)$ where $|U \cap T|$ has odd cardinality, i.e. a T-cut of D. The clutter of T-cuts in D is ideal by the Edmonds-Johnson theorem (Theorem 3.1) and therefore so is the clutter of odd cycles in G. $\square$

When $G = K_5$, the complete graph on 5 nodes, the clutter $\mathcal{H}$ of odd cycles is not ideal since $x_j = \frac{1}{3}$ for $j = 1, \ldots, 10$ is a fractional extreme point of the polyhedron $\{x \in R_+^{10} : \ M(\mathcal{H})x \geq \mathbf{1}\}$.

Barahona [1] observed that Theorem 5.1 has the following generalization.

Theorem 5.2 *In a graph not contractible to K_5, the clutter of odd cycles is ideal.*

This follows from a famous theorem of Wagner [67] stating that any edge-maximal graph not contractible to K_5 can be constructed recursively by pasting plane triangulations and copies of V_8 along K_3's and K_2's, where V_8 is the cycle $v_1, v_2, \ldots, v_8, v_1$ with chords $v_i v_{i+4}$ for $i = 1, 2, 3, 4$.

Is there a converse to Barahona's theorem? In particular, is it true that, if the clutter of odd cycles is ideal in a graph G, then G is not contractible to K_5? The answer to the second question is no. For example, insert a node of degree 2 on every edge of K_5. The graph is now bipartite and the clutter of odd cycles has become the trivial clutter, which is ideal! The problem is that contraction of an edge changes odd cycles into even cycles and vice versa. To get a converse to Barahona's theorem, one needs to redefine contraction appropriately. It is convenient to work in the more general context of signed graphs.

5.2 Signed Graphs

Consider a graph G and a subset S of its edges. The pair (G, S) is called a *signed graph*. A subset X of edges of G is *odd* (resp. *even*) if $|X \cap S|$ is odd (resp. even). A set $S' \subseteq E(G)$ is a *signature* of (G, S) if (G, S') has the same odd cycles as (G, S).

Consider a signed graph (G, S) and let $\delta(U)$ be a cut of G. Since $\delta(U)$ intersects every cycle with even parity, $S \triangle \delta(U)$ is a signature of (G, S). We call the operation which consists of replacing S by $S \triangle \delta(U)$ a *signature-exchange*. In a signed graph (G, S), *deleting* an edge means removing it from the graph. *Contracting* an edge e means first (if necessary) doing a signature-exchange so that the edge e is even (i.e. not in the signature) and then removing the edge and identifying its endnodes.

Let E' and E'' be disjoint edge sets. One can readily verify that all the signed graphs obtained by deleting the edges in E' and contracting the edges in E'' are identical (up to signature-exchanges), no matter in which order the contractions and deletions are performed. A signed graph obtained from (G, S) by a sequence of contractions and deletions and signature-exchanges is called a *minor* of (G, S).

Let $\mathcal{H}$ denote the clutter of odd cycles of a signed graph (G, S). It is easy to check that every minor of $\mathcal{H}$ is the clutter of odd cycles of a signed graph (G', S') obtained as a minor of (G, S). A signed complete graph K_r on r nodes is called an *odd-K_r* if all its edges are odd. Guenin proved the following theorem.

Theorem 5.3 (Guenin [28]) *The clutter of odd cycles of a signed graph (G, S) is ideal if and only if (G, S) has no odd-K_5 minor.*

A clutter is *binary* (see Section 6) if its edges and its minimal transversals intersect in an odd number of vertices. The clutter of odd cycles in a signed graph is a binary clutter. Theorem 5.3 is a special case of a famous conjecture of Seymour [57], [60] (Conjecture 6.9) on ideal binary clutters. In [57], Seymour characterized the binary clutters that have the MFMC property. Specialized to the clutter of odd cycles, this theorem is the following.

Theorem 5.4 (Seymour [57]) *The clutter of odd cycles of a signed graph (G, S) has the MFMC property if and only if (G, S) has no odd-K_4 minor.*

5.3 Schrijver's proof of Guenin's Theorem

One direction of Guenin's theorem is easy: If the clutter of odd cycles is ideal for a signed graph (G, S), then (G, S) has no odd-K_5 minor. Thus the essence of Theorem 5.3

is the converse. Schrijver [53] obtained a shorter proof for this result, which curtails the technical and case-checking part of Guenin's proof.

Schrijver's proof which we give next (albeit with a different presentation along the lines of the proof of Theorem 5.8 see [25]) relies on the following two lemmas on mni binary clutters. These lemmas were also used in Guenin's original proof. Observe at the outset that $\mathcal{J}_t$ is not binary.

Lemma 5.5 *Let $\mathcal{H}$ be a mni binary clutter and C_1, C_2 be edges in $core(\mathcal{H})$. If $C \subseteq C_1 \cup C_2$ and C is an edge of $\mathcal{H}$ then $C = C_1$ or $C = C_2$.*

Proof: Let C be an edge of $\mathcal{H}$ contained in $C_1 \cup C_2$. Then (Proposition 6.1) $C_1 \triangle C_2 \triangle C$ contains an edge of $\mathcal{H}$, say C'. This implies that $C \cup C' \subseteq C_1 \cup C_2$ and $C \cap C' \subseteq C_1 \cap C_2$ (for if $e \in C \cap C'$ then $e \notin C_1 \triangle C_2$). Hence $|C| + |C'| \leq |C_1| + |C_2|$. So C, C' are also of minimum cardinality, and C, C' are edges of $core(\mathcal{H})$. Let B be the mate of C. Since $\mathcal{H}$ is binary, $|C \cap B|$ is odd, hence at least 3. It follows that either, $|C_1 \cap B| \geq 2$ or $|C_2 \cap B| \geq 2$. This implies that C_1 or C_2 is the mate of B, i.e. $C = C_1$ or $C = C_2$. $\square$

Lemma 5.6 *Let $\mathcal{H}$ be a mni binary clutter. For any $e \in V(\mathcal{H})$ there exist edges C_1, C_2, C_3 of $core(\mathcal{H})$ and edges B_1, B_2, B_3 of $core(b(\mathcal{H}))$ such that*

(i) $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \{e\}$

(ii) $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{e\}$

(iii) For distinct $i, j \in \{1, 2, 3\}$ we have $C_i \cap B_j = \{e\}$. For $i \in \{1, 2, 3\}$ we have $|C_i \cap B_i| = d + 1$ where $d + 1$ is odd and $d + 1 \geq 3$.

Proof: Corollary 4.6 states that there exist s edges $C_1, \ldots, C_s$ of $core(\mathcal{H})$ such that $C_1 - \{e\}, \ldots, C_s - \{e\}$ are pairwise disjoint. Moreover, exactly $d+1 \geq 2$ of these edges, say $C_1, \ldots, C_d$, contain vertex e. As $\mathcal{H}$ is binary, $d+1$ is odd (since $d+1 = |C \cap B|$ for any pair of mates C, B). Thus $d+1 \geq 3$ and (i) follows. Let B_1, B_2, B_3 be the mates of C_1, C_2, C_3. For $i \in \{1, 2, 3\}$ we have: $|C_i \cap B_i| = d+1 > 1$; $C_1 - \{e\}, \ldots, C_s - \{e\}$ disjoint; and $|B_i| = s$. Then $e \in B_i$ as B_i intersects each $C_1, \ldots, C_s$. Since $e \in C_1 \cap C_2 \cap C_3$, it follows from Remark 4.7 that $B_i \cap B_j \subseteq \{e\}$ for all distinct $i, j \in \{1, 2, 3\}$. Hence, (ii) holds. Finally (iii) holds since B_1, B_2, B_3 are the mates of C_1, C_2, C_3. $\square$

A key ingredient in Schrijver's proof is the following lemma. The particular version presented here was given in [25].

Lemma 5.7 *Let $G = (V, E)$ be a graph, let e be an edge of G with endnodes x and y, let (Y_0, Y_1, Y_2, Y_3) be disjoint subsets of V, and let P_1, P_2, and P_3 be internally node disjoint xy–paths in $G \backslash e$. Moreover, suppose that*

(1) $x, y \in Y_0$ and, for $i \in \{0, 1, 2, 3\}$, Y_i is a stable set of $G \backslash e$,

(2) for $i \in \{1,2,3\}$, $V(P_i) \subseteq Y_0 \cup Y_i$, and

(3) for distinct $i, j \in \{1,2,3\}$, there exists a path from $V(P_i)$ to $V(P_j)$ in $G[Y_i \cup Y_j]$.

Then $(G, E(G))$ has a minor isomorphic to odd–K_5.

Proof: Suppose otherwise, and let G be a counterexample minimizing $|V(G)|+|E(G)|$. For distinct $i, j \in \{1,2,3\}$, let P_{ij} be a path from $V(P_i)$ to $V(P_j)$ in $G[Y_i \cup Y_j]$. (We assume that $P_{ij} = P_{ji}$.) By the minimality of G, we have $E(G) := \{e\} \cup P_1 \cup P_2 \cup P_3 \cup P_{12} \cup P_{23} \cup P_{13}$, and $V(G) := V(P_1) \cup V(P_2) \cup V(P_3) \cup V(P_{12}) \cup V(P_{23}) \cup V(P_{13})$.

Suppose that G has a node v of degree 2, and define $G' := G/\delta_G(v)$. Note that, $(G, E(G))/\delta_G(v) = (G', E(G'))$, and that G' satisfies the conditions of the lemma. However, this contradicts the minimality of G, and, hence, G has no nodes of degree 2. Thus, we see that $Y_0 = \{x, y\}$, and, for each $i \in \{1,2,3\}$, P_i has exactly one internal node, say v_i. Now, the neighbors of x are v_1, v_2, v_3, and y, and the neighbors of y are v_1, v_2, v_3, and x. Moreover, since G has no nodes of degree 2, we also conclude that $Y_1 = V(P_{12}) \cap V(P_{13})$, $Y_2 = V(P_{12}) \cap V(P_{23})$, and $Y_3 = V(P_{13}) \cap V(P_{23})$. Therefore, $|Y_1| = |Y_2| = |Y_3|$.

If $|Y_1| = 1$, then $(G, E(G))$ is isomorphic to odd–K_5, so we may assume that $|Y_1| > 1$. For distinct $i, j \in \{1,2,3\}$, let e_{ij} be the edge on P_{ij} that is incident with v_i. Let $G' := G \backslash \{e_{13}, e_{32}, e_{21}\}/\{e_{12}, e_{23}, e_{31}\}$, and, for distinct $i, j \in \{1,2,3\}$, let $P'_{ij} := P_{ij} - \{e_{ij}, e_{ji}\}$. Now let $Y'_1 := V(P'_{12}) \cap V(P'_{13})$, let $Y'_2 := V(P'_{12}) \cap V(P'_{23})$, let $Y'_3 := V(P'_{13}) \cap V(P'_{23})$, and let $Y'_0 := \{x, y\}$. Note that, $(G', E(G'))$ is a minor of $(G, E(G))$ and that G' satisfies the conditions of the lemma. However, this contradicts the minimality of G. □

Given a graph G and $U \subseteq V(G)$, the subgraph of G induced by U is denoted $G[U]$.

Proof of Theorem 5.3: Let $\mathcal{H}$ be a mni clutter of odd cycles of a signed graph (G, S). We will show that (G, S) contains an odd-K_5 minor. Fix an edge $e \in E(G)$, with endnodes say x and y. Let C_1, C_2, C_3 be the sets of *core*$(\mathcal{H})$ and let B_1, B_2, B_3 be the sets of *core*$(b(\mathcal{H}))$ given in Lemma 5.6.

Claim 1: For distinct $i, j \in \{1,2,3\}$ the odd cycles C_i and C_j have no common node other than x, y.

Proof of claim: Otherwise $(C_i \cup C_j) - \{e\}$ contains a path P from x to y different from $C_i - \{e\}$ and $C_j - \{e\}$. By Lemma 5.5, $(C_i \cup C_j) - \{e\}$ contains no odd cycle. Hence, P and $C_i - \{e\}$ have the same parity and so $P \cup \{e\}$ is an odd cycle in $C_i \cup C_j$, contradicting Lemma 5.5. ◇

Since $\mathcal{H}$ is binary, B_i $(i = 1,2,3)$ is a signature. It follows that for distinct $i, j \in \{1,2,3\}$ $B_i \triangle B_j$ intersects all cycles with even parity; i.e. $B_i \triangle B_j$ is a cut of G. Moreover, $e \notin B_i \triangle B_j$. Therefore, for distinct $i, j \in \{1,2,3\}$, there exists $U_{ij} \subseteq V(G)$ such that $\delta(U_{ij}) = B_i \triangle B_j$ and $x, y \notin U_{ij}$. Note that

$$\delta(U_{12} \triangle U_{13} \triangle U_{23}) = \delta(U_{12}) \triangle \delta(U_{13}) \triangle \delta(U_{23}) = \emptyset.$$

Moreover, $x, y \notin U_{12} \triangle U_{13} \triangle U_{23}$ and G is connected. Therefore, $U_{12} \triangle U_{13} \triangle U_{23} = \emptyset$. Let $Y_1 := U_{12} \cap U_{13}, Y_2 := U_{12} \cap U_{23}, Y_3 := U_{13} \cap U_{23}$, and let $Y_0 = V(G) - (Y_1 \cup Y_2 \cup Y_3)$.

Claim 2: For distinct $i, j, k \in \{1, 2, 3\}$, the edge set $B_i - \{e\}$ consists of all edges with one endnode in Y_0 and the other in Y_i and all edges with one endnode in Y_j and the other in Y_k.

Proof of claim: We may assume $i = 1$. Since $B_1 - \{e\}, B_2 - \{e\}, B_3 - \{e\}$ are pairwise disjoint, $B_1 - \{e\} = (B_1 \triangle B_2) \cap (B_1 \triangle B_3)$. But $\delta(U_{12}) = B_1 \triangle B_2$ and $\delta(U_{13}) = B_1 \triangle B_3$. Thus the edges of $B_1 - \{e\}$ are exactly the edges in both $\delta(U_{12})$ and $\delta(U_{13})$. $\diamond$

For each $i \in \{1, 2, 3\}$ let $P_i := C_i - \{e\}$, thus P_i is an xy-path. Recall that for distinct $i, j, k \in \{1, 2, 3\}$, $C_i \cap (B_j \cup B_k) = \{e\}$. It follows together with Claim 2 that for each $i \in \{1, 2, 3\}$, $V(P_i) \subseteq Y_0 \cup Y_i$. Moreover, since $|C_i \cap B_i| > 1$, $P_i \cap V(Y_i) \neq \emptyset$.

Claim 3: For distinct $i, j \in \{1, 2, 3\}$, there exists a path P_{ij} from $V(P_i)$ to $V(P_j)$ in $G[Y_i \cup Y_j]$.

Proof of claim: Recall, $U_{ij} = Y_i \cup Y_j$. It suffices to prove that $G[U_{ij}]$ is connected. If not, there is an $X \subseteq U_{ij}$ such that $\delta(X)$ is a non-empty proper subset of $\delta(U_{ij})$. Then $B_i \triangle \delta(X)$ is contained in $B_i \cup B_j$ but is distinct from B_i and B_j. Since $B_i \triangle \delta(X)$ is a signature, it contains an element of $b(\mathcal{H})$, a contradiction with Lemma 5.5. $\diamond$

Let $B := B_1 \triangle B_2 \triangle B_3$. Then B is a signature for (G, S). Let $T := \{e\} \cup P_1 \cup P_2 \cup P_3 \cup P_{12} \cup P_{13} \cup P_{23}$. Each edge in $T - \{e\}$ is in at most one of the sets B_1, B_2, B_3. Therefore, the odd edges of $(G, B)[T]$ are e and any edge whose endnodes are in different parts of (Y_0, Y_2, Y_2, Y_3). Let (G', S') be the signed graph obtained from $(G, B)[T]$ by contracting the edges in $T - B$; thus $S' = E(G')$. For $i \in \{1, 2, 3\}$, let $P'_i = P_i \cap B$; for distinct $i, j \in \{1, 2, 3\}$, let $P'_{ij} = P_{ij} \cap B$; and for $l \in \{0, 1, 2, 3\}$ let Y'_l be the set of nodes of G' corresponding to Y_i. Now by Lemma 5.7, we see that (G', S') contains an odd-K_5 minor, as required. $\square$

5.3.1 Cycling

Let (G, S) be a signed graph. Weights $w \in Z_+^{E(G)}$ are called *Eulerian* if $w(\delta(v))$ is even for every $v \in V(G)$. We say that the clutter of odd cycles of (G, S) is *cycling* [62] if (6) and (5) have both optimum integer solutions for all Eulerian edge-weights. Note that the clutter of odd cycles of odd-K_5 is not cycling. However, it is the only obstruction to the property.

Theorem 5.8 (Geelen and Guenin [25]) *The clutter of odd cycles of a signed graph (G, S) is cycling if and only if (G, S) has no minor isomorphic to odd-K_5.*

Let $\mathcal{H}$ be the clutter of odd cycles of a signed graph (G, S). Suppose that $\mathcal{H}$ is cycling and let $w \in Z_+^{E(G)}$. Now, $2w$ is Eulerian, so there exists an integral optimal solution x to (5) with respect to the weights $2w$. Clearly, x is also optimal with respect to w. Hence, if $\mathcal{H}$ is cycling it is also ideal (see Corollary 4.5). Thus Theorem 5.8 implies Theorem 5.3 and the fact that for clutters of odd cycles, the property of being ideal is the same as cycling. Using the same trick as above, of doubling the edge–capacities, we also obtain the following result.

Corollary 5.9 *If the clutter of odd cycles of a signed graph is ideal then it has the 1/2-MFMC property.*

5.3.2 Odd st-Walks

Guenin [30] considers the following generalization of the odd cycle clutter. Let (G, S) be a signed graph and let s, t be two nodes of G. A subset of edges of G is an *odd st-walk* if it is an odd st-path or the union of an even st-path P and an odd cycle C where P and C share at most one node. The odd cycle clutter is obtained when $s = t$.

Guenin characterized exactly when this clutter is ideal. This generalizes Theorem 5.3.

6 Binary Clutters

A clutter is *binary* if its edges and its minimal transversals intersect in an odd number of vertices. It follows from the definition that a clutter is binary if and only if its blocker is binary. An equivalent formulation is given by Lehman.

Proposition 6.1 (Lehman [34], see also Seymour [55]) *A clutter $\mathcal{C}$ is binary if and only if, for any three edges S_1, S_2, S_3 of $\mathcal{C}$, the set $S_1 \triangle S_2 \triangle S_3$ contains an edge of $\mathcal{C}$.*

Proof: Let $\mathcal{C}$ be a binary clutter and $S = S_1 \triangle S_2 \triangle S_3$ where $S_1, S_2, S_3 \in E(\mathcal{C})$. Since every minimal transversal T has an odd intersection with S_1, S_2 and S_3, we have $S \cap T \neq \emptyset$. Therefore S contains an edge of $\mathcal{C}$.

Conversely, assume that for any three edges S_1, S_2, S_3 of $\mathcal{C}$, the set $S_1 \triangle S_2 \triangle S_3$ contains an edge of $\mathcal{C}$. We leave it as an exercise to show that, for any odd number of edges $S_1, \ldots, S_k$ of $\mathcal{C}$, the set $S_1 \triangle \ldots \triangle S_k$ contains an edge of $\mathcal{C}$. Now consider any $S \in E(\mathcal{C})$, $T \in E(b(\mathcal{C}))$ and let $S \cap T = \{x_1, \ldots, x_k\}$. Since $T - x_i$ is not a transversal of $\mathcal{C}$, there exists an edge S_i of $\mathcal{C}$ such that $T \cap S_i = \{x_i\}$. It follows that $T \cap (S \triangle S_1 \triangle \ldots \triangle S_k) = \emptyset$. Therefore $S \triangle S_1 \triangle \ldots \triangle S_k$ does not contain an edge of $\mathcal{C}$. It follows that k is odd. $\square$

Let $\mathcal{P}_4$ be the clutter with four vertices and the following three edges:

$$E(\mathcal{P}_4) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}.$$

One can easily show that neither $\mathcal{P}_4$ nor $\mathcal{J}_t$ is a binary clutter, for $t \geq 2$. Seymour proved the following.

Theorem 6.2 (Seymour [55]) *$\mathcal{C}$ is a binary clutter if and only if $\mathcal{C}$ has no minor $\mathcal{P}_4$ or $\mathcal{J}_t$, for $t \geq 2$.*

The following clutters (and their blockers!) are examples of binary clutters.

Example 6.3 *The clutter of st-cuts in a graph.*

Example 6.4 *The clutter of two-commodity cuts in a graph.*

Example 6.5 *The clutter of T-joins in a graft (G, T).*

Example 6.6 *The clutter of odd cycles in a signed graph.*

Example 6.7 *The clutter of st-T-cuts.*

Example 6.8 *The clutter of odd st-walks.*

6.1 Seymour's Conjecture

Recall (Section 4.2) that $\mathcal{F}_7$ denotes the clutter with 7 vertices and 7 edges corresponding to points and lines of the Fano plane (finite projective geometry on 7 points). It is easy to verify that $\mathcal{F}_7$ is binary, *mni* and that $b(\mathcal{F}_7) = \mathcal{F}_7$.

Let K_5 denote the complete graph on five vertices. We let $\mathcal{O}_{K_5}$ denote the binary clutter whose vertices are the edges of K_5 and whose edges are the odd cycles of K_5. So $\mathcal{O}_{K_5}$ has 10 edges of cardinality three and 12 edges of cardinality five. $\mathcal{O}_{K_5}$ is binary and *mni*. It follows that $b(\mathcal{O}_{K_5})$ is binary and *mni*.

Conjecture 6.9 (Seymour [57]) *A binary clutter is ideal if and only if it contains no $\mathcal{F}_7$, $\mathcal{O}_{K_5}$ or $b(\mathcal{O}_{K_5})$ minor.*

6.2 Binary Matroids

In the remainder of this section, we present results of Novick-Sebö [43] and Cornuéjols-Guenin [11] on ideal binary clutters. We adopt a matroidal point of view. See Oxley's excellent textbook [46] on matroid theory for background material.

A matroid is *binary* if it can be represented over $GF(2)$.

Example 6.10 *The Fano matroid F_7 has the following binary representation.*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Given a matroid M, the dual matroid is denoted by M^*. A binary matroid is *regular* if it has no F_7 or F_7^* minor (Tutte [66]).

Let M be a matroid with element set U and let k be a positive integer. A *k-separation* of M is a partition (U_1, U_2) of U such that $|U_1| \geq k$, $|U_2| \geq k$ and $r(U_1) + r(U_2) \leq r(U) + k - 1$. The matroid M is *k-connected* if it has no $(k-1)$-separation. The k-separation is *strict* if $|U_1| > k$, $|U_2| > k$. A matroid is *internally k-connected* if it has no strict $(k-1)$-separation.

Theorem 6.11 (Seymour [59]) *Every 3-connnected, internally 4-connected regular matroid is graphic, cographic or a 10-element matroid R_{10}.*

Theorem 6.12 (Seymour [60]) *Let M be a 3-connected binary matroid with no F_7 minor. Then M is regular or $M = F_7^*$.*

6.3 Signed Matroid

Let M be a binary matroid and $S \subseteq V(M)$ a subset of its elements. The pair (M, S) is called a *signed matroid*, and S is called the *signature* of M. We say that a circuit C of M is *odd* (resp. *even*) if $|C \cap S|$ is odd (resp. even).

Proposition 6.13 *The odd circuits of a signed matroid form a binary clutter.*

Proof: Consider a signed matroid (M, S) and let C_1, C_2, C_3 be three odd circuits. Since S intersects each of C_1, C_2, C_3 with odd parity, so does $L = C_1 \triangle C_2 \triangle C_3$. Since M is binary, L is a disjoint union of circuits (see for example Oxley [46] Theorem 9.1.2). One of these circuits must be odd since $|L \cap S|$ is odd. The result now follows from Proposition 6.1. □

Let M be a binary matroid. Any nontrivial binary clutter obtained as the odd circuit clutter of the signed matroid (M, S), for some S, is called a *source* of M. Any nontrivial binary clutter $\mathcal{H}$ such that every circuit of M is of the form $T_1 \Delta T_2$, for $T_1, T_2 \in E(\mathcal{H})$, is called a *lift* of M. One can show that a lift of M is the blocker of a source of M^*.

In a binary matroid, any circuit C and cocircuit D have an even intersection (see for example Oxley [46] Theorem 9.1.2). So, if D is a cocircuit, then (M, S) and $(M, S \Delta D)$ have exactly the same odd circuits.

Remark 6.14 *Let (M, S) be a signed matroid and $\mathcal{H}$ the clutter of its odd circuits.*

- *$\mathcal{H} \setminus e$ is the clutter of odd circuits of the signed matroid $(M \setminus e,\ S - \{e\})$.*
- *If $e \notin S$, then $\mathcal{H}/e$ is the clutter of odd circuits of the signed matroid $(M/e,\ S)$.*

- *If $e \in S$ is not a loop of M, then $\mathcal{H}/e$ is the clutter of odd circuits of the signed matroid $(M/e,\ S\Delta D)$ where D is any cocircuit containing e.*
- *If $e \in S$ is a loop of M, then $\mathcal{H}/e$ is a trivial clutter.*

Given a nontrivial binary clutter $\mathcal{H}$, the minimal sets in $E(\mathcal{H}) \cup \{T_1 \Delta T_2 : T_1, T_2 \in E(\mathcal{H})\}$ form the circuits of a binary matroid $u(\mathcal{H})$. This binary matroid is called the *up matroid* of $\mathcal{H}$. Since $\mathcal{H}$ is binary, the minimal transversals of $\mathcal{H}$ intersect with odd parity exactly the circuits of $u(\mathcal{H})$ that are edges of $\mathcal{H}$. It follows that $\mathcal{H}$ is the clutter of odd circuits of the signed matroid $(u(\mathcal{H}), S)$ where S is any minimal transversal of $\mathcal{H}$. Moreover, this representation is essentially unique (see for example [11]):

Proposition 6.15 *Let (M, S) and (M', S') be signed matroids that have the same clutter of odd circuits $\mathcal{H}$. If M is a 2-connected matroid and $\mathcal{H}$ is a nontrivial clutter, then $M = M' = u(\mathcal{H})$.*

To prove this, we use the following result of Lehman [34] (see Oxley [46] Theorem 4.3.2 or Exercise 9 of Section 9.3).

Theorem 6.16 (Lehman [34]) *Let t be an element of a 2-connected binary matroid M. The circuits of M not containing t are of the form $C_1 \Delta C_2$ where C_1 and C_2 are circuits of M containing t.*

Proof of Proposition 6.15: Let N be the binary matroid with elements $V(M) \cup \{t\}$ and circuits $\Gamma = C$ when C is an even circuit of (M, S) and $\Gamma = C \cup \{t\}$ when C is an odd circuit of (M, S). Define N' similarly from (M', S'). Since $\mathcal{H}$ is nontrivial, at least one circuit of N contains t and some $x \neq t$. Since M is 2-connected, for every pair of elements in $V(M)$, there is a circuit of M containing both. So for x and any $v \in V(N)$, there is a circuit of N containing both. It follows that, for any pair of elements in $V(N)$, there is a circuit containing both. So N is 2-connected. Furthermore, every $v \in V(\mathcal{H})$ belongs to an edge of $\mathcal{H}$. So N' is 2-connected as well. By Theorem 6.16, a 2-connected matroid is uniquely determined by the set of circuits containing any fixed element. In particular, N and N' are uniquely determined by the circuits containing t. This implies $N = N'$. Since $M = N/t$ and $M' = N'/t$, it follows that $M = M' = u(\mathcal{H})$. $\square$

Proposition 6.17 (Novick and Sebö [43])

A binary clutter $\mathcal{H}$ is the odd cycle clutter of a signed graph if and only if $u(\mathcal{H})$ is a graphic matroid.

A binary clutter $\mathcal{H}$ is the T-cut clutter of a graft if and only if $u(\mathcal{H})$ is a cographic matroid.

The next result relates the minors of the matroid $u(\mathcal{H})$ to the minors of the clutter $\mathcal{H}$. For a clutter $\mathcal{H}$ and $v \notin V(\mathcal{H})$, the clutter $\mathcal{H}^+$ has vertex set $V(\mathcal{H}) \cup \{v\}$ and edge set $\{A \cup \{v\} : A \in E(\mathcal{H})\}$.

Theorem 6.18 (Cornuéjols and Guenin [11]) *Let $\mathcal{H}$ be a nontrivial binary clutter such that its up matroid $u(\mathcal{H})$ is 2-connected, and let N be a 2-connected binary matroid. Then $u(\mathcal{H})$ has N as a minor if and only if $\mathcal{H}$ has $\mathcal{H}_1$ or $\mathcal{H}_2^+$ as a minor, where $\mathcal{H}_1$ is a source of N and $\mathcal{H}_2$ is a lift of N.*

To prove this, we use the following result of Brylawski [3] and Seymour [56] (see Oxley [46] Proposition 4.3.6).

Theorem 6.19 (Brylawski [3], Seymour [56]) *Let M be a 2-connected matroid and N a 2-connected minor of M. For any $i \in V(M) - V(N)$, at least one of $M \setminus i$ or M/i is 2-connected and has N as a minor.*

Proof of Theorem 6.18: $\mathcal{H}$ is the clutter of odd circuits of the signed matroid (M, S) where $M = u(\mathcal{H})$ and S is a minimal transversal of $\mathcal{H}$.

Suppose first that $\mathcal{H}$ has a minor $\mathcal{H}_1$ that is a source of N. Then $\mathcal{H}_1$ is nontrivial and it follows from Remark 6.14 that $\mathcal{H}_1$ is the clutter of odd circuits of a signed matroid (N', S') where N' is a minor of M. Since $\mathcal{H}_1$ is nontrivial and N is 2-connected, $N = N' = u(\mathcal{H}_1)$ by Proposition 6.15. So N is a minor of M.

Suppose now that $\mathcal{H}$ has a minor $\mathcal{H}_2^+$ where $\mathcal{H}_2$ is a lift of N. Let t be the vertex of $V(\mathcal{H}_2^+) - V(\mathcal{H}_2)$. Since $\mathcal{H}_2^+$ is a nontrivial minor of $\mathcal{H}$, it is the clutter of odd circuits of a signed matroid (N', S') where N' is a minor of M. Since $u(\mathcal{H}_2^+)$ is 2-connected, $N' = u(\mathcal{H}_2^+)$ by Proposition 6.15. So N' is 2-connected. Therefore, by Theorem 6.16 and the definition of lift, $N = N' \setminus t$. So N is a minor of M.

Now we prove the converse. Suppose that M has N as minor and does not satisfy the theorem. Let $\mathcal{H}$ be such a counterexample with smallest number of vertices. Clearly, N is a proper minor of M as otherwise $u(\mathcal{H}) = N$, i.e. $\mathcal{H}$ is a source of N. By Theorem 6.19, for every $i \in V(M) - V(N)$, one of $M \setminus i$ and M/i is 2-connected and has N as a minor. Suppose first that M/i is 2-connected and has an N minor. Since M is 2-connected, i is not a loop of M and therefore $\mathcal{H}/i$ is nontrivial by Remark 6.14, a contradiction to the choice of $\mathcal{H}$ with smallest number of vertices. Thus, for every $i \in V(M) - V(N)$, $M \setminus i$ is 2-connected and has an N minor. By minimality, $\mathcal{H} \setminus i$ must be trivial. It follows from Remark 6.14 that all odd circuits of (M, S) use i. As $M = u(\mathcal{H})$, even circuits of M do not use i.

We claim that $V(M) - V(N) = \{i\}$. Suppose not and let $j \neq i$ be an element of $V(M) - V(N)$. The set of circuits of (M, S) using j is exactly the set of odd circuits. It follows that the elements i, j must be in series in M. But then $M \setminus i$ is not connected, a contradiction.

Therefore $V(M) - V(N) = \{i\}$ and $M \setminus i = N$. As the circuits of (M, S) using i are exactly the odd circuits of (M, S), it follows that column i of $\mathcal{H}$ consists of all 1's, i.e. $\mathcal{H} = \mathcal{H}_2^+$. By Theorem 6.16 applied to i and M, every circuit of N is of the form $T_1 \Delta T_2$ where $T_1, T_2 \in E(\mathcal{H}_2)$. So $\mathcal{H}_2$ is a lift of N. □

6.4 k-Connectedness of Binary Clutters

A binary clutter $\mathcal{H}$ has a *k-separation* if $u(\mathcal{H})$ has a k-separation, i.e. there exists a partition (U_1, U_2) of $V(\mathcal{H})$ such that $|U_1| \geq k$, $|U_2| \geq k$ and $r(U_1) + r(U_2) \leq r(V(\mathcal{H})) + k - 1$. The k-separation is *strict* if $|U_1| > k$, $|U_2| > k$. The binary clutter $\mathcal{H}$ is *k-connected* if it has no $(k-1)$-separation. It is *internally k-connected* if it has no strict $(k-1)$-separation.

Theorem 6.20 [11] *Minimally nonideal binary clutters are 3-connected.*

The minimally nonideal binary clutter F_7 has a 3-separation. So minimally nonideal clutters are not 4-connected in general. However they are internally 4-connected.

Theorem 6.21 [11] *Minimally nonideal binary clutters are internally 4-connected.*

Conjecture 6.22 *Minimally nonideal binary clutters are internally 5-connected.*

Let Q_6 be the clutter where $V(Q_6)$ is the set of edges of K_4 and $E(Q_6)$ the set of triangles of K_4. The next result proves Seymour's conjecture (Conjecture 6.9) for the class of clutters that do not have $\mathcal{Q}_6^+$ or $b(\mathcal{Q}_6)^+$ as a minor.

Theorem 6.23 (Cornuéjols and Guenin [11]) *A binary clutter is ideal if it does not have $\mathcal{F}_7$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, $\mathcal{Q}_6^+$, or $b(\mathcal{Q}_6)^+$ as a minor.*

Proof: It suffices to show that every *mni* clutter $\mathcal{H}$ contains one of the minors in the statement of the theorem.

Claim 1: The result holds if $u(\mathcal{H})$ has no F_7^* minor.

Proof of claim: When $u(\mathcal{H}) = R_{10}$, then $\mathcal{H}$ is one of the sources of R_{10}. We leave it as an exercise to show that R_{10} has 6 sources. One such source is $b(\mathcal{O}_{K_5})$ and the other five are ideal.

When $u(\mathcal{H})$ is graphic, then $\mathcal{H}$ is ideal if and only if $\mathcal{H}$ has no $\mathcal{O}_{K_5}$ minor, by Proposition 6.17 of Novick-Sebö and Guenin's theorem (Theorem 5.3).

When $u(\mathcal{H})$ is cographic, then $\mathcal{H}$ is ideal, by Proposition 6.17 of Novick-Sebö and the Edmonds-Johnson theorem (Theorem 3.1).

By the connectivity results (Theorems 6.20 and 6.21), $u(\mathcal{H})$ is 3-connected and internally 4-connected. So, by Seymour's theorem (Theorem 6.11), the result holds when $u(\mathcal{H})$ is a regular matroid.

Now consider the case when $u(\mathcal{H})$ is not regular. Another theorem of Seymour (Theorem 6.12) shows that $u(\mathcal{H}) = F_7$. So $\mathcal{H}$ is a source of F_7. It is easy to verify that F_7 has three sources. Two of these sources are ideal and the third is the clutter $\mathcal{F}_7$. So the result holds. ◇

Claim 2: The result holds if $u(\mathcal{H})$ has an F_7^* minor.

Proof of claim: By Theorem 6.18, $u(\mathcal{H})$ has an F_7^* minor if and only if $\mathcal{H}$ has $\mathcal{H}_1$ or $\mathcal{H}_2^+$ as a minor, where $\mathcal{H}_1$ is a source of F_7^* and $\mathcal{H}_2$ is a lift of F_7^*. On can easily verify that F_7^* has one source and three lifts. The source is Q_6^+, which is one of the excluded minors in the statement of the theorem. For the three lifts $\mathcal{H}_2$ of F_7^*, one can check that $\mathcal{H}_2^+$ contains $\mathcal{F}_7$, $\mathcal{Q}_6^+$ and $b(\mathcal{Q}_6)^+$ as minors, respectively, which are excluded minors in the statement of the theorem. $\square$

The class of clutters of T-cuts is closed under minor taking. Moreover, it is not hard to check that none of the five excluded minors of Theorem 6.23 are clutters of T-cuts. Thus Theorem 6.23 implies that clutters of T-cuts are ideal, and thus that their blocker, the clutters of T-joins are ideal. Hence Theorem 6.23 implies the Edmonds-Johnson theorem (Theorem 3.1). Similarly, the class of clutters of odd circuits is closed under minor taking. Moreover, it can be shown that $\mathcal{O}_{K_5}$ is the only clutter of odd circuits among the five excluded minors. It follows that Theorem 6.23 also implies Guenin's theorem (Theorem 5.3). Note, however, that the proof of Theorem 6.23 uses these two results.

7 Ideal $0, \pm 1$ Matrices

The concept of ideal 0,1 matrix can be extended to a $0, \pm 1$ matrix. Given a $0, \pm 1$ matrix A, denote by $n(A)$ the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A. The $0, \pm 1$ matrix A is *ideal* if its fractional generalized set covering polytope $Q(A) = \{x : \ Ax \geq \mathbf{1} - n(A), \ 0 \leq x \leq \mathbf{1}\}$ only has integral extreme points.

7.1 Propositional Logic

In propositional logic, *atomic propositions* $x_1, \dots, x_j, \dots, x_n$ can be either *true* or *false*. A *truth assignment* is an assignment of "true" or "false" to every atomic proposition. A *literal* is an atomic proposition x_j or its negation $\neg x_j$. A *clause* is a disjunction of literals and is *satisfied* by a given truth assignment if at least one of its literals is true.

A survey of the connections between propositional logic and integer programming can be found in Hooker [31], Truemper [65] or Chandru and Hooker [5].

A truth assignment satisfies the set S of clauses

$$\bigvee_{j \in P_i} x_j \vee (\bigvee_{j \in N_i} \neg x_j) \ \ for\ all\ i \in S$$

if and only if the corresponding $0, 1$ vector satisfies the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} x_j \geq 1 - |N_i| \ \ for\ all\ i \in S.$$

The above system of inequalities is of the form

$$Ax \geq \mathbf{1} - n(A). \tag{13}$$

Given a set S of clauses, the *satisfiability problem* (SAT) consists in finding a truth assignment that satisfies all the clauses in S or show that none exists. Equivalently, SAT consists in finding a $0,1$ solution x to (13) or show that none exists.

Given a set S of clauses (the premises) and a clause C (the conclusion), *logical inference* in propositional logic consists of deciding whether every truth assignment that satisfies all the clauses in S also satisfies the conclusion C.

To the clause C, using transformation (13), we associate an inequality

$$cx \geq 1 - n(c),$$

where c is a $0, +1, -1$ vector. Therefore C cannot be deduced from S if and only if the integer program

$$min\ \{cx : Ax \geq \mathbf{1} - n(A),\ x \in \{0,1\}^n\} \tag{14}$$

has a solution with value $-n(c)$.

The above problems are NP-hard in general but can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [4],[64]. A set S of clauses is *ideal* if the corresponding $0, \pm 1$ matrix A defined in (13) is ideal. If S is ideal, it follows from the definition that the satisfiability and logical inference problems can be solved by linear programming.

Remark 7.1 *Let S be an ideal set of clauses. If every clause of S contains more than one literal then, for every atomic proposition x_j, there exist at least two truth assignments satisfying S, one in which x_j is true and one in which x_j is false.*

Proof: Since the point $x_j = 1/2,\ j = 1, \ldots, n$ belongs to the polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ and $Q(A)$ is an integral polytope, then the above point can be expressed as a convex combination of $0,1$ vectors in $Q(A)$. Clearly, for every index j, there exists in the convex combination a $0,1$ vector with $x_j = 0$ and another with $x_j = 1$. $\square$

Let S be an ideal set of clauses. A consequence of Remark 7.1 is that the satisfiability problem can be solved more efficiently than by general linear programming.

Theorem 7.2 (Conforti and Cornuéjols [7]) *Let S be an ideal set of clauses. Then S is satisfiable if and only if a recursive application of the following procedure stops with an empty set of clauses.*

Recursive Step

If $S = \emptyset$, then S is satisfiable.

If S contains a clause C with a single literal (unit clause), *set the corresponding atomic proposition x_j so that C is satisfied. Eliminate from S all clauses that become satisfied and remove x_j from all the other clauses. If a clause becomes empty, then S is not satisfiable* (unit resolution).

If every clause in S contains at least two literals, choose any atomic proposition x_j appearing in a clause of S and add to S an arbitrary clause x_j or $\neg x_j$.

It is easy to modify the above algorithm in order to solve the logical inference problem when S is an ideal set of clauses.

7.2 Relating Ideal $0, \pm 1$ Matrices to Ideal 0,1 Matrices

This section follows [8]. Hooker [32] was the first to relate idealness of a $0, \pm 1$ matrix to that of a family of 0,1 matrices. These results were strengthened by Guenin [27] and by Nobili, Sassano [41].

A *prime implication* of $Q(A)$ is a generalized set covering inequality $ax \geq 1 - n(a)$ that is satisfied by all the $0,1$ vectors in $Q(A)$ but is not dominated by any other such generalized set covering inequality. A *row monotonization* of A is any $0,1$ matrix obtained from a row submatrix of A by multiplying some of its columns by -1. A row monotonization of A is *maximal* if it is not a proper submatrix of any row monotonization of A.

Theorem 7.3 (Hooker [32]) *If A is a $0, \pm 1$ matrix such that $Q(A)$ contains all of its prime implications, then A is ideal if and only if all the maximal row monotonizations of A are ideal $0,1$ matrices.*

In [27], the idealness of a $0, \pm 1$ matrix A is linked to the idealness of a single $0,1$ matrix as follows. Given a $0, \pm 1$ matrix A, let P and R be $0,1$ matrices of the same dimension as A, such that $P_{ij} = 1$ if and only if $A_{ij} = 1$, and $R_{ij} = 1$ if and only if $A_{ij} = -1$. The matrix

$$D_A = \left[\begin{array}{c|c} P & R \\ \hline I & I \end{array}\right],$$

is the $0,1$ *extension* of A. Note that the transformation $x^+ = x$ and $x^- = 1 - x$ maps every vector x in $Q(A)$ into a vector in $\{(x^+, x^-) \geq 0 : \ Px^+ + Rx^- \geq 1, \ x^+ + x^- = 1\}$. So $Q(A)$ corresponds to the face of $Q(D_A)$, obtained by setting the inequalites $x^+ + x^- \geq 1$ at equality.

Theorem 7.4 (Guenin [27]) *Let A be a $0, \pm 1$ matrix such that $Q(A)$ contains all of its prime implications. Then A is ideal if and only if the $0,1$ matrix D_A is ideal.*

Furthermore A is ideal if and only if $min\{cx : \ x \in Q(A)\}$ has an integer optimum for every vector $c \in \{0, \pm 1, \pm\infty\}^n$.

In Nobili, Sassano [41], a condition for a $0, \pm 1$ matrix A to be ideal, without assuming that $Q(A)$ contains all of its prime implications is given as follows. Given a $0, \pm 1$ matrix A, let a^1 and a^2 be two rows of A, such that there is one index k such that $a_k^1 a_k^2 = -1$ and, for all $j \neq k$, $a_j^1 a_j^2 = 0$. A *disjoint implication* of A is the $0, \pm 1$ vector $a^1 + a^2$. The matrix A^+ obtained by recursively adding all disjoint implications and removing all dominated rows is called the *disjoint completion* of A.

Theorem 7.5 (Nobili and Sassano [41]) *Let A be a $0, \pm 1$ matrix. Then A is ideal if and only if D_{A^+} is an ideal $0, 1$ matrix, where A^+ is the disjoint completion of A.*

Let J be a subset of columns of a $0, \pm 1$ matrix A. The *deletion* of J consists of removing all columns in J, all rows with at least one 1 in a column of J and rows that become dominated. The *contraction* of J consists of removing all columns in J, all rows with at least one -1 in a column of J and rows that become dominated. The *semi-deletion* of J consists of removing all rows with a 1 in at least one column of J and then all zero columns. The *semi-contraction* of J consists of removing all rows with at least one -1 in a column of J and then all zero columns.

Nobili and Sassano define a *weak minor* of a $0, \pm 1$ matrix A to be any submatrix that can be obtained from A by a sequence of deletions, contractions, semi-deletions and semi-contractions. They define A to be *minimally nonideal* if A is not ideal but every weak minor of A is ideal. The usefulness of this concept comes from the fact that a $0, \pm 1$ matrix A is minimally nonideal if and only D_A is a minimally nonideal $0, 1$ matrix.

For $n \geq 3$, the following $n \times n$ $0, \pm 1$ matrix, denoted $\tilde{J}_n$, is minimally nonideal:

$$\tilde{J}_n = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Nobili and Sassano [41] give the following characterization of minimally nonideal $0, \pm 1$ matrices.

Theorem 7.6 (Nobili and Sassano [41]) *Let A be a $0, \pm 1$ matrix with n columns. Then A is minimally nonideal if and only if A is a switching of $\tilde{J}_n$, after permutation of rows and columns, or A is a switching of a minimally nonideal $0, 1$ matrix or A contains an $n \times n$ submatrix B with two nonzeroes per row and per column and $det(B) = \pm 2$ and all rows in A but not in B have at least three nonzeroes.*

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