

A Counterexample to a Conjecture of Gomory and Johnson

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Abstract

In *Mathematical Programming* 2003, Gomory and Johnson conjecture that the facets of the infinite group problem are always generated by piecewise linear functions. In this paper we give an example showing that the Gomory-Johnson conjecture is false.

1 Introduction

Let $f \in]0, 1[$ be given. Consider the following infinite group problem (Gomory and Johnson [3]) with a single equality constraint and a nonnegative integer variable s_r associated with each real number $r \in [0, 1[$.

$$\begin{aligned} \sum_{r \in [0, 1[} r s_r &= f \\ s_r &\in \mathbb{Z}_+ \quad \forall r \in [0, 1[\\ s &\text{ has finite support,} \end{aligned} \tag{1}$$

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where additions are performed modulo 1. Vector s has *finite support* if $s_r \neq 0$ for a finite number of distinct $r \in [0, 1[$.

Gomory and Johnson [3] say that a function $\pi : [0, 1[\rightarrow \mathbb{R}$ is a *valid function* for (1) if π is nonnegative, $\pi(0) = 0$, and every solution of (1) satisfies

$$\sum_{r \in [0, 1[} \pi(r) s_r \geq 1.$$

Note that, if π is a valid function, then $\pi(f) \geq 1$.

For any valid function π , let $P(\pi)$ be the set of solutions which satisfy $\sum_{r \in [0, 1[} \pi(r) s_r \geq 1$ at equality. A valid function π is a *facet* for (1) if, for any valid function π^* , $P(\pi^*) \supset P(\pi)$ implies $\pi^* = \pi$.

A function is *piecewise linear* if there are finitely many values $0 = r_0 < r_1 < \dots < r_k = 1$ such that the function is of the form $\pi(r) = a_j r + b_j$ in interval $[r_{j-1}, r_j[$, for $j = 1, \dots, k$. The *slopes* of a piecewise linear function are the different values of a_j for $j = 1, \dots, k$.

Gomory and Johnson [5] gave many examples of continuous facets, and in all their examples these functions are piecewise linear. This led them to formulate the following conjecture, which they describe as “important and challenging”.

Conjecture 1.1 (Facet Conjecture). *Every continuous facet for (1) is piecewise linear.*

We show that the above conjecture is false by exhibiting a facet for (1) that is continuous but not piecewise linear.

1.1 Literature overview

The definition of valid function we stated, which is the one given by Gomory and Johnson in [3], differs from the one adopted later by the same two authors in [5], where they included in the definition the further assumption that the function π be continuous. So a “valid function” in [5] corresponds to a “continuous valid function” from [3] and the present paper. Dey et al. [2] show that there are facets that are not continuous.

The definition of facet given in our paper is identical to the one given by Gomory and Johnson in [5], with the caveat that in [5] valid functions are required to be continuous. Thus, given a continuous valid function π , π would be a facet according to the definition in [5] if and only if $P(\pi^*) \supset P(\pi)$ implies $\pi^* = \pi$ for every *continuous* valid function π^* . For clarity, we refer to a continuous valid function satisfying the latter property as a *facet in the sense of Gomory-Johnson*. By definition, if a continuous valid function is a facet, it is also a facet in the sense of Gomory-Johnson. Thus, the conjecture they state in [5] is actually that *every facet in the sense of Gomory-Johnson is piecewise linear*. The above discussion shows that a counterexample to Conjecture 1.1 also disproves the conjecture in [5].

In earlier work, Gomory and Johnson [3] emphasized extreme functions rather than facets. A valid function π is *extreme* if it cannot be expressed as a convex combination of two distinct valid functions. It follows from the definition that facets are extreme. Therefore our counterexample also provides an extreme function that is continuous but not piecewise linear. Gomory and Johnson [5] write in a footnote that the definition of facet “is different from, although eventually equivalent to,” the definition of extreme function. The statement

that extreme functions are facets appears to be quite nontrivial to prove, and to the best of our knowledge there is no proof in the literature. We therefore cautiously treat extreme functions and facets as distinct concepts, and leave their equivalence as an open question.

We obtain the counterexample to Conjecture 1.1 by exhibiting a sequence of piecewise linear functions and then considering the pointwise limit of this sequence of functions. These functions were discovered by Kianfar and Fathi [6] who show that they are facets for the infinite group problem. They call them the n -step MIR (Mixed-Integer Rounding) functions in their paper. Our treatment and analysis in this paper is different from theirs. The emphasis in [6] was on deriving valid inequalities for MILPs which are generalizations of the standard MIR inequalities. In this paper, we use this class of functions primarily to construct a counterexample to Conjecture 1.1.

2 Preliminaries

A valid function $\pi : [0, 1[\rightarrow \mathbb{R}$ is *minimal* if there is no valid function π' such that 1) $\pi'(a) \leq \pi(a)$ for all $a \in [0, 1[$, and 2) the inequality is strict for at least one a . If π is a minimal valid function, then $\pi(r) \leq 1$ for every $r \in [0, 1[$, as follows immediately from the fact that π is nonnegative.

When convenient, we will extend the domain of definition of the function π to the whole real line \mathbb{R} by making the function periodic: $\pi(x) = \pi(x + k)$ for any $x \in [0, 1[$ and $k \in \mathbb{Z}$.

A function $\pi : \mathbb{R} \rightarrow \mathbb{R}$ is *subadditive* if for every $a, b \in \mathbb{R}$

$$\pi(a + b) \leq \pi(a) + \pi(b).$$

Given $f \in]0, 1[$, a function $\pi : [0, 1[\rightarrow \mathbb{R}$ is *symmetric* if for every $a \in [0, 1[$

$$\pi(a) + \pi(f - a) = 1.$$

Gomory and Johnson prove the following result in [3].

Theorem 2.1 (Minimality Theorem). *Let $\pi : [0, 1[\rightarrow \mathbb{R}$ be such that $\pi(0) = 0$ and $\pi(f) = 1$. A necessary and sufficient condition for π to be valid and minimal is that π is subadditive and symmetric.*

Any facet for (1) is minimal. Therefore if π is a facet for (1), then π is subadditive and symmetric. The following two facts are well-known and will be useful in our arguments.

Fact 2.2. *If π_1 and π_2 are subadditive, then $\pi_1 + \pi_2$ is subadditive.*

Fact 2.3. *Let π be a subadditive function and define $\pi'(x) = \alpha\pi(\beta x)$ for some constants $\alpha > 0$ and β . Then π' is subadditive.*

Proof. $\pi'(a + b) = \alpha\pi(\beta(a + b)) \leq \alpha(\pi(\beta a) + \pi(\beta b)) = \pi'(a) + \pi'(b)$. □

A function π' defined as $\pi'(x) = \alpha\pi(\beta x)$ will be referred to as a *scaling* of π .

Let $E(\pi)$ denote the set of all possible of inequalities $\pi(u_1) + \pi(u_2) \geq \pi(u_1 + u_2)$ that are satisfied as *equalities* by π . Here u_1 and u_2 are any real numbers. The following theorem is proved in Gomory and Johnson [5] and is used in this paper to prove that certain inequalities are facets.

Theorem 2.4 (Facet Theorem). *Let π be a minimal valid function. If there is no minimal valid function that satisfies the equalities in $E(\pi)$ other than π itself, then π is a facet.*

We remark that, even though Theorem 2.4 is proved in [5] under the assumptions that valid functions are continuous, the continuity assumption is not needed in the proof, thus the statement remains true even in the setting of the present paper.

In the paper we need the following lemma, which is a variant of the Interval Lemma stated in Gomory and Johnson [5]. They prove the lemma under the assumption that the function in the statement is continuous, whereas we only require the function to be bounded on every interval. Other variants of the Interval Lemma that do not require the function to be continuous have been given by Dey et al. [2]. The proof we give is in the same spirit of the solution of Cauchy's Equation (see for example Chapter 2 of Aczél [1]).

Lemma 2.5 (Interval Lemma). *Let $\pi : \mathbb{R} \rightarrow \mathbb{R}$ be a function bounded on every bounded interval. Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$.*

If $\pi(u) + \pi(v) = \pi(u + v)$ for every $u \in U$ and $v \in V$, then there exists $c \in \mathbb{R}$ such that $\pi(u) = \pi(u_1) + c(u - u_1)$ for every $u \in U$, $\pi(v) = \pi(v_1) + c(v - v_1)$ for every $v \in V$, $\pi(w) = \pi(u_1 + v_1) + c(w - u_1 - v_1)$ for every $w \in U + V$.

Proof. We first show the following.

Claim 1. *Let $u \in U$, and let $\varepsilon > 0$ such that $v_1 + \varepsilon \in V$. For every nonnegative integer p such that $u + p\varepsilon \in U$, we have $\pi(u + p\varepsilon) - \pi(u) = p(\pi(v_1 + \varepsilon) - \pi(v_1))$.*

For $h = 1, \dots, p$, by hypothesis $\pi(u + h\varepsilon) + \pi(v_1) = \pi(u + h\varepsilon + v_1) = \pi(u + (h-1)\varepsilon) + \pi(v_1 + \varepsilon)$. Thus $\pi(u + h\varepsilon) - \pi(u + (h-1)\varepsilon) = \pi(v_1 + \varepsilon) - \pi(v_1)$, for $h = 1, \dots, p$. By summing the above p equations, we obtain $\pi(u + p\varepsilon) - \pi(u) = p(\pi(v_1 + \varepsilon) - \pi(v_1))$. This concludes the proof of Claim 1.

Let $\bar{u}, \bar{u}' \in U$ such that $\bar{u} - \bar{u}' \in \mathbb{Q}$ and $\bar{u} > \bar{u}'$. Define $c := \frac{\pi(\bar{u}) - \pi(\bar{u}')}{\bar{u} - \bar{u}'}$.

Claim 2. *For every $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $\pi(u) - \pi(u') = c(u - u')$.*

We only need to show that, given $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $\pi(u) - \pi(u') = c(u - u')$. We may assume $u > u'$. Choose a positive rational ε such that $\bar{u} - \bar{u}' = \bar{p}\varepsilon$ for some integer \bar{p} , $u - u' = p\varepsilon$ for some integer p , and $v_1 + \varepsilon \in V$. By Claim 1,

$$\pi(\bar{u}) - \pi(\bar{u}') = \bar{p}(\pi(v_1 + \varepsilon) - \pi(v_1)) \quad \text{and} \quad \pi(u) - \pi(u') = p(\pi(v_1 + \varepsilon) - \pi(v_1)).$$

Dividing the last equality by $u - u'$ and the second to last by $\bar{u} - \bar{u}'$, we get

$$\frac{\pi(v_1 + \varepsilon) - \pi(v_1)}{\varepsilon} = \frac{\pi(\bar{u}) - \pi(\bar{u}')}{\bar{u} - \bar{u}'} = \frac{\pi(u) - \pi(u')}{u - u'} = c.$$

Thus $\pi(u) - \pi(u') = c(u - u')$. This concludes the proof of Claim 2.

Claim 3. *For every $u \in U$, $\pi(u) = \pi(u_1) + c(u - u_1)$.*

Let $\delta(x) = \pi(x) - cx$. We show that $\delta(u) = \delta(u_1)$ for all $u \in U$ and this proves the claim. Since π is bounded on every bounded interval, δ is bounded over U, V and $U + V$. Let M be a number such that $|\delta(x)| \leq M$ for all $x \in U \cup V \cup (U + V)$.

Suppose by contradiction that, for some $u^* \in U$, $\delta(u^*) \neq \delta(u_1)$. Let N be a positive integer such that $|N(\delta(u^*) - \delta(u_1))| > 2M$.

By Claim 2, $\delta(u^*) = \delta(u)$ for every $u \in U$ such that $u^* - u$ is rational. Thus there exists \bar{u} such that $\delta(\bar{u}) = \delta(u^*)$, $u_1 + N(\bar{u} - u_1) \in U$ and $v_1 + \bar{u} - u_1 \in V$. Let $\bar{u} - u_1 = \varepsilon$. By Claim 1,

$$\delta(u_1 + N\varepsilon) - \delta(u_1) = N(\delta(v_1 + \varepsilon) - \delta(v_1)) = N(\delta(u_1 + \varepsilon) - \delta(u_1)) = N(\delta(\bar{u}) - \delta(u_1))$$

Thus $|\delta(u_1 + N\varepsilon) - \delta(u_1)| = |N(\delta(\bar{u}) - \delta(u_1))| = |N(\delta(u^*) - \delta(u_1))| > 2M$, which implies $|\delta(u_1 + N\varepsilon)| + |\delta(u_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between U and V , Claim 3 implies that there exists some constant c' such that, for every $v \in V$, $\pi(v) = \pi(v_1) + c'(v - v_1)$. We show $c' = c$. Indeed, given $\varepsilon > 0$ such that $u_1 + \varepsilon \in U$ and $v_1 + \varepsilon \in V$, $c\varepsilon = \pi(u_1 + \varepsilon) - \pi(u_1) = \pi(v_1 + \varepsilon) - \pi(v_1) = c'\varepsilon$, where the second equality follows from Claim 1.

Therefore, for every $v \in V$, $\pi(v) = \pi(v_1) + c\pi(v - v_1)$. Finally, since $\pi(u) + \pi(v) = \pi(u + v)$ for every $u \in U$ and $v \in V$, for every $w \in U + V$, $\pi(w) = \pi(u_1 + v_1) + c(w - u_1 - v_1)$. \square

The following theorem, due to Gomory and Johnson [5], gives a class of facets.

Theorem 2.6. *Let $\pi : [0, 1[\rightarrow \mathbb{R}$ be a minimal valid function that is piecewise linear. If π has only two distinct slopes, then π is a facet.*

Again, we note that Theorem 2.6 is proved in [5] under the assumptions that valid functions are continuous. However, in [5] the continuity assumption is used in the proof only when applying the Interval Lemma. Since our version of the Interval Lemma (Lemma 2.5) applies to any bounded function, and since minimal valid functions are bounded, Theorem 2.6 is valid also in the setting of the present paper. Gomory and Johnson give in [4] a very similar statement to the one of Theorem 2.6, namely that piecewise linear minimal valid functions with only two distinct slopes are extreme.

3 The construction

We first define a sequence of valid functions $\psi_i : [0, 1[\rightarrow \mathbb{R}$ that are piecewise linear, and then consider the limit ψ of this sequence. We will then show that ψ is a facet but not piecewise linear.

Let $0 < \alpha < 1$. ψ_0 is the triangular function given by

$$\psi_0(x) = \begin{cases} \frac{1}{\alpha}x & 0 \leq x \leq \alpha \\ \frac{1-x}{1-\alpha} & \alpha \leq x < 1. \end{cases}$$

Notice that the corresponding inequality $\sum_{r \in [0, 1[} \psi_0(r)s_r \geq 1$ defines the Gomory mixed-integer inequality if one views (1) as a relaxation of the simplex tableau of an integer program.

We first fix a nonincreasing sequence of positive real numbers ϵ_i , for $i = 1, 2, 3, \dots$, such that $\epsilon_1 \leq 1 - \alpha$ and

$$\sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i < \alpha. \tag{2}$$

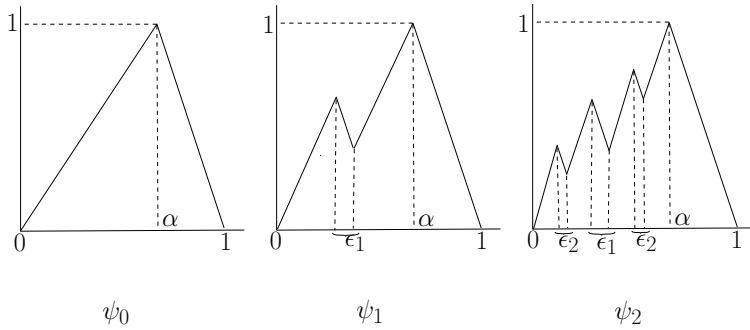


Figure 1: First two steps in the construction of the limit function

For example, $\epsilon_i = \alpha(\frac{1}{4})^i$ is such a sequence when $0 < \alpha \leq \frac{4}{5}$. The upper bound of $\frac{4}{5}$ for α is implied by the fact that $\epsilon_1 \leq 1 - \alpha$.

We construct ψ_{i+1} from ψ_i by modifying each segment with positive slope in the graph of ψ_i as follows.

For every maximal (with respect to set inclusion) interval $[a, b] \subseteq [0, \alpha]$ where ψ_i has constant positive slope we replace the line segment from $(a, \psi_i(a))$ to $(b, \psi_i(b))$ with the following three segments.

- The segment connecting $(a, \psi_i(a))$ and $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- The segment connecting $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- The segment connecting $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \psi_i(b))$.

Figure 1 shows the transformation of ψ_0 to ψ_1 and ψ_1 to ψ_2 .

The function ψ which we show to be a facet but not piecewise linear is defined as the limit of this sequence of functions, namely

$$\psi(x) = \lim_{i \rightarrow \infty} \psi_i(x) \quad (3)$$

This limit is well defined when (2) holds, as shown in Section 5.

In the next section we show that each function ψ_i is well defined and is a facet. In Section 5 we analyze the limit function ψ , showing that it is well defined, is a facet, but is not piecewise linear.

As discussed in the Introduction, the sequence ψ_i defines a class of facets which was also discovered independently by Kianfar and Fathi in [6] where they are referred to as n -step MIR functions. Their constructions are a little more general than the class of facets defined by the ψ_i 's. Our analysis in the next section is different from their treatment of these functions.

4 Analysis of the function ψ_i

Fact 4.1. For $i \geq 0$, ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope. Furthermore:

1. There is one negative slope interval of length $1 - \alpha$ and there are 2^{k-1} negative slope intervals of length ϵ_k for $k = 1, \dots, i$;
2. The negative slope pieces have slope $-\frac{1}{1-\alpha}$;
3. Each positive slope interval has length $\frac{\gamma_i}{2^i}$, where $\gamma_i = \alpha - \sum_{k=1}^i 2^{k-1}\epsilon_k$;
4. The positive slope pieces have slope $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$;
5. The function ψ_i is well-defined.

Proof. The fact that ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope, and facts 1. and 2. are immediate by construction.

Therefore the sum of the lengths of the negative slope intervals is $1 - \alpha + \sum_{k=1}^i 2^{k-1}\epsilon_k$. Since ψ_i contains 2^i positive slope intervals with the same length, this proves 3.

The total decrease of ψ_i in $[0, 1]$ is $\frac{-1}{1-\alpha}(1 - \gamma_i)$. Since ψ_i is continuous, piecewise linear, all positive slope intervals have the same slope and $\psi_i(0) = \psi_i(1) = 0$, then a positive slope interval has slope $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$ and this proves 4.

Finally, by (2), $\gamma_i > 0$ for every $i \geq 0$, thus ψ_i is a well-defined function. □

We now demonstrate that each function ψ_i is subadditive. Note that the function ψ_i depends only upon the choice of parameters $\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i$. It will sometimes be convenient to denote the function ψ_i by $\psi_i^{\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i}$ in this section.

The key observation is the following lemma. Figure 2 illustrates this for the function $\psi_2^{\alpha, \epsilon_1, \epsilon_2}$.

Lemma 4.2. For $x \in [0, \alpha + \epsilon_1]$ and $i \geq 1$, $\psi_i^{\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i}(x) = \lambda x + \mu \psi_{i-1}^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \frac{2\epsilon_3}{\alpha+\epsilon_1}, \dots, \frac{2\epsilon_i}{\alpha+\epsilon_1}}\left(\frac{2x}{\alpha+\epsilon_1}\right)$, where

$$\lambda = \frac{1 - \alpha - \epsilon_1}{(\alpha + \epsilon_1)(1 - \alpha)} \text{ and } \mu = \frac{\epsilon_1}{(\alpha + \epsilon_1)(1 - \alpha)}.$$

Proof. Notice that λ is the slope of the line passing through the points $(0, 0)$ and $(\frac{\alpha+\epsilon_1}{2}, \psi_1^{\alpha, \epsilon_1}(\frac{\alpha+\epsilon_1}{2})) = (\frac{\alpha+\epsilon_1}{2}, \frac{1-\epsilon_1}{2})$.

For $x \in [0, \alpha + \epsilon_1]$, let $\phi_i(x) = \psi_i^{\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i}(x) - \lambda x$. Notice that the graph of ϕ_1 in the interval $[0, \alpha + \epsilon_1]$ is comprised of two identical triangles, one with basis $[0, \frac{\alpha+\epsilon_1}{2}]$ and apex $(\frac{\alpha-\epsilon_1}{2}, \mu)$ and the other with basis $[\frac{\alpha+\epsilon_1}{2}, \alpha+\epsilon_1]$ and apex (α, μ) , where $\mu = \psi_1(\frac{\alpha-\epsilon_1}{2}) - \lambda(\frac{\alpha-\epsilon_1}{2})$. Therefore, for $x \in [0, 1]$,

$$\mu^{-1} \phi_1\left(\frac{(\alpha + \epsilon_1)x}{2}\right) = \psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(x)$$

thus $\phi_1(x) = \mu \psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$ because $\phi_1(x) = \phi_1(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$.

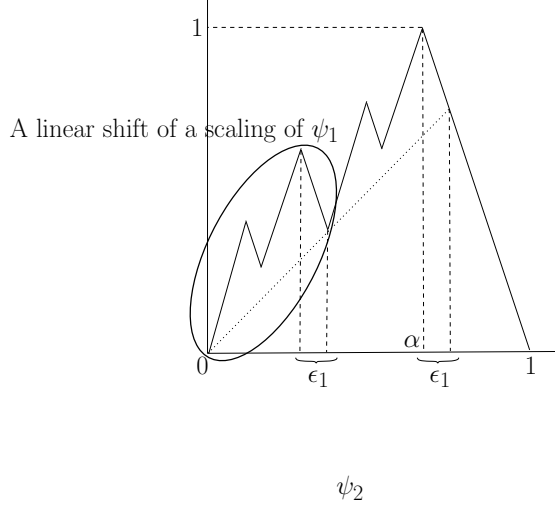


Figure 2: Illustrating the proof of subadditivity of ψ_2

Assume by induction that $\phi_i(x) = \phi_i(x + \frac{\alpha + \epsilon_1}{2})$ for every $x \in [0, \frac{\alpha + \epsilon_1}{2}[$, and that, for $x \in [0, 1[$, $\mu^{-1}\phi_i\left(\frac{(\alpha + \epsilon_1)x}{2}\right) = \psi_{i-1}^{\frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}, \frac{2\epsilon_2}{\alpha + \epsilon_1}, \dots, \frac{2\epsilon_i}{\alpha + \epsilon_1}}(x)$.

Notice that, by Fact 4.1, the slope of ψ_i in the intervals of positive slope is always greater than λ , hence ϕ_i has positive slope exactly in the same intervals where ψ_i has positive slope.

Therefore, by construction of $\psi_{i+1}^{\alpha, \epsilon_1, \dots, \epsilon_{i+1}}$, the function ϕ_{i+1} is obtained from ϕ_i by replacing each maximal positive slope segment $[(a, \phi_i(a)), (b, \phi_i(b))]$ with:

- the segment connecting $(a, \phi_i(a))$ and $(\frac{(a+b) - \epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b) - \epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b) + \epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b) + \epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \phi_i(b))$.

Thus, by induction, $\phi_{i+1}(x) = \phi_{i+1}(x + \frac{\alpha + \epsilon_1}{2})$ for every $x \in [0, \frac{\alpha + \epsilon_1}{2}[$ and, for $x \in [0, 1[$, we have that $\mu^{-1}\phi_{i+1}\left(\frac{(\alpha + \epsilon_1)x}{2}\right) = \psi_i^{\frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}, \frac{2\epsilon_2}{\alpha + \epsilon_1}, \dots, \frac{2\epsilon_i}{\alpha + \epsilon_1}, \frac{2\epsilon_{i+1}}{\alpha + \epsilon_1}}(x)$. Therefore, for $x \in [0, \alpha + \epsilon_1[$, we have $\phi_{i+1}(x) = \mu\psi_i^{\frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}, \frac{2\epsilon_2}{\alpha + \epsilon_1}, \dots, \frac{2\epsilon_{i+1}}{\alpha + \epsilon_1}}\left(\frac{2x}{\alpha + \epsilon_1}\right)$. \square

Remark 4.3. Given any $0 < \alpha < 1$ and any non-increasing sequence of positive real numbers ϵ_i satisfying (2) and $\epsilon_1 \leq 1 - \alpha$, let $\alpha' = \frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}$, $\epsilon'_i = \frac{2\epsilon_{i+1}}{\alpha + \epsilon_1}$, $i \geq 1$. Then $\epsilon'_1 \leq 1 - \alpha'$, $\{\epsilon'_i\}$ is a non-increasing sequence, and $\sum_{i=1}^{+\infty} 2^{i-1}\epsilon'_i < \alpha'$.

We next prove that each ψ_i is a non-negative function.

Fact 4.4. $\psi_i^{\alpha, \epsilon_1, \dots, \epsilon_i}(x) \geq 0$ for all x , and for all parameters such that $\epsilon_1 \leq 1 - \alpha$ and ϵ_i is a non-increasing sequence.

Proof. The proof is by induction on i . ψ_0 is non-negative by definition.

Consider ψ_{i+1} . Clearly $\psi_{i+1}(x) \geq 0$ for $x \in [\alpha + \epsilon_1, 1[$, since $\psi_{i+1}(x) = \psi_0(x)$ in this interval. Note that in Lemma 4.2, $\lambda \geq 0$ because $1 - \alpha \geq \epsilon_1$. So, when $x \in [0, \alpha + \epsilon_1[$

Lemma 4.2 implies that ψ_{i+1} is non-negative, because ψ_i is non-negative. Note that the parameters for ψ_i also satisfy the hypothesis by Remark 4.3, so we can use the induction hypothesis. \square

Lemma 4.5. *Given any $0 < \alpha < 1$ and any nonincreasing sequence of positive real numbers ϵ_i satisfying (2) and $\epsilon_1 \leq 1 - \alpha$, the function $\psi_i^{\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i}$ is subadditive for all i .*

Proof. The proof is by induction. ψ_0^α is subadditive, since it is a valid and minimal Gomory function. By the induction hypothesis, $\psi_k^{\alpha, \epsilon_1, \dots, \epsilon_k}$ is subadditive and we wish to show this implies that $\psi_{k+1}^{\alpha, \epsilon_1, \dots, \epsilon_{k+1}}$ is subadditive.

By Remark 4.3 and induction, the function $\psi_k^{\frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}, \frac{2\epsilon_2}{\alpha + \epsilon_1}, \frac{2\epsilon_3}{\alpha + \epsilon_1}, \dots, \frac{2\epsilon_{k+1}}{\alpha + \epsilon_1}}$ is subadditive.

We define the function $\psi'_k(x) = \mu \psi_k^{\frac{\alpha - \epsilon_1}{\alpha + \epsilon_1}, \frac{2\epsilon_2}{\alpha + \epsilon_1}, \frac{2\epsilon_3}{\alpha + \epsilon_1}, \dots, \frac{2\epsilon_{k+1}}{\alpha + \epsilon_1}}\left(\frac{2x}{\alpha + \epsilon_1}\right)$ where μ is defined in the statement of Lemma 4.2. Note that ψ'_k has a period of $\frac{\alpha + \epsilon_1}{2}$ and ψ'_k is subadditive by Fact 2.3.

In the remaining part of the proof, we will not need the extended notation $\psi_k^{\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_k}$ and we shall refer to this function as simply ψ_k . We now prove the subadditivity of ψ_{k+1} assuming the subadditivity of ψ_k , i.e. $\psi_{k+1}(a+b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$. Assume without loss of generality that $a \leq b$. We then have the following two cases.

Case 1 : b is in the range $[0, \alpha + \epsilon_1]$.

If $a + b \in [0, \alpha + \epsilon_1]$, $\psi_{k+1}(a + b) = \psi'_k(a + b) + \lambda(a + b)$ by Lemma 4.2. Now Fact 2.2 shows that ψ_{k+1} is subadditive.

If $a + b$ is in the range $[\alpha + \epsilon_1, 1]$, then $\psi_{k+1}(a + b) \leq \lambda(a + b)$. On the other hand, $\psi_{k+1}(a) \geq \lambda a$ and $\psi_{k+1}(b) \geq \lambda b$, hence $\psi_{k+1}(a + b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$.

If $a + b$ is greater than 1, then $(a + b) \bmod 1 < \alpha + \epsilon_1$. Let $x = \alpha + \epsilon_1 - b$ and $y = 1 - \alpha - \epsilon_1$. So $(a + b) \bmod 1 = a - x - y$. Then

$$\begin{aligned} \psi_{k+1}(a) + \psi_{k+1}(b) &= \psi'_k(a) + \psi'_k(b) + \lambda a + \lambda b \\ &\geq \psi'_k(a + b) + \lambda a + \lambda b && ((a + b) \bmod 1 < \alpha + \epsilon_1) \\ &= \psi'_k(a - x) + \lambda a + \lambda b && (a - x = (a + b) - (\alpha + \epsilon_1)) \end{aligned} \quad (4)$$

because ψ'_k has period $\alpha + \epsilon_1$. Also,

$$\begin{aligned} \psi_{k+1}(a + b) &= \psi_{k+1}(a - x - y) \\ &\leq \psi_{k+1}(a - x) + \frac{y}{1 - \alpha} && (\text{All negative slopes in } \psi_{k+1} \text{ are } -\frac{1}{1 - \alpha}) \\ &= \psi_{k+1}(a - x) + \lambda(b + x) && (\text{by definition of } \lambda, x, y) \\ &= \psi'_k(a - x) + \lambda(a - x) + \lambda(b + x) && (\text{by Lemma 4.2 because } 0 \leq a - x \leq \alpha + \epsilon_1) \\ &= \psi'_k(a - x) + \lambda a + \lambda b \end{aligned} \quad (5)$$

From (4) and (5) we get that $\psi_{k+1}(a) + \psi_{k+1}(b) \geq \psi_{k+1}(a + b)$.

Case 2 : b is in the range $[\alpha + \epsilon_1, 1]$

If $a + b$ is also in the range $[\alpha + \epsilon_1, 1]$, then $\psi_{k+1}(a + b) \leq \psi_{k+1}(b)$. Therefore, $\psi_{k+1}(a + b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$.

Now we consider the case where $a + b > 1$. Since every line segment with negative slope in the graph of ψ_{k+1} has slope $-\frac{1}{1-\alpha}$, then in the range $[0, a]$ the line of slope $-\frac{1}{1-\alpha}$ passing through $(a, \psi_{k+1}(a))$ lies above the graph of ψ_{k+1} . Formally, for every $x \in [0, a]$,

$$-\frac{1}{1-\alpha}x + \psi_{k+1}(a) + \frac{a}{1-\alpha} \geq \psi_{k+1}(x). \quad (6)$$

Now

$$\psi_{k+1}(a) + \psi_{k+1}(b) = \psi_{k+1}(a) + \frac{1-b}{1-\alpha} \geq -\frac{a+b-1}{1-\alpha} + \psi_{k+1}(a) + \frac{a}{1-\alpha} \geq \psi_{k+1}(a+b-1)$$

where the first equality is because $\psi_{k+1}(b) = \frac{1}{1-\alpha}(1-b)$, and the last inequality follows by (6). Therefore, we get $\psi_{k+1}(a) + \psi_{k+1}(b) \geq \psi_{k+1}(a+b-1) = \psi_{k+1}(a+b)$. \square

Fact 4.6. $\psi_i(x)$ is a symmetric function.

Proof. It is straightforward to show that ψ_0 is symmetric. Notice that, by construction, the function $\psi_{i+1} - \psi_i$ satisfies

$$(\psi_{i+1} - \psi_i)(x) + (\psi_{i+1} - \psi_i)(\alpha - x) = 0.$$

Therefore, if ψ_i is symmetric, also ψ_{i+1} is symmetric. \square

Theorem 4.7. For $i \geq 0$, the function ψ_i is a facet.

Proof. Since ψ_i is a function that is piecewise linear, subadditive, symmetric and has only two slopes, then, by Theorems 2.1 and 2.6, ψ_i is a facet. \square

5 Analysis of the limit function

Recall that ψ is the function defined by

$$\psi(x) = \lim_{i \rightarrow \infty} \psi_i(x)$$

for every $x \in [0, 1[$.

Fact 4.1 implies the following.

Fact 5.1. Let $\gamma = \alpha - \sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i$. Then $\gamma > 0$ by (2) and $\gamma < \gamma_i$ for all i , and the value s_i of the positive slope in ψ_i is bounded above by $\frac{1-\gamma}{(1-\alpha)\gamma}$.

We can now show the following lemma.

Lemma 5.2. For any x , the sequence $\{\psi_i(x)\}_{i=1,2,3,\dots}$ is a Cauchy sequence, and therefore it converges. Moreover, the sequence of functions $\{\psi_i\}_{i=1,2,3,\dots}$ converges uniformly to ψ .

Proof. By Fact 4.1, there are 2^i intervals where ψ_i has positive slope, each of length $\frac{\gamma_i}{2^i}$. Note that $|\psi_i(x) - \psi_{i+1}(x)| \leq s_{i+1} \frac{\gamma_i}{2^i}$ since the values of the two functions match at the ends of the positive-slope intervals of ψ_i .

By Fact 5.1, $s_{i+1} \leq \frac{1-\gamma}{(1-\alpha)\gamma}$ and we know that $\gamma_i < \alpha$. So $|\psi_i(x) - \psi_{i+1}(x)| \leq C \frac{1}{2^i}$ where $C = \alpha \frac{1-\gamma}{(1-\alpha)\gamma}$. Therefore, $|\psi_n(x) - \psi_m(x)| \leq \sum_{i=n}^{m-1} C \frac{1}{2^i}$ if $n < m$. We can bound this expression using

$$\sum_{i=n}^{m-1} C \frac{1}{2^i} \leq \sum_{i=n}^{\infty} C \frac{1}{2^i} = C \frac{1}{2^{n-1}}$$

This implies that the sequence is Cauchy and hence convergent. Moreover, since the bound on $|\psi_n(x) - \psi_m(x)|$ does not depend on x , the above argument immediately implies that the sequence of functions ψ_i converges uniformly to ψ . \square

This also implies the following corollary.

Corollary 5.3. *The function ψ is continuous.*

Proof. ψ_i is continuous for each $i \in \{1, 2, 3, \dots\}$ by construction. Since this sequence of functions converges uniformly to ψ , ψ is continuous [7]. \square

For each integer $i \geq 0$, define S_i to be the subset of points of $]0, 1[$ over which the function ψ_i has negative slope. By Fact 4.1, S_i is the union of 2^i open intervals. Furthermore, by construction $S_i \subseteq S_{i+1}$ for every $i \in \mathbb{N}$. The set $S \subseteq [0, 1]$ defined by

$$S = \cup_{i=0}^{\infty} S_i,$$

is the set of points over which ψ has negative slope, and it is an open set since it is the union of open intervals.

Fact 5.4. *The set S is dense in $[0, 1]$.*

Proof. Let $a \in [0, 1]$. We need to show that, for any $\delta > 0$, there exists $b \in S$ such that $|a - b| < \delta$. Choose i such that $\frac{1}{2^i} < \delta$. If ψ_i has negative slope in a , then $a \in S$ and we are done. Thus a is in a positive slope interval of ψ_i . By Fact 4.1, such an interval has length $\frac{\gamma_i}{2^i}$, hence there exists a point b in a negative slope interval of ψ_i , and thus in S , such that $|a - b| \leq \frac{\gamma_i}{2^i} < \delta$, since $\gamma_i < 1$. \square

Fact 5.5. *The function ψ is not piecewise linear.*

Proof. Suppose by contradiction that ψ is piecewise linear. Then, for some $\varepsilon > 0$, the restriction of ψ to $]0, \varepsilon[$ is linear. By Fact 5.4, $]0, \varepsilon[$ contains a point of S , therefore ψ has constant negative slope in $]0, \varepsilon[$. Since $\psi(0) = 0$, then $\psi(x) < 0$ for every $x \in]0, \varepsilon[$, a contradiction. \square

Lemma 5.6. *The function ψ is a minimal valid function.*

Proof. By Theorem 2.1, we only need to show that ψ is subadditive and symmetric. This follows from pointwise convergence. \square

We finally show that the function ψ is a facet.

Theorem 5.7. *The function ψ is a facet for the problem (1).*

Proof. We will use the Facet Theorem (Theorem 2.4). We show that if a minimal valid function satisfies the set of equalities $E(\psi)$, then it coincides with ψ everywhere.

Consider any minimal valid function ϕ that satisfies the set of equalities $E(\psi)$. Therefore, if $\psi(u) + \psi(v) = \psi(u + v)$, then $\phi(u) + \phi(v) = \phi(u + v)$.

We first show the following.

Claim 1. $\phi(x) = \psi(x)$ for all $x \in S$

For any maximal interval $]a, b[$ over which the graph of ψ has a negative slope, consider the following intervals : $U = [(a+b)/2, b]$, $V = [1 - ((b-a)/2), 1]$ and therefore $U+V = [a, b]$. It is easy to see that $\psi(u) + \psi(v) = \psi(u+v)$ for $u \in U, v \in V$. This implies $\phi(u) + \phi(v) = \phi(u+v)$. Now Lemma 2.5 (the Interval Lemma) implies that ϕ are straight lines over U, V and $U + V$.

We now use an inductive argument to prove that not only do the slopes of ψ and ϕ coincide on intervals where the slope of ψ is negative, in fact $\psi(x) = \phi(x)$ for all x in these intervals.

Every maximal segment s with negative slope in ψ also appears in ψ_i for some i (i.e. $s \subseteq S_i$ for some i). Let $index(s)$ be the least such i . We prove that $\psi(x) = \phi(x)$ for every s with negative slope by induction on $index(s)$. $\phi(\alpha) = \psi(\alpha) = 1$ and $\phi(0) = \psi(0) = 0$ since ϕ is assumed to be a valid inequality. This implies that ϕ is the same as ψ in the range $[\alpha, 1]$. This proves the base case of the induction.

By the induction hypothesis, we assume the claim is true for negative-slope segments s with $index(s) = k$. Consider all negative-slope segments s with $index(s) = k + 1$. Amongst these consider the segment s_c which is closest to the origin. Let the midpoint of this segment be m . We know that $2m$ is the start of a negative-slope segment s' in ψ with $index(s') = k$. By construction, $\psi(m) + \psi(m) = \psi(2m)$. So $\phi(m) + \phi(m) = \phi(2m)$. From the induction hypothesis, we know that $\psi(2m) = \phi(2m)$ and so $\phi(m) = \frac{1}{2}\phi(2m) = \frac{1}{2}\psi(2m) = \psi(m)$.

Now consider any other negative-slope segment s with $index(s) = k + 1$ and let its midpoint be m_s . Note that $m_s + m$ is the start of a negative-slope segment s' with $index(s') = k$. So

$$\phi(m_s + m) = \psi(m_s + m) \tag{7}$$

because of the inductive hypothesis. Note that $\psi(m_s + m) = \psi(m_s) + \psi(m)$ by construction. So, $\phi(m_s + m) = \phi(m_s) + \phi(m)$. Since we showed that $\phi(m) = \psi(m)$, (7) implies that $\phi(m_s) = \psi(m_s)$. Since the values coincide at the midpoints of these segments and the slopes of the segments are the same, $\phi(x) = \psi(x)$ for any x in the domain of these segments. This concludes the proof of Claim 1.

We now use Claim 1 to show that

$$\phi(x) \leq \psi(x) \text{ for all } x \in [0, 1[. \tag{8}$$

By Lemma 5.6 ψ is a minimal valid function and ϕ is assumed to be minimal, thus symmetry combined with (8) would imply that $\phi(x) = \psi(x)$ for all $x \in [0, 1[$.

Claim 2. Let $\bar{x} \in]0, 1[$. For every positive integer n , there exists $y_n, z_n \in S$ such that $0 < z_n < \frac{1}{n}$ and $y_n + z_n = \bar{x}$.

Since S is open and, by Fact 5.4, it is dense in $[0, 1]$, for any positive integer n there exists an open interval $I \subseteq S$ such that $0 < \bar{x} - y < \frac{1}{n}$ for all $y \in I$. Let $I =]u_1, u_2[$ with $u_1 \neq u_2$. Also, since S is dense in $[0, 1]$, there exists $z \in S \cap]\bar{x} - u_2, \bar{x} - u_1[$. Let this z be z_n and $y_n = \bar{x} - z \in I$. Since $I \subseteq S$, we have that $y_n \in S$. This concludes the proof of Claim 2.

Note that the sequence $\{y_n\}$ converges to \bar{x} and $\{z_n\}$ converges to 0. For every positive integer n we have

$$\begin{aligned} \phi(\bar{x}) &= \phi(y_n + z_n) \\ &\leq \phi(y_n) + \phi(z_n) && \text{(By subadditivity of } \phi) \\ &= \psi(y_n) + \psi(z_n) \\ \Rightarrow \phi(\bar{x}) &\leq \lim_{n \rightarrow \infty} (\psi(y_n) + \psi(z_n)) \\ &= \psi(\bar{x}) && \text{(By continuity of } \psi) \end{aligned}$$

□

6 Conjectures

The referees propose a variation of the Gomory-Johnson conjecture (Conjecture 1.1) which is not disproved by our counterexample ψ and a generalization of the Gomory-Johnson two-slope theorem that includes our function ψ .

Conjecture 6.1. *If $\pi : [0, 1[\rightarrow \mathbb{R}$ is a facet of (1), then it is either piecewise linear or else it is the limit of facet-defining piecewise linear functions.*

Conjecture 6.2. *Let $\pi : [0, 1[\rightarrow \mathbb{R}$ be a minimal valid function. If π is absolutely continuous (thus differentiable almost everywhere) and has only two slopes, then π is a facet.*

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