Cut-Generating Functions for Integer Variables

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Abstract

For an integer linear program, Gomory's corner relaxation is obtained by ignoring the nonnegativity of the basic variables in a tableau formulation. In this paper, we do not relax these nonnegativity constraints. We generalize a classical result of Gomory and Johnson characterizing minimal cut-generating functions in terms of subadditivity, symmetry, and periodicity. Our result is based on a new concept, the notion of generalized symmetry condition. We also prove a 2-Slope Theorem for extreme cut-generating functions in our setting, in the spirit of the 2-Slope Theorem of Gomory and Johnson.

1 Introduction

An ongoing debate in integer linear programming centers on the value of general-purpose cuts (Gomory cuts are a famous example) versus facet-defining inequalities for special problem structures (for example, comb inequalities for the traveling salesman problem). Both have been successful in practice. In this paper we focus on the former type of cuts, which are attractive for their wide applicability. Nowadays, state-of-the-art integer programming solvers routinely use several classes of general-purpose cuts. Recently, there has been a renewed interest in the *theory* of general-purpose cuts. This was sparked by a beautiful paper of Andersen, Louveaux, Weismantel, and Wolsey [ALWW07] on 2-row cuts that illuminated their connection to lattice-free convex sets. This line of research focused on cut coefficients for the continuous nonbasic variables in a tableau form, and lifting properties for the integer nonbasic variables [BC09, DW10b, DW10a, CCZ11b, BCC$^+$13, CCD$^+$15, BP14]. Decades earlier, Gomory and Johnson [GJ72a, GJ72b] and Johnson [Joh74] had studied cut coefficients for the integer nonbasic variables directly. Although their characterization involves concepts that are not always easy to verify algorithmically (such as subadditivity), it provides a useful framework for the study of cutting-planes. Jeroslow [Jer79], Blair [Bla78], and Bachem, Johnson, and Schrader [BJS82] extended the work of Gomory and Johnson on minimal valid inequalities for the corner relaxation to general integer linear programs. In this paper, we pursue the study of general-purpose cuts in integer programming, further extending the framework introduced by Gomory and Johnson. Our focus is also on the cut coefficients of the integer variables.

Consider a pure integer linear program and the optimal simplex tableau of its linear programming relaxation. We select n rows of the tableau, corresponding to n basic variables $\{x_i\}_{i=1}^n$. Let

$\{y_j\}_{j=1}^m$ denote the nonbasic variables. The tableau restricted to these n rows is of the form

$$\begin{aligned} x =& f + \sum_{j=1}^m r^j y_j, \\ & x \in \mathbb{Z}_+^n, \\ & y_j \in \mathbb{Z}_+, \forall j \in \{1, \dots, m\}, \end{aligned} \tag{1}$$

where $f \in \mathbb{R}_+^n$ and $r^j \in \mathbb{R}^n$ for all $j \in \{1, \dots, m\}$. We assume $f \notin \mathbb{Z}^n$; therefore, the basic solution $x = f$, $y = 0$ is not feasible. We would like to generate cutting-planes that cut off this infeasible solution.

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (1) if the inequality $\sum_{j=1}^m \pi(r^j) y_j \geq 1$ holds for all feasible solutions (x, y) to (1) for any possible number m of nonbasic variables and any choice of nonbasic columns r^j. Gomory and Johnson [GJ72a, GJ72b] and Johnson [Joh74] characterized such functions for the corner relaxation of (1) obtained by relaxing $x \in \mathbb{Z}_+^n$ to $x \in \mathbb{Z}^n$. They also introduced the infinite group relaxation

$$\begin{aligned} x =& f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in \mathbb{Z}^n, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support}, \end{aligned} \tag{2}$$

where an infinite-dimensional vector is said to have *finite support* if it has a finite number of nonzero entries.

Here we consider the following generalization of the Gomory-Johnson model:

$$\begin{aligned} x =& f + \sum_{r \in \mathbb{R}^n} r y_r, \\ & x \in S, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & y \text{ has finite support}, \end{aligned} \tag{3}$$

where S can be any nonempty subset of the Euclidean space. This flexibility in the choice of S makes (3) a relevant model for 1) integer convex and conic programs and 2) integer programs with complementarity constraints, as well as integer linear programs; see [CCD+15]. The Gomory-Johnson model (2) is the special case of (3) where $S = \mathbb{Z}^n$. The models studied in [Jer79, Bla78, BJS82] are closely related to the case $S = \{0\}$. The case where $S = \mathbb{Z}_+^n$, or more generally where S is the set of integer points in a full-dimensional rational polyhedron, is of particular interest in integer linear programming due to its connection to (1) above. It is a main focus of this paper. In the context of *mixed*-integer linear programming, the model (3) with continuous as well as integer variables is also interesting; we will discuss it in Section 3.4 (where we let $S = \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$ for $p \in \{0, 1, \dots, n\}$) and Section 6 (where we consider a generalization of model (3) with both integer and continuous nonbasic variables).

Note that (3) is nonempty since for any $\bar{x} \in S$, the solution $x = \bar{x}$, $y_{\bar{x}-f} = 1$, and $y_r = 0$ for all $r \neq \bar{x} - f$ is feasible. In the remainder of the paper, we assume that $f \in \mathbb{R}^n \setminus S$. Therefore, the basic solution $x = f$, $y = 0$ is not a feasible solution of (3). We are interested in valid inequalities for (3) that cut off the above infeasible basic solution.

We can generalize the notion of cut-generating function as follows. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (3) if the inequality $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ holds for all feasible solutions (x, y) to (3). For example, the function that takes the value 1 for all $r \in \mathbb{R}^n$ is a cut-generating function because every feasible solution of (3) satisfies $y_r \geq 1$ for at least one $r \in \mathbb{R}^n$. When $S = \mathbb{Z}^n_+$, we recover the earlier definition of cut-generating function for (1).

A key feature that distinguishes the cut-generating functions for model (3) from those that were studied by Gomory and Johnson for model (2) is that they need not be nonnegative even if we assume continuity. In fact, they can take any real value, positive and negative, as the following examples illustrate.

Example 1. Consider the model (3) where $n = 1$, $0 < f < 1$, and $S = \mathbb{Z}_+$. Cornuéjols, Kis, and Molinaro [CKM13] showed that for $0 < \alpha \leq 1$, the following family of functions are cut-generating functions:

$$\pi^1_\alpha(r) \;=\; \min\left\{\frac{r - \lfloor \alpha r \rfloor}{1-f},\; \frac{-r}{f} + \frac{\lceil \alpha r \rceil (1-\alpha f)}{\alpha f(1-f)}\right\}.$$

Note that when $\alpha = 1$, the function $\pi^1_1(r) = \min\{\frac{r-\lfloor r \rfloor}{1-f}, \frac{\lceil r \rceil - r}{f}\}$ is the well-known Gomory function. This function is periodic and takes its values in the interval $[0, 1]$. However, when $\alpha < 1$, this is not the case any more: The function π^1_α takes all real values between $-\infty$ and $+\infty$, and it is not periodic in the usual sense. See Figure 1.

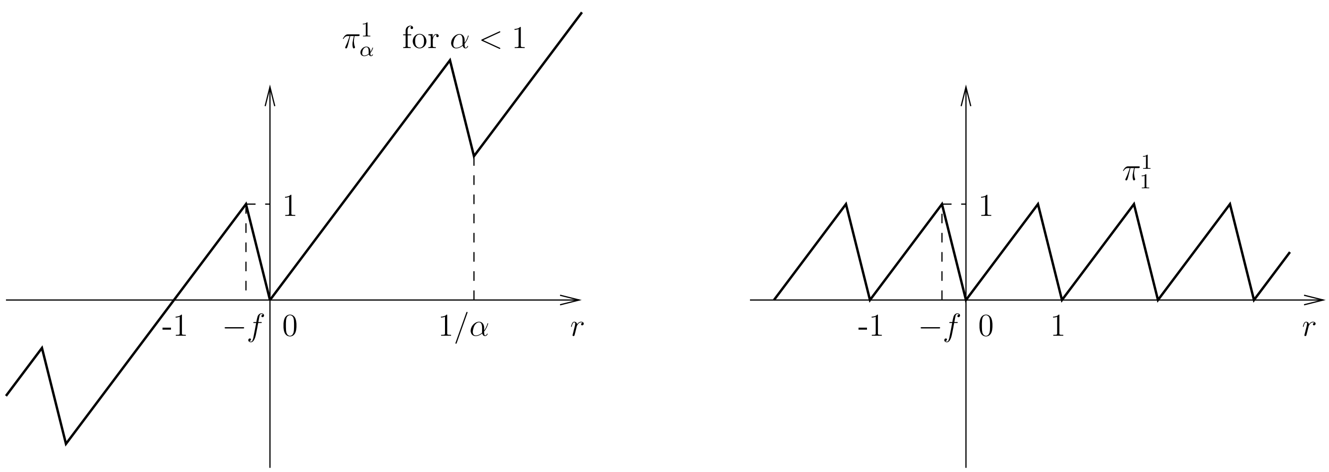

Figure 1: Two cut-generating functions: π^1_α for some $\alpha < 1$ and π^1_1.

The next example is mostly of theoretical interest. It illustrates another property of model (3) that does not arise in the Gomory-Johnson model (2).

Example 2. Consider the model (3) where $n = 1$, $f > 0$, and $S = \{0\}$. In this case, the model (3) reduces to $\sum_{r\in\mathbb{R}} ry_r = -f$, $y_r \in \mathbb{Z}_+$ for $r \in \mathbb{R}$, and y has finite support. For any $\alpha \leq -\frac{1}{f} < 0$, the linear function $\pi^2_\alpha(r) = \alpha r$ is a cut-generating function. This can be seen by observing that for any y feasible to (3), we have $\sum_{r\in\mathbb{R}} \pi^2_\alpha(r)y_r = \sum_{r\in\mathbb{R}}(\alpha r)y_r = \alpha \sum_{r\in\mathbb{R}} ry_r = -\alpha f \geq 1$.

1.1 Minimal Cut-Generating Functions

We say that a cut-generating function π' for (3) *dominates* another cut-generating function π if $\pi \geq \pi'$, that is, $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. A cut-generating function π is *minimal* if there is no cut-generating function π' distinct from π that dominates π. When $n = 1$, $S = \mathbb{Z}_+$, and $0 < f < 1$, the cut-generating functions π^1_α of Example 1 are minimal [CKM13]. Later in Section 1.2, we will show that the linear cut-generating functions π^2_α of Example 2 are also minimal. The following theorem shows that minimal cut-generating functions indeed always exist when $S \neq \emptyset$ in (3). This result also appears in a recent paper of Basu and Paat [BP14].

Theorem 1. *Every cut-generating function for* (3) *is dominated by a minimal cut-generating function.*

Proof. Let π be a cut-generating function for (3). Denote by Π the set of cut-generating functions π' that dominate π. Let $\{\pi_\ell\}_{\ell \in L} \subset \Pi$ be a nonempty family of cut-generating functions such that for any pair $\ell', \ell'' \in L$, we have $\pi_{\ell'} \leq \pi_{\ell''}$ or $\pi_{\ell'} \geq \pi_{\ell''}$. To prove the claim, it is enough to show by Zorn's Lemma (see, e.g., [Cie97]) that there exists a cut-generating function that is a lower bound on $\{\pi_\ell\}_{\ell \in L}$.

Define the function $\bar{\pi} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ as $\bar{\pi}(r) = \inf_\ell \{\pi_\ell(r) : \ell \in L\}$. Clearly, the function $\bar{\pi}$ is a lower bound on $\{\pi_\ell\}_{\ell \in L}$. We show that it is a cut-generating function for (3). First we prove that $\bar{\pi}$ is finite everywhere. Choose $\bar{x} \in S$. For any $\bar{r} \in \mathbb{R}^n$, let $\bar{y}$ be defined as $\bar{y}_{\bar{r}} = 1$, $\bar{y}_{\bar{x}-f-\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise. The solution $(\bar{x}, \bar{y})$ is feasible to (3). Then for any $\ell \in L$, the cut-generating function π_ℓ satisfies $\sum_{r \in \mathbb{R}^n} \pi_\ell(r)\bar{y}_r = \pi_\ell(\bar{r}) + \pi_\ell(\bar{x} - f - \bar{r}) \geq 1$. Moreover, we have $\pi_\ell \leq \pi$ because $\pi_\ell \in \Pi$; hence,

$$\pi_\ell(\bar{r}) \geq 1 - \pi_\ell(\bar{x} - f - \bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r}).$$

Therefore, $\bar{\pi}(\bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r})$. This shows that $\bar{\pi}(r)$ is finite for all $r \in \mathbb{R}^n$. That is, $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}$. Now consider any feasible solution (x, y) of (3). Note that $\{\pi_\ell\}_{\ell \in L}$ is a totally ordered set, $\bar{\pi}$ is finite everywhere, and only a finite number of the terms y_r are nonzero. Combining these facts, we can write

$$\sum_{r \in \mathbb{R}^n} \bar{\pi}(r) y_r = \sum_{r \in \mathbb{R}^n} \inf_\ell \{\pi_\ell(r) : \ell \in L\} y_r = \inf_\ell \left\{ \sum_{r \in \mathbb{R}^n} \pi_\ell(r) y_r : \ell \in L \right\} \geq 1.$$

This proves that $\bar{\pi}$ is a cut-generating function. $\square$

Theorem 1 shows that one can focus on minimal cut-generating functions since non-minimal ones are not needed in the description of the convex hull of feasible solutions to (3).

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$; it is *symmetric* or satisfies the *symmetry condition* if $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$; and it is *periodic with respect to* $\mathbb{Z}^n$ if $\pi(r) = \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$. When $S = \mathbb{Z}^n$, cut-generating functions are traditionally assumed to be nonnegative, and subadditivity, symmetry, and periodicity with respect to $\mathbb{Z}^n$ are necessary and sufficient for a nonnegative cut-generating function to be minimal [GJ72a, Joh74, CCZ11a]. However, for general S, Examples 1 and 2 show that minimal cut-generating functions do not necessarily satisfy periodicity with respect to $\mathbb{Z}^n$, nor symmetry. We define a new condition, which we call the generalized symmetry condition, to replace symmetry and periodicity in the characterization of minimal cut-generating functions for

(3). Let $\mathbb{Z}_{++}$ be the set of strictly positive integers. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *generalized symmetry condition* if

$$\pi(r) = \sup_{x,k}\left\{\frac{1}{k}\left(1 - \pi(x - f - kr)\right) : x \in S, k \in \mathbb{Z}_{++}\right\} \text{ for all } r \in \mathbb{R}^n. \tag{4}$$

The functions π^1_α and π^2_α of Examples 1 and 2 satisfy the generalized symmetry condition. We briefly outline the proof in each case.

Let $S = \mathbb{Z}_+$ and $0 < f < 1$ in (3), and consider the function π^1_α defined in Example 1. The inequality $\bar{k}\pi^1_\alpha(\bar{r}) + \pi^1_\alpha(\bar{x} - f - \bar{k}\bar{r}) \geq 1$ holds for any $\bar{r} \in \mathbb{R}$, $\bar{k} \in \mathbb{Z}_{++}$, and $\bar{x} \in \mathbb{Z}_+$ because π^1_α is a cut-generating function [CKM13] and the solution $x = \bar{x}$, $y_{\bar{r}} = \bar{k}$, $y_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $y_r = 0$ otherwise is feasible to (3). Hence, $\pi^1_\alpha(r) \geq \frac{1}{k}(1 - \pi^1_\alpha(x - f - kr))$ for all $r \in \mathbb{R}$, $k \in \mathbb{Z}_{++}$, and $x \in \mathbb{Z}_+$. Furthermore, the graph of π^1_α is symmetric relative to the point $(-f/2, 1/2)$. In other words, the symmetry condition holds: $\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r)$ for all $r \in \mathbb{R}$. Therefore, for all $r \in \mathbb{R}$, we have

$$\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r) \leq \sup_{x,k}\left\{\frac{1}{k}\left(1 - \pi^1_\alpha(x - f - kr)\right) : x \in \mathbb{Z}_+, k \in \mathbb{Z}_{++}\right\} \leq \pi^1_\alpha(r).$$

This shows that π^1_α satisfies the generalized symmetry condition.

Now let $f > 0$ and $S = \{0\}$ in (3), and consider the function π^2_α of Example 2. Because $S = \{0\}$, the term x disappears in (4). Using $\alpha \leq \frac{-1}{f}$, for any $r \in \mathbb{R}$, we can write

$$\sup_{k\in\mathbb{Z}_{++}}\left\{\frac{1}{k}\left(1 - \pi^2_\alpha(-f - kr)\right)\right\} = \alpha r + \sup_{k\in\mathbb{Z}_{++}}\left\{\frac{1 + \alpha f}{k}\right\} = \alpha r = \pi^2_\alpha(r).$$

This shows that π^2_α satisfies the generalized symmetry condition.

Our main result about minimal cut-generating functions for (3) is the following theorem which holds for any choice of $S \neq \emptyset$.

Theorem 2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (3) *if and only if $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition.*

This theorem will be proved in Section 2.

1.2 Strengthening the Notion of Minimality

The notion of minimality that we defined above can be unsatisfactory for certain choices of S. We illustrate this in the next proposition and remark.

Proposition 3. *If a cut-generating function for* (3) *is linear, then it is minimal.*

Proof. Let π be a linear cut-generating function for (3). By Theorem 1, there exists a minimal cut-generating function π' such that $\pi' \leq \pi$. By Theorem 2, π' is subadditive and $\pi'(0) = 0$. For any $r \in \mathbb{R}^n$, the inequality $\pi' \leq \pi$ implies $\pi(r)+\pi(-r) \geq \pi'(r)+\pi'(-r) \geq \pi'(0) = 0 = \pi(r)+\pi(-r)$ where the last equality follows from the linearity of π. Hence, $\pi' = \pi$. $\square$

Remark 4. *For a minimal cut-generating function* π*, it is possible that the inequality* $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ *is implied by an inequality* $\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq 1$ *arising from some other cut-generating function* π'*. Indeed, for* $n = 1$*,* $f > 0$*, and* $S = \{0\}$*, consider again the cut-generating functions* π^2_α *of Example 2 with* $\alpha \leq -\frac{1}{f}$*. These are minimal by Proposition 3. However, the inequalities* $|\alpha| f \sum_{r\in\mathbb{R}} \frac{-r}{f} y_r \geq 1$ *generated from* π^2_α *for* $\alpha < -\frac{1}{f}$ *are implied by the inequality* $\sum_{r\in\mathbb{R}} \frac{-r}{f} y_r \geq 1$ *generated for* $\alpha = -\frac{1}{f}$*.*

Therefore, it makes sense to define a stronger notion of minimality as follows. A cut-generating function π' for (3) *implies* another cut-generating function π *via scaling* if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. Note that when the function π' is nonnegative, this notion is identical to the notion of domination introduced earlier; however, the two notions are distinct when π' can take negative values. A cut-generating function π is *restricted minimal* if there is no cut-generating function π' distinct from π that implies π via scaling. This notion was the one used by Jeroslow [Jer79], Blair [Bla78], and Bachem, Johnson, and Schrader [BJS82]; they just called it minimality. In this paper, we call it restricted minimality to distinguish it from the notion of minimality introduced in Section 1.1. The next proposition shows that restricted minimal cut-generating functions are the minimal cut-generating functions which enjoy an additional “tightness” property.

Proposition 5. *A cut-generating function* π *for* (3) *is restricted minimal if and only if it is minimal and* $\inf_x\{\pi(x - f) : x \in S\} = 1$*.*

The proof of this proposition will be presented at the end of Section 2.

The next proposition shows that restricted minimal cut-generating functions exist and they are always sufficient to separate the infeasible basic solution $x = f$, $y = 0$ from the closed convex hull of feasible solutions to (3).

Proposition 6. *Every cut-generating function for* (3) *is implied via scaling by a restricted minimal cut-generating function.*

Proof. Let π be a cut-generating function. Let $\mu = \inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi(r)y_r : (x, y) \text{ satisfies } (3)\}$; note that $\mu \geq 1$. Define $\pi' = \frac{\pi}{\mu}$. The function π' is also a cut-generating function, and it satisfies $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi'(r)y_r : (x, y) \text{ satisfies } (3)\} = 1$. By Theorem 1, there exists a minimal cut-generating function π^* that dominates π'. The function π^* implies π via scaling since $\mu\pi^* \leq \mu\pi' = \pi$. We claim that π^* is restricted minimal. First note that $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi^*(r)y_r : (x, y) \text{ satisfies } (3)\} = 1$. Now consider $\beta \geq 1$ and a cut-generating function π^{**} such that $\pi^* \geq \beta\pi^{**}$. We must have $\beta = 1$ since $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n} \pi^{**}(r)y_r : (x, y) \text{ satisfies } (3)\} \geq 1$. Then because π^* is minimal, we get $\pi^{**} = \pi^*$. This proves the claim. □

For the case $S = \{0\}$, Bachem, Johnson, and Schrader [BJS82] showed that restricted minimal cut-generating functions satisfy the symmetry condition. This can be generalized as in the next theorem. Let us say that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *nondecreasing with respect to a set* $S \subset \mathbb{R}^n$ if $\pi(r) \leq \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in S$.

Theorem 7. *Let* $K \subset \mathbb{R}^n$ *be a closed convex cone and* $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$*. Let* $\pi : \mathbb{R}^n \to \mathbb{R}$*. The function* π *is a restricted minimal cut-generating function for* (3) *if and only if* $\pi(0) = 0$*,* π *is subadditive, nondecreasing with respect to* S*, and satisfies the symmetry condition.*

This theorem will be proved in Section 3.

The notion of minimality can be strengthened even further by taking into consideration the linear inequalities that are valid for S. Let $\alpha^\top(x - f) \geq \alpha_0$ be a valid inequality for $S \subset \mathbb{R}^n$.

Because $f + \sum_{r\in\mathbb{R}^n} r y_r = x \in S$ for any (x, y) feasible to (3), such a valid inequality can be translated to the space of the nonbasic variables y as $\sum_{r\in\mathbb{R}^n} \alpha^\top r y_r \geq \alpha_0$. We say that a cut-generating function π' for (3) *implies* another cut-generating function π for (3) if there exists a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for S and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \alpha^\top r + \beta\pi'(r)$ for all $r \in \mathbb{R}^n$. This definition makes sense because if $\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq 1$ is a valid inequality for (3), then $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq \sum_{r\in\mathbb{R}^n} \alpha^\top r y_r + \beta \sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq \alpha_0 + \beta \geq 1$ is also valid for (3). When the closed convex hull of S, $\overline{\mathrm{conv}}(S)$, is equal to the whole of $\mathbb{R}^n$, the only inequalities that are valid for S are those that have $\alpha = 0$ and $\alpha_0 \leq 0$; in this case, a cut-generating function may imply another only via scaling. However, the two notions may be different when $\overline{\mathrm{conv}}(S) \subsetneq \mathbb{R}^n$. We say that a cut-generating function π is *strongly minimal* if there does not exist a cut-generating function π' distinct from π that implies π. Note that strongly minimal cut-generating functions are restricted minimal. Indeed, if π is a cut-generating function that is not restricted minimal, there exists a cut-generating function $\pi' \neq \pi$ and $\beta \geq 1$ such that $\pi \geq \beta\pi'$; but then π' implies π by taking $\alpha = 0$ and $\alpha_0 = 0$ which shows that π is not strongly minimal.

For a set which is contained in the nonnegative orthant, minimality of a valid inequality is usually defined with respect to the nonnegative orthant. In a model for disjunctive conic programs, Kılınç-Karzan [KK] generalized this notion broadly by defining and studying the minimality of a valid inequality with respect to an arbitrary regular cone which contains the feasible solution set. The various definitions of the minimality of a cut-generating function that we explore in this paper could be viewed from a similar perspective. Our results demonstrate how strengthening the notion of minimality imposes additional structure on cut-generating functions for (3).

Let $[k] = \{1, \ldots, k\}$ for $k \in \mathbb{Z}_{++}$; let e^i denote the i^{th} standard unit vector in $\mathbb{R}^n$ for $i \in [n]$. In Section 4.1, we prove the following theorem about strongly minimal cut-generating functions for (3) when $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$.

Theorem 8. *Let $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ and $f \in \mathbb{R}^n_+ \setminus S$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a strongly minimal cut-generating function for (3) if and only if $\pi(0) = 0$, $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies the symmetry condition.*

In Section 4.2, we give an example showing that strongly minimal cut-generating functions do not always exist. On the other hand, we can show that they always exist when the closed convex hull of S is a full-dimensional polyhedron.

Theorem 9. *Suppose $\overline{\mathrm{conv}}(S)$ is a full-dimensional polyhedron. Let $f \in \overline{\mathrm{conv}}(S)$. Then every cut-generating function for (3) is implied by a strongly minimal cut-generating function.*

The proof will be given in Section 4.2.

A yet stronger notion is that of extreme cut-generating function. A cut-generating function π is *extreme* if whenever cut-generating functions π_1, π_2 satisfy $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, we have $\pi = \pi_1 = \pi_2$. It follows from this definition that extreme cut-generating functions are minimal. We will show in Section 5 that extreme cut-generating functions must in fact be strongly minimal. The main result of Section 5 is a 2-Slope Theorem for extreme cut-generating functions for (3) when $S = \mathbb{Z}_+$, in the spirit of the Gomory-Johnson 2-Slope Theorem for $S = \mathbb{Z}$.

Section 6 extends some of the earlier results to a mixed-integer model where nonbasic continuous and nonbasic integer variables are both present.

2 Characterization of Minimal Cut-Generating Functions

In this section, we characterize minimal cut-generating functions for (3) under only the basic assumption that $S \neq \emptyset$. In the next three lemmas, we state necessary conditions that are satisfied by all minimal cut-generating functions.

Lemma 10. *If π is a minimal cut-generating function for* (3)*, then $\pi(0) = 0$.*

Proof. Suppose $\pi(0) < 0$, and let $(\bar{x}, \bar{y})$ be a feasible solution of (3). Then there exists some $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(0)\bar{k} < 1 - \sum_{r \in \mathbb{R}^n \setminus \{0\}} \pi(r)\bar{y}_r$ since the right-hand side of the inequality is a constant. Define $\tilde{y}$ as $\tilde{y}_0 = \bar{k}$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Note that $(\bar{x}, \tilde{y})$ is a feasible solution of (3). This contradicts the assumption that π is a cut-generating function since $\sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r < 1$. Thus, $\pi(0) \geq 0$.

Let $(\bar{x}, \bar{y})$ be a feasible solution of (3), and consider $\tilde{y}$ defined as $\tilde{y}_0 = 0$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Then $(\bar{x}, \tilde{y})$ is a feasible solution of (3). Now define the function π' as $\pi'(0) = 0$ and $\pi'(r) = \pi(r)$ for all $r \neq 0$. Observe that $\sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$ where the inequality follows because π is a cut-generating function. This implies that π' is also a cut-generating function for (3). Since π is minimal and $\pi' \leq \pi$, we must have $\pi = \pi'$ and $\pi(0) = 0$. $\square$

The proof of the next lemma is similar to the one presented by Gomory and Johnson [GJ72a] for the case $S = \mathbb{Z}$ and Johnson [Joh74] for the case $S = \mathbb{Z}^n$. It is included here for the sake of completeness.

Lemma 11. *If π is a minimal cut-generating function for* (3)*, then π is subadditive.*

Proof. Let $r^1, r^2 \in \mathbb{R}^n$. We need to show $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$. This inequality holds when $r^1 = 0$ or $r^2 = 0$ by Lemma 10.

Assume now that $r^1 \neq 0$ and $r^2 \neq 0$. Define the function π' as $\pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$ and $\pi'(r) = \pi(r)$ for $r \neq r^1 + r^2$. We show that π' is a cut-generating function. Since π is minimal, it then follows that $\pi(r^1 + r^2) \leq \pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$.

Consider any feasible solution $(\bar{x}, \bar{y})$ to (3). Define $\tilde{y}$ as $\tilde{y}_{r^1} = \bar{y}_{r^1} + \bar{y}_{r^1+r^2}$, $\tilde{y}_{r^2} = \bar{y}_{r^2} + \bar{y}_{r^1+r^2}$, $\tilde{y}_{r^1+r^2} = 0$, and $\tilde{y}_r = \bar{y}_r$ otherwise. Note that $\tilde{y}$ is well-defined since $r^1 \neq 0$ and $r^2 \neq 0$. It is easy to verify that $\tilde{y}$ has finite support, $\tilde{y}_r \in \mathbb{Z}_+$ for all $r \in \mathbb{R}^n$, and $\sum_{r \in \mathbb{R}^n} r\tilde{y}_r = \sum_{r \in \mathbb{R}^n} r\bar{y}_r$. These together show that $(\bar{x}, \tilde{y})$ is a feasible solution of (3). Furthermore, we have $\sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r$ which is greater than or equal to 1 since π is a cut-generating function. This proves that π' is a cut-generating function. $\square$

The next lemma shows that all minimal cut-generating functions satisfy the generalized symmetry condition (4).

Lemma 12. *If π is a minimal cut-generating function for* (3)*, then it satisfies the generalized symmetry condition.*

Proof. Let $\bar{r} \in \mathbb{R}^n$. For any $\bar{x} \in S$ and $\bar{k} \in \mathbb{Z}_{++}$, define $\bar{y}$ as $\bar{y}_{\bar{r}} = \bar{k}$, $\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise. Since $(\bar{x}, \bar{y})$ is feasible to (3) and π is a cut-generating function for (3), we have $\pi(\bar{r}) \geq \frac{1}{\bar{k}}(1 - \pi(\bar{x} - f - \bar{k}\bar{r}))$. Then the definition of supremum implies $\pi(\bar{r}) \geq \sup_{x,k} \left\{ \frac{1}{k}(1 - \pi(x - f - k\bar{r})) : x \in S, k \in \mathbb{Z}_{++} \right\}$. Note that the value on the right-hand side is bounded from above since π is a real-valued function and the left-hand side is finite.

Let the function $\rho : \mathbb{R}^n \to \mathbb{R}$ be defined as $\rho(r) = \sup_{x,k}\{\frac{1}{k}(1-\pi(x-f-kr)) : x \in S, k \in \mathbb{Z}_{++}\}$. Note that $\pi \geq \rho$ by the first part. Now suppose π does not satisfy the generalized symmetry condition. Then there exists $\tilde{r} \in \mathbb{R}^n$ such that $\pi(\tilde{r}) > \rho(\tilde{r})$. Define the function π' as $\pi'(\tilde{r}) = \rho(\tilde{r})$ and $\pi'(r) = \pi(r)$ for all $r \neq \tilde{r}$. Consider any feasible solution $(\tilde{x}, \tilde{y})$ to (3). If $\tilde{y}_{\tilde{r}} = 0$, we get $\sum_{r\in\mathbb{R}^n} \pi'(r)\tilde{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$. Otherwise, $\tilde{y}_{\tilde{r}} \geq 1$ and we have $\pi'(\tilde{r})\tilde{y}_{\tilde{r}} + \sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi'(r)\tilde{y}_r \geq 1-\pi(\tilde{x}-f-\tilde{y}_{\tilde{r}}\tilde{r})+\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi(r)\tilde{y}_r \geq 1$ where we use $\pi'(\tilde{r}) = \rho(\tilde{r}) \geq \frac{1}{\tilde{y}_{\tilde{r}}}(1-\pi(\tilde{x}-f-\tilde{y}_{\tilde{r}}\tilde{r}))$ to obtain the first inequality and the subadditivity of π and $\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} r\tilde{y}_r = \tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}$ to obtain the second inequality. Thus, π' is a cut-generating function for (3). Since $\pi' \leq \pi$ and $\pi'(\tilde{r}) = \rho(\tilde{r}) < \pi(\tilde{r})$, this contradicts the minimality of π. $\square$

We now prove Theorem 2 stated in the introduction.

Theorem 2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (3) *if and only if $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition.*

Proof. The necessity of these conditions has been proven in Lemmas 10, 11, and 12. We now prove their sufficiency.

Assume that $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition. Since $\pi(0) = 0$, the generalized symmetry condition implies $\pi(\bar{x} - f) \geq 1$ for all $\bar{x} \in S$ by taking $r = 0$, $x = \bar{x}$, and $k = 1$ in (4). We first show that π is a cut-generating function for (3). To see this, note that any feasible solution $(\bar{x}, \bar{y})$ for (3) satisfies $\sum_{r\in\mathbb{R}^n} r\bar{y}_r = \bar{x} - f$, and using the subadditivity of π, we can write $\sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq \pi(\sum_{r\in\mathbb{R}^n} r\bar{y}_r) = \pi(\bar{x} - f) \geq 1$.

If π is not minimal, then by Theorem 1, there exists a minimal cut-generating function π' such that $\pi' \leq \pi$ and $\pi'(\bar{r}) < \pi(\bar{r})$ for some $\bar{r} \in \mathbb{R}^n$. Let $\epsilon = \pi(\bar{r}) - \pi'(\bar{r})$. Because π satisfies the generalized symmetry condition, there exists $\bar{x} \in S$ and $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(\bar{r}) - \frac{\epsilon}{2} \leq \frac{1}{\bar{k}}(1 - \pi(\bar{x} - f - \bar{k}\bar{r}))$. Rearranging the terms and using $\pi' \leq \pi$ and $\pi(\bar{r}) - \pi'(\bar{r}) = \epsilon$, we obtain

$$1 \geq \bar{k}\left(\pi(\bar{r}) - \frac{\epsilon}{2}\right) + \pi\left(\bar{x} - f - \bar{k}\bar{r}\right) \geq \bar{k}\left(\pi'(\bar{r}) + \frac{\epsilon}{2}\right) + \pi'\left(\bar{x} - f - \bar{k}\bar{r}\right)$$

which implies $\bar{k}\pi'(\bar{r})+\pi'(\bar{x}-f-\bar{k}\bar{r}) < 1$. This contradicts the hypothesis that π' is a cut-generating function because the solution $x = \bar{x}$, $\bar{y}_{\bar{r}} = \bar{k}$, $\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise is feasible to (3). $\square$

Next we state two properties of subadditive functions that will be used later in the paper. The first lemma below shows that if the supremum is achieved in the generalized symmetry condition, it must be achieved for $k = 1$.

Lemma 13. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function that satisfies the generalized symmetry condition. Suppose $r \in \mathbb{R}^n$ is a point for which the supremum in* (4) *is achieved. Then the supremum is achieved when $k = 1$, that is, $\pi(r) = 1 - \pi(x - f - r)$ for some $x \in S$.*

Proof. Consider a vector $r \in \mathbb{R}^n$ for which the supremum in (4) is achieved. Namely, there exists $x \in S$ and $k \in \mathbb{Z}_{++}$ such that $\pi(r) = \frac{1}{k}(1 - \pi(x - f - kr))$. This equation can be rewritten as

$$k\pi(r) + \pi(x - f - kr) = 1. \tag{5}$$

We also have

$$k\pi(r) + \pi(x - f - kr) = \pi(r) + (k - 1)\pi(r) + \pi\left(x - f - kr\right) \geq \pi(r) + \pi(x - f - r) \geq 1$$

where the first inequality follows from the subadditivity of π and the second from $\pi(r) \geq 1 - \pi(x - f - r)$ by the generalized symmetry condition. Using (5), we see that equality holds throughout. In particular, $\pi(r) + \pi(x - f - r) = 1$. Thus, the supremum in (4) is achieved when $k = 1$. $\square$

For a subadditive function $\pi : \mathbb{R}^n \to \mathbb{R}$, we have $\pi(r) \geq \frac{\pi(kr)}{k}$ for all $r \in \mathbb{R}^n$ and $k \in \mathbb{Z}_{++}$. Hence, $\pi(r) \geq \sup_k\{\frac{\pi(kr)}{k} : k \in \mathbb{Z}_{++}\}$. In fact, we have $\pi(r) = \sup_k\{\frac{\pi(kr)}{k} : k \in \mathbb{Z}_{++}\}$ because equality holds for $k = 1$. When $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k}$ for some $r \in \mathbb{R}^n$, Bachem, Johnson, and Schrader [BJS82] showed that π is actually linear in $k \in \mathbb{Z}_{++}$.

Lemma 14 (Bachem, Johnson, and Schrader [BJS82])**.** *If a subadditive function $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k}$ for some $r \in \mathbb{R}^n$, then $\pi(kr) = k\pi(r)$ for all $k \in \mathbb{Z}_{++}$.*

We close this section with a proof of Proposition 5, which was stated in the introduction.

Proposition 5. *A cut-generating function π for* (3) *is restricted minimal if and only if it is minimal and* $\inf_x\{\pi(x - f) : x \in S\} = 1$.

Proof. If π is a cut-generating function, we have $\pi(\bar{x} - f) \geq 1$ for any $\bar{x} \in S$. To see this, note that the solution $x = \bar{x}$, $y_{\bar{x}-f} = 1$, and $y_r = 0$ for all $r \neq \bar{x} - f$ is feasible to (3) and the inequality $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ reduces to $\pi(\bar{x} - f) \geq 1$.

To prove the "only if" part, let π be a restricted minimal cut-generating function. Then there does not exist any cut-generating function $\pi' \neq \pi$ that implies π via scaling by $\beta \geq 1$. By taking $\beta = 1$, we note that no cut-generating function $\pi' \neq \pi$ dominates π. Thus, π is minimal. Let $\nu = \inf_x\{\pi(x - f) : x \in S\}$. By the above observation, we have $\nu \geq 1$. Suppose $\nu > 1$, and let $\pi' = \frac{\pi}{\nu}$. For any feasible solution (x, y) to (3), we have $\sum_{r \in \mathbb{R}^n} \pi'(r) y_r = \frac{1}{\nu} \sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq \frac{1}{\nu}\pi(\sum_{r \in \mathbb{R}^n} r y_r) = \frac{1}{\nu}\pi(x - f) \geq 1$ where the first inequality follows from the subadditivity of π (Theorem 2) and the second from the definition of ν. Therefore, π' is a cut-generating function. Since π' is distinct from π and implies π via scaling, this contradicts the hypothesis that π is restricted minimal. Therefore, $\nu = \inf_x\{\pi(x - f) : x \in S\} = 1$.

For the converse, let π be a minimal cut-generating function such that $\inf_x\{\pi(x - f) : x \in S\} = 1$. Suppose π is not restricted minimal. Then there exists a cut-generating function $\pi' \neq \pi$ that implies π via scaling. That is, there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. Because π is minimal, we must have $\beta > 1$, but then $\inf_x\{\pi'(x - f) : x \in S\} = \frac{1}{\beta} \inf_x\{\pi(x - f) : x \in S\} < 1$. This implies that there exists $x \in S$ such that $\pi'(x - f) < 1$, contradicting the choice of π' as a cut-generating function. $\square$

3 Specializing the Set S

In this section, we turn our attention to sets S that arise in the context of integer programming. The majority of the results in this section consider $S = C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ where $C \subset \mathbb{R}^n$ is a closed convex set and p is an integer between 0 and n. The case $p = n$ and $C = \mathbb{R}^n_+$ is of particular interest since it corresponds to the pure integer linear programming case. At the other extreme, when $p = 0$ and C is a closed convex cone, we recover the infinite relaxation of a mixed-integer conic programming model studied by Morán, Dey, and Vielma [MDV12]. In their model, Morán, Dey, and Vielma presented an extension of the duality theory to mixed-integer conic programs and showed that subadditive functions that are nondecreasing with respect to C can be used to generate valid inequalities.

For a set $S \in \mathbb{R}^n$, let $\text{cl}(S)$ represent the closure of S, $\text{conv}(S)$ represent the convex hull of S, $\overline{\text{conv}}(S)$ represent the closed convex hull of S, $\text{rec}(S)$ represent the recession cone of S, and $\text{lin}(S)$ represent the lineality space of S.

3.1 The Case $S = C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a Convex Set C

We first show that when S is the set of mixed-integer points in a closed convex set, a function that satisfies the generalized symmetry condition is monotone in a certain sense. Let K be a closed convex cone and L be a linear subspace in $\mathbb{R}^n$. Recall that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *nondecreasing with respect to* $K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $\pi(r) \leq \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. We say that the function π is *periodic with respect to* $L \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $\pi(r) = \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in L \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Note that when $L = \mathbb{R}^n$ and $p = n$, this definition of periodicity reduces to the earlier definition of periodicity with respect to $\mathbb{Z}^n$.

Proposition 15. *Let $C \subset \mathbb{R}^n$ be a closed convex set, $S = C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and $f \in \mathbb{R}^n$. If $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies the generalized symmetry condition, then it is nondecreasing with respect to $\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. In particular, it is periodic with respect to $\text{lin}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.*

Proof. Suppose π satisfies the generalized symmetry condition. Then for any $r \in \mathbb{R}^n$ and $\epsilon > 0$, there exist $x^\epsilon \in S$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $\frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r)) > \pi(r) - \epsilon$. Let $w \in \text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Observing that $x^\epsilon + k^\epsilon w \in C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) = S$, condition (4) implies

$$\pi(r + w) \geq \frac{1}{k^\epsilon} \left(1 - \pi\left((x^\epsilon + k^\epsilon w) - f - k^\epsilon(r + w)\right)\right) = \frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r)) > \pi(r) - \epsilon.$$

Taking limits of both sides as $\epsilon \downarrow 0$, we get $\pi(r+w) \geq \pi(r)$. The second statement follows from the observation that $w, -w \in \text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $w \in \text{lin}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. In this case, repeating the same argument with both w and $-w$ gives us the equality necessary to establish the periodicity of π. $\square$

Proposition 16. *Let $C \subset \mathbb{R}^n$ be a closed convex set, $S = C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and $f \in \mathbb{R}^n$. Let $X \subset S$ be such that $S = X + (\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$. The function $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies the generalized symmetry condition if and only if it is nondecreasing with respect to $\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and satisfies the condition*

$$\pi(r) = \sup_{x,k} \left\{ \frac{1}{k}(1 - \pi(x - f - kr)) : x \in X, k \in \mathbb{Z}_{++} \right\} \text{ for all } r \in \mathbb{R}^n. \tag{6}$$

Proof. Suppose π satisfies the generalized symmetry condition. By Proposition 15, π is nondecreasing with respect to $\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $r \in \mathbb{R}^n$ and $\epsilon > 0$. For any $x \in X$ and $k \in \mathbb{Z}_{++}$, we have $k\pi(r) + \pi(x - f - kr) \geq 1$. Because π satisfies the generalized symmetry condition, there exist $x^\epsilon \in S$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $k^\epsilon \pi(r) + \pi(x^\epsilon - f - k^\epsilon r) < 1 + k^\epsilon \epsilon$. Let $\bar{x} \in X$ be such that $x^\epsilon \in \bar{x} + (\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$. Because π is nondecreasing with respect to $\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, we get $k^\epsilon \pi(r) + \pi(\bar{x} - f - k^\epsilon r) \leq k^\epsilon \pi(r) + \pi(x^\epsilon - f - k^\epsilon r) < 1 + k^\epsilon \epsilon$. This shows that π satisfies (6).

To prove the converse, suppose π is nondecreasing with respect to $\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and satisfies (6). Let $r \in \mathbb{R}^n$ and $\epsilon > 0$. For any $x \in S$ and $k \in \mathbb{Z}_{++}$, there exists $\bar{x} \in X$ such that $x \in \bar{x} + (\text{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$ and we have $k\pi(r) + \pi(x - f - kr) \geq k\pi(r) + \pi(\bar{x} - f - kr) \geq 1$. Furthermore, there exist $x^\epsilon \in X \subset S$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $\pi(r) - \epsilon < \frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r))$. This shows that π satisfies the generalized symmetry condition. $\square$

When the set X in the statement of Proposition 16 can be chosen to be finite, (6) further implies that

$$\forall r \in \mathbb{R}^n \quad \exists x^r \in X \text{ such that } \pi(r) = \sup_k \left\{ \frac{1}{k}(1 - \pi(x^r - f - kr)) : k \in \mathbb{Z}_{++} \right\}. \tag{7}$$

A finite set X satisfying the hypothesis of Proposition 16 exists for two choices of unbounded sets S which are important in integer programming. When S is the set of pure integer points in a rational (possibly unbounded) polyhedron, the existence of such a finite set X follows from Meyer's Theorem and its proof [Mey74]. When S is the set of mixed-integer points in a closed convex cone K, one can simply let $X = \{0\}$. Then (6) can be stated as

$$\pi(r) = \sup_k \left\{ \frac{1}{k}(1 - \pi(-f - kr)) : k \in \mathbb{Z}_{++} \right\} \text{ for all } r \in \mathbb{R}^n. \tag{8}$$

In general, (8) is a weaker requirement than symmetry on subadditive functions. However, the next proposition shows that (8) implies symmetry if the supremum is achieved for all $r \in \mathbb{R}^n$.

Proposition 17. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function.*

(i) Let $X \subset S$ be a finite set, and suppose π satisfies (7). *Fix $r \in \mathbb{R}^n$, and choose $x^r \in X$ as in* (7)*. The supremum in* (7) *is attained if and only if $\pi(r) + \pi(x^r - f - r) = 1$.*

(ii) Suppose π satisfies (8)*. Fix $r \in \mathbb{R}^n$. The supremum in* (8) *is attained if and only if $\pi(r) + \pi(-f - r) = 1$. Furthermore, the supremum in* (8) *is attained for all $r \in \mathbb{R}^n$ if and only if π satisfies the symmetry condition.*

Proof. We first prove (i). Fix $r \in \mathbb{R}^n$, and choose $x^r \in X$ as in (7). Suppose the supremum on the right-hand side of (7) is attained. Let $k^* \in \mathbb{Z}_{++}$ be such that $\frac{1}{k^*}(1 - \pi(x^r - f - k^*r)) \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \in \mathbb{Z}_{++}$. Because π satisfies (7), we have $\pi(r) \geq 1 - \pi(x^r - f - r)$ and $\pi(r) = \frac{1}{k^*}(1 - \pi(x^r - f - k^*r))$. Using the subadditivity of π, we can write

$$1 = k^*\pi(r) + \pi(x^r - f - k^*r) = \pi(r) + (k^* - 1)\pi(r) + \pi(x^r - f - k^*r) \geq \pi(r) + \pi(x^r - f - r) \geq 1.$$

This shows $\pi(r) + \pi(x^r - f - r) = 1$. To prove the converse, suppose $\pi(r) + \pi(x^r - f - r) = 1$. Then $1 - \pi(x^r - f - r) = \pi(r) = \sup_k\{\frac{1}{k}(1 - \pi(x^r - f - kr)) : k \in \mathbb{Z}_{++}\}$ which shows that the supremum is attained for $k = 1$. This concludes the proof of (i).

Claim (ii) follows from (i) by noting that (8) is equivalent to (7) with $X = \{0\}$. In this case, $x^r \in X$ in (7) is necessarily equal to zero for any $r \in \mathbb{R}^n$. Let $r \in \mathbb{R}^n$. By (i), the supremum in (8) is attained if and only if $\pi(r) + \pi(-f - r) = 1$. If the supremum is attained for all $r \in \mathbb{R}^n$, then $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$, which is the symmetry condition on π. □

Proposition 18. *Let $X \subset S$ be a finite set, and let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$ and π satisfies* (7)*. Fix $r \in \mathbb{R}^n$, and choose $x^r \in X$ as in* (7)*. If the supremum in* (7) *is not attained, then*

$$\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k} = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k}.$$

Furthermore, $\pi(kr) = k\pi(r)$ for all $k \in \mathbb{Z}_{++}$.

Proof. Fix $r \in \mathbb{R}^n$, and choose $x^r \in X$ as in (7). Suppose the supremum in (7) is not attained. Since π satisfies (7), $\pi(r) \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \in \mathbb{Z}_{++}$. It follows that $\pi(r) \geq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{1}{k}(1 - \pi(x^r - f - kr))$. Let $\epsilon = \pi(r) - \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{1}{k}(1 - \pi(x^r - f - kr))$, and suppose $\epsilon > 0$. By the definition of limit supremum, there exists $k_0 \in \mathbb{Z}_{++}$ such that $\pi(r) - \frac{\epsilon}{2} \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \geq k_0$. It follows that the supremum in (7) must be attained for some $k < k_0$, a contradiction. Therefore, $\epsilon = 0$. Using $\pi(0) = 0$ and the subadditivity of π, we can write

$$\begin{aligned}
\pi(r) &= \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{1 - \pi(x^r - f - kr)}{k} = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(x^r - f - kr)}{k} \\
&\leq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr) + \pi(-x^r + f)}{k} = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k} \\
&\leq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k} \leq \pi(r).
\end{aligned}$$

In particular, $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k} = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k}$. It follows from Lemma 14 that $\pi(kr) = k\pi(r)$ for all $k \in \mathbb{Z}_{++}$. □

When the set X in the statement of Proposition 16 is finite, we can obtain a simplified version of (6) in which the double supremum over x and k is decoupled through Propositions 17 and 18.

Corollary 19. *Let $C \subset \mathbb{R}^n$ be a closed convex set, $S = C \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and $f \in \mathbb{R}^n$. Let $X \subset S$ be a finite set such that $S = X + (\operatorname{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$. The function π satisfies the generalized symmetry condition if and only if it is nondecreasing with respect to $\operatorname{rec}(C) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and satisfies the condition*

$$\pi(r) = \max\left\{ \max_{x \in X}\{1 - \pi(x - f - r)\},\ \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k} \right\} \text{ for all } r \in \mathbb{R}^n. \tag{9}$$

Proof. By Proposition 16, it will be enough to show that π satisfies (6) if and only if it satisfies (9). Suppose π satisfies (6). Fix $r \in \mathbb{R}^n$. By (6), we have $\pi(r) \geq \max_{x \in X}\{1 - \pi(x - f - r)\}$. By the subadditivity of π, $\pi(r) \geq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k} \geq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k}$. The "only if" part then follows from Propositions 17 and 18 by observing that X is finite and π satisfies (7). To prove the "if" part, suppose π satisfies (9). Fix $r \in \mathbb{R}^n$. Observe that (9) implies $\pi(r) \geq 1 - \pi(x - f - r)$ for all $x \in X$. By the subadditivity of π, $k\pi(r) + \pi(x - f - kr) \geq \pi(r) + \pi(x - f - r) \geq 1$ for all $x \in X$ and $k \in \mathbb{Z}_{++}$. In particular, $\pi(r) \geq \sup_{x,k}\{\frac{1}{k}(1 - \pi(x - f - kr)) : x \in X, k \in \mathbb{Z}_{++}\}$. If there exists $x^r \in X$ such that $\pi(r) = 1 - \pi(x^r - f - r)$, then (7) holds for that x^r. If $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k}$, then (7) holds for any $x \in X$ since

$$\limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{1 - \pi(x - f - kr)}{k} \geq \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{1 - \pi(x - f) - \pi(-kr)}{k} = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{-\pi(-kr)}{k} = \pi(r).$$

In either case, (6) is satisfied. □

3.2 The Case $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a Convex Cone K

In this section, we consider the case where S is the set of mixed-integer points in a closed convex cone K. The following theorem recapitulates the results of Theorem 2 and Proposition 16 for this case.

Theorem 20. *Let $K \subset \mathbb{R}^n$ be a closed convex cone and $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (3) *if and only if $\pi(0) = 0$, π is subadditive, nondecreasing with respect to S, and satisfies* (8).

When $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a closed convex cone K, we can choose $X = \{0\}$ in Corollary 19. Then (8) in the statement of Theorem 20 can be replaced without any loss of generality with (9) which now reads $\pi(r) = \max\{1 - \pi(-f - r),\ \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}\}$ for all $r \in \mathbb{R}^n$. This condition simplifies further to just $\pi(r) = 1 - \pi(-f - r)$, the symmetry condition, when we consider *restricted* minimal cut-generating functions. This will be proved next in Theorem 7, which was already stated in the introduction. Theorem 7 generalizes to $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ a result of Bachem, Johnson, and Schrader [BJS82] for $S = \{0\}$.

Theorem 7. *Let $K \subset \mathbb{R}^n$ be a closed convex cone and $S = K \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a restricted minimal cut-generating function for* (3) *if and only if $\pi(0) = 0$, π is subadditive, nondecreasing with respect to S, and satisfies the symmetry condition.*

Proof. We first prove the "if" part. Assume $\pi(0) = 0$, π is subadditive, nondecreasing with respect to S, and satisfies the symmetry condition. Since condition (8) is a weaker requirement than symmetry, it follows from Theorem 20 that π is a minimal cut-generating function. Because π is nondecreasing with respect to S, we have $\pi(x - f) \geq \pi(-f)$ for all $x \in S$. Furthermore, by taking $r = -f$, the symmetry condition implies $\pi(-f) = 1$. It follows that $\min\{\pi(x - f) : x \in S\} = \pi(-f) = 1$. Then by Proposition 5, π is restricted minimal.

We now prove the "only if" part. Assume that π is a restricted minimal cut-generating function. By Proposition 5, π is a minimal cut-generating function and satisfies $\inf_x\{\pi(x - f) : x \in S\} = 1$. Since π is minimal, Theorem 20 implies that $\pi(0) = 0$, π is subadditive, nondecreasing with respect to S, and satisfies (8). Because π is nondecreasing with respect to S, we have $\pi(-f) = \inf_x\{\pi(x - f) : x \in S\} = 1$. Now suppose that there exists $\bar{r} \in \mathbb{R}^n$ such that $\pi(\bar{r}) > 1 - \pi(-f - \bar{r})$. Letting $X = \{0\}$ and using Proposition 17(i), we see that the supremum in (8) is not attained. By Proposition 18, $\pi(k\bar{r}) = k\pi(\bar{r})$ for all $k \in \mathbb{Z}_{++}$. By the subadditivity of π, $\pi(-f + k(f + \bar{r})) + (k - 1)\pi(-f) \geq \pi(k\bar{r}) = k\pi(\bar{r})$ for all $k \in \mathbb{Z}_{++}$. Rearranging terms and using $\pi(-f) = 1$, we get $k(1 - \pi(\bar{r})) \geq 1 - \pi(-f + k(f + \bar{r}))$. Thus, $1 - \pi(\bar{r}) \geq \frac{1-\pi(-f+k(f+\bar{r}))}{k}$ for all $k \in \mathbb{Z}_{++}$. This implies

$$1 - \pi(\bar{r}) \geq \sup_k \left\{ \frac{1 - \pi(-f - k(-f - \bar{r}))}{k} : k \in \mathbb{Z}_{++} \right\} = \pi(-f - \bar{r})$$

where the equality follows from (8). This contradicts the hypothesis that $\pi(\bar{r}) > 1 - \pi(-f - \bar{r})$. $\square$

Let $K_1, K_2 \in \mathbb{R}^n$ be two closed convex cones such that $K_2 \subset K_1$. Because $K_2 \subset K_1$, every cut-generating function for (3) when $S = K_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is a cut-generating function for (3) when $S = K_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. However, it is rather surprising that every *restricted minimal* cut-generating function for (3) when $S = K_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is also a restricted minimal cut-generating function for (3) when $S = K_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. A similar statement is also true for minimal cut-generating functions. We show this in the next proposition.

Proposition 21. *Let $K_1, K_2 \in \mathbb{R}^n$ be two closed convex cones such that $K_2 \subset K_1$. If π is a (restricted) minimal cut-generating function for* (3) *when $S = K_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, then π is also a (restricted) minimal cut-generating function for* (3) *when $S = K_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.*

Proof. We prove the statement for the case of restricted minimality only. A similar claim on minimal cut-generating functions follows by using Theorem 20 instead of Theorem 7.

Assume π is a restricted minimal cut-generating function for (3) when $S = K_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. By Theorem 7, $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $K_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and satisfies the symmetry condition. Because $K_2 \subset K_1$, π is also nondecreasing with respect to $K_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Therefore, again by Theorem 7, π is a restricted minimal cut-generating function for (3) when $S = K_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. ☐

In particular, Proposition 21 implies that a (restricted) minimal cut-generating function for (3) when $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ is still (restricted) minimal for (3) when $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, and a (restricted) minimal cut-generating function for (3) when $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ is still (restricted) minimal for (3) when $S = \{0\}$. We focus on the cases $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ and $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ in the next two sections.

3.3 The Case $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$

In their seminal papers, Gomory and Johnson [GJ72a] and Johnson [Joh74] characterized minimal cut-generating functions for (2) in terms of subadditivity, symmetry, and periodicity with respect to $\mathbb{Z}^n$. In this section, we relate our Theorems 7 and 20 to their results.

For the model (2), Theorem 20 states that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function if and only if $\pi(0) = 0$, π is subadditive, periodic with respect to $\mathbb{Z}^n$, and satisfies (8). For the same model, Theorem 7 shows that π is restricted minimal if and only if it satisfies the symmetry condition as well as the conditions for minimality above. In the context of model (2), cut-generating functions are conventionally required to be nonnegative; therefore, the minimal ones take values in the interval $[0, 1]$ only. (See [GJ72a, Joh74, CCZ11a].) While the above implications of Theorems 7 and 20 hold without this additional assumption, the notions of minimality and restricted minimality coincide for nonnegative cut-generating functions for (2). To see this, note that any nonnegative minimal cut-generating function π for (2) satisfies $\pi(-f) \geq 1$ because $0 \in S$ and $\pi(-f) \leq 1$ because it takes values in $[0, 1]$ only. The periodicity of π with respect to $\mathbb{Z}^n$ then implies $\min_x\{\pi(x - f) : x \in \mathbb{Z}^n\} = \pi(-f) = 1$. It follows from Proposition 5 that any nonnegative minimal cut-generating function for (2) is in fact restricted minimal. Hence, by taking $K = \mathbb{R}^n$ and $p = n$ in the statement of Theorem 7, we can recover the well-known result of Gomory and Johnson on nonnegative minimal cut-generating functions for (2).

Theorem 22 (Gomory and Johnson [GJ72a], Johnson [Joh74])**.** *Let $\pi : \mathbb{R}^n \to \mathbb{R}_+$. The function π is a minimal cut-generating function for* (2) *if and only if $\pi(0) = 0$, π is subadditive, symmetric, and periodic with respect to $\mathbb{Z}^n$.*

Note that when $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, a minimal cut-generating function $\pi : \mathbb{R}^n \to \mathbb{R}$ for (3) has to be periodic with respect to $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ by Theorem 20. In particular, the value of π cannot depend on the last $n - p$ entries of its argument. This shows a simple bijection between minimal cut-generating functions for $S = \mathbb{Z}^p$ and those for $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$: Let $\mathrm{proj}_{\mathbb{R}^p} : \mathbb{R}^n \to \mathbb{R}^p$ denote the orthogonal projection onto the first p coordinates. The function $\pi' : \mathbb{R}^p \to \mathbb{R}$ is a minimal cut-generating function for $S = \mathbb{Z}^p$ if and only if $\pi = \pi' \circ \mathrm{proj}_{\mathbb{R}^p}$ is a minimal cut-generating function for $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$. Using the same arguments, one can also show that such a bijection exists between restricted minimal cut-generating functions for $S = \mathbb{Z}^p$ and those for $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$.

3.4 The Case $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$

In this section, we focus on the case where $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ which is of particular importance in integer linear programming. We simplify the statement of Theorems 7 and 20 for this special case exploiting the fact that $\mathbb{R}^n_+$ has the finite generating set $\{e^i\}_{i=1}^n$. However, we first prove a simple lemma.

Lemma 23. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function. For any $\alpha > 0$ and $r \in \mathbb{R}^n$, $\frac{\pi(\alpha r)}{\alpha} \leq \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$.*

Proof. Consider $\epsilon = \frac{\alpha}{k}$ for $k \in \mathbb{Z}_{++}$. We have $k\pi(\frac{\alpha}{k} r) \geq \pi(\alpha r)$ by the subadditivity of π. Thus, $\frac{\pi(\alpha r)}{\alpha} \leq \frac{\pi(\frac{\alpha}{k} r)}{\frac{\alpha}{k}}$. Letting $k \to +\infty$, this implies $\frac{\pi(\alpha r)}{\alpha} \leq \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$. □

Proposition 24. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$. The function π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ if and only if $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$.*

Proof. Suppose π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. Because $\pi(0) = 0$, π has to have $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$. Now suppose $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$. For any $w \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, using the subadditivity of π and Lemma 23 with $\alpha = w_i$ for $i \in [n] \setminus [p]$, we can write

$$\pi(-w) \leq \sum_{i=1}^{n} \pi(-w_i e^i) \leq \sum_{i=1}^{p} w_i \pi(-e^i) + \sum_{i=p+1}^{n} w_i \limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0.$$

Thus, for any $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, $\pi(r + w) \geq \pi(r) - \pi(-w) \geq \pi(r)$. This shows that π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. □

Theorem 20 and Proposition 24 thus show the following: A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for (3) when $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ if and only if $\pi(0) = 0$, $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies (8). Similarly, Theorem 7 and Proposition 24 show the following.

Theorem 25. *Let $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ and $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a restricted minimal cut-generating function for (3) if and only if $\pi(0) = 0$, $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies the symmetry condition.*

4 Strongly Minimal Cut-Generating Functions

The following example illustrates the distinction between restricted minimal and strongly minimal cut-generating functions.

Example 3. Consider the model (3) where $n = 1$, $0 < f < 1$, and $S = \mathbb{Z}_+$. The Gomory function $\pi^1_1(r) = \min\{\frac{r - \lfloor r \rfloor}{1-f}, \frac{\lceil r \rceil - r}{f}\}$ is a cut-generating function in this setting. For any $\alpha \geq 0$, we define perturbations of the Gomory function as $\pi^3_\alpha(r) = \alpha r + (1 + \alpha f)\pi^1_1(r)$. One can easily verify that $\pi^3_\alpha(0) = 0$ and $\pi^3_\alpha(-1) = -\alpha \leq 0$. Furthermore, π^3_α is symmetric and subadditive since π^1_1 is. By Theorem 25, π^3_α is a restricted minimal cut-generating function. However, for $\alpha > 0$, π^3_α is not strongly minimal because it is implied by the Gomory function π^1_1.

When $f \notin \overline{\operatorname{conv}}(S)$, any valid inequality that strictly separates f from S can be used to cut off the infeasible solution $x = f$, $y = 0$. Therefore, when we analyze strongly minimal cut-generating functions, our focus will be on the case $f \in \overline{\operatorname{conv}}(S)$.

Lemma 26. *Suppose $f \in \overline{\operatorname{conv}}(S)$. Let π be a (restricted) minimal cut-generating function for* (3)*. Any cut-generating function for* (3) *that implies π is also (restricted) minimal.*

Proof. We will prove the claim for the case of restricted minimality only. The proof for minimality is similar.

Let π be a restricted minimal cut-generating function for (3). Let π' be a cut-generating function that implies π. Then there exist a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for S and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\operatorname{conv}}(S)$, the inequality $\alpha^\top(x-f) \geq \alpha_0$ is also valid for $x = f$. Hence, $\alpha_0 \leq 0$, and $\beta \geq 1$. We claim that π' is restricted minimal.

Let $\bar\pi'$ be a restricted minimal cut-generating function that implies π' via scaling. Such a function $\bar\pi'$ always exists by Proposition 6. Then there exists $\nu \geq 1$ such that $\pi' \geq \nu\bar\pi'$. By Proposition 5 and Theorem 2, $\bar\pi'$ is subadditive. We first show that $\bar\pi : \mathbb{R}^n \to \mathbb{R}$, defined as $\bar\pi(r) = \beta\bar\pi'(r) + \frac{\alpha^\top r}{\nu}$, is also a cut-generating function. Indeed, for any feasible solution (x, y) to (3), we can use the validity of $\alpha^\top(x-f) \geq \alpha_0$ for S and the subadditivity of $\bar\pi'$ to write

$$\sum_{r\in\mathbb{R}^n} \bar\pi(r)y_r = \sum_{r\in\mathbb{R}^n} \frac{\alpha^\top r}{\nu} y_r + \beta \sum_{r\in\mathbb{R}^n} \bar\pi'(r)y_r \geq \frac{\alpha^\top(x-f)}{\nu} + \beta\bar\pi'(x-f) \geq \frac{\alpha_0}{\nu} + \beta \geq \alpha_0 + \beta \geq 1.$$

Therefore, $\bar\pi$ is a cut-generating function. Because $\nu \geq 1$, so is $\nu\bar\pi$. Furthermore, for all $r \in \mathbb{R}^n$, we have

$$\nu\bar\pi(r) = \alpha^\top r + \beta\nu\bar\pi'(r) \leq \alpha^\top r + \beta\pi'(r) \leq \pi(r). \tag{10}$$

Since π is a restricted minimal cut-generating function, $\nu\bar\pi = \bar\pi = \pi$, $\nu = 1$, and equality holds throughout (10). In particular, the first inequality in (10) is tight. Using this, $\nu = 1$, and $\beta \geq 1$, we get $\bar\pi' = \pi'$. This proves that π' is restricted minimal. ☐

The next proposition characterizes strongly minimal cut-generating functions as a certain subset of restricted minimal cut-generating functions.

Proposition 27. *Suppose S is full-dimensional and $f \in \overline{\operatorname{conv}}(S)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a strongly minimal cut-generating function for* (3) *if and only if it is a restricted minimal cut-generating function for* (3) *and for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for S such that $\alpha \neq 0$, there exists $x^* \in S$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$.*

Proof. We first prove the "only if" part of the statement. Let π be a strongly minimal cut-generating function for (3). It follows by setting $\alpha = 0$ and $\alpha_0 = 0$ in the definition of strong minimality that π is restricted minimal. In particular, it is subadditive by Theorem 2 and Proposition 5. Suppose there exists a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for S such that $\alpha \neq 0$ and $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0} \geq 1$ for all $x \in S$. Because $f \in \overline{\operatorname{conv}}(S)$, we must have $\alpha_0 \leq 0$. Define the function $\pi' : \mathbb{R}^n \to \mathbb{R}$ by letting $\pi'(r) = \frac{\pi(r)-\alpha^\top r}{1-\alpha_0}$. We claim that π' is a cut-generating function. To see this, first note that π' is subadditive because π is. Also, $\pi'(x-f) \geq 1$ for all $x \in S$ by our hypothesis. Then for any feasible solution (x, y) to (3), we can write $\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq \pi'(\sum_{r\in\mathbb{R}^n} ry_r) = \pi'(x-f) \geq 1$. Thus, π' is indeed a cut-generating function for (3). Furthermore, it is not difficult to show that π' is distinct from π. Consider $\bar{x} \in S$ such that $\alpha^\top(\bar{x}-f) > \alpha_0$; such a point exists because S is full-dimensional. Because π is a cut-generating function, $\pi(\bar{x}-f) \geq 1$. Then $\pi'(\bar{x}-f) = \frac{\pi(\bar{x}-f)-\alpha^\top(\bar{x}-f)}{1-\alpha_0} < \pi(\bar{x}-f)$ because $\alpha^\top(\bar{x}-f) > \alpha_0 \geq \alpha_0 \pi(\bar{x}-f)$. Finally, note that π' implies π since $\pi(r) \geq (1-\alpha_0)\pi'(r)+\alpha^\top r$ for all $r \in \mathbb{R}^n$. Because π' is distinct from π, this contradicts the strong minimality of π.

Now we prove the "if" part. Let π be a restricted minimal cut-generating function for (3). Suppose that for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for S such that $\alpha \neq 0$, there exists $x^* \in S$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$. Let π' be a cut-generating function that implies π. Then there exists a valid inequality $\mu^\top(x-f) \geq \mu_0$ and $\nu \geq 0$ for S such that $\mu_0+\nu \geq 1$ and $\pi(r) \geq \nu\pi'(r)+\mu^\top r$ for all $r \in \mathbb{R}^n$. Note that $\mu_0 \leq 0$ because $f \in \overline{\text{conv}}(S)$. We will show $\pi' = \pi$, proving that π is strongly minimal. First suppose $\mu \neq 0$. Then by our hypothesis, there exists $x^* \in S$ such that $1 > \frac{\pi(x^*-f)-\mu^\top(x^*-f)}{1-\mu_0} \geq \frac{\nu\pi'(x^*-f)}{1-\mu_0}$. Rearranging the terms, we get $\pi'(x^*-f) < \frac{1-\mu_0}{\nu} \leq 1$. This contradicts the fact that π' is a cut-generating function because the solution $x = x^*$, $y_{x^*-f} = 1$, and $y_r = 0$ otherwise is feasible to (3). Hence, we can assume $\mu = 0$. Then we actually have $\pi \geq \nu\pi'$ for some $\nu \geq 1$. Because π is restricted minimal, it must be that $\pi' = \pi$. □

4.1 Strongly Minimal Cut-Generating Functions for $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$

The main result of this section is Theorem 8, which was already stated in the introduction.

Theorem 8. *Let $S = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ and $f \in \mathbb{R}^n_+ \setminus S$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a strongly minimal cut-generating function for (3) if and only if $\pi(0) = 0$, $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies the symmetry condition.*

Proof. Let π be a restricted minimal cut-generating function. By Theorem 25 and Proposition 27, it will be enough to show that $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n]\setminus[p]$ if and only if, for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\alpha \neq 0$, there exists $x^* \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$.

We first prove the "if" part of the statement above. Because π is restricted minimal, Theorem 25 implies that $\pi(-e^i) \leq 0$ for all $i \in [p]$, $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, π is subadditive and symmetric. The symmetry condition implies in particular that $\pi(-f) = 1$. Suppose in addition that for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ with $\alpha \neq 0$, there exists $x^* \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$. Let $\alpha \in \mathbb{R}^n$ be such that $\alpha_i = -\pi(-e^i)$ for all $i \in [p]$ and $\alpha_i = -\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon}$ for all $i \in [n] \setminus [p]$. Note that α is well-defined since π is subadditive and $\pi(-e^i) \leq \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = -\alpha_i \leq 0$ for all $i \in [n] \setminus [p]$ by Lemma 23. Now consider the inequality $\alpha^\top(x-f) \geq -\alpha^\top f$ which is valid for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ because $\alpha \in \mathbb{R}^n_+$. Note that for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we can write

$$\begin{aligned}
\pi(x-f) - \alpha^\top x &= \pi(x-f) + \sum_{i=1}^{p} \pi(-e^i)x_i + \sum_{i=p+1}^{n} \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} x_i \\
&\geq \pi(x-f) + \sum_{i=1}^{p} \pi(-e^i)x_i + \sum_{i=p+1}^{n} \pi(-e^i x_i) \geq \pi(-f) = 1
\end{aligned}$$

by using Lemma 23 and the subadditivity of π to obtain the first and second inequality, respectively. Because $\alpha, f \in \mathbb{R}^n_+$ and $\pi(x-f) - \alpha^\top x \geq 1$ for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we have $\frac{\pi(x-f)-\alpha^\top(x-f)}{1+\alpha^\top f} \geq 1$ for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. Then by our hypothesis, we must have $\alpha = 0$.

We now prove the "only if" part. Via Theorem 25, the restricted minimality of π implies that $\pi(-e^i) \leq 0$ for all $i \in [p]$, $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, and π is subadditive. Suppose in addition that $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$. Let $\alpha^\top(x-f) \geq \alpha_0$ be a valid inequality for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0} \geq 1$ for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. We are going to show $\alpha = 0$. First observe that because the inequality $\alpha^\top(x-f) \geq \alpha_0$ is valid for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we must have $\alpha \in \mathbb{R}^n_+$ and $\alpha_0 \leq 0$. Define the function $\pi' : \mathbb{R}^n \to \mathbb{R}$ by letting $\pi'(r) = \frac{\pi(r)-\alpha^\top r}{1-\alpha_0}$. Then π' is subadditive because π is. Furthermore, $\pi'(x-f) \geq 1$ for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ by our choice of the inequality $\alpha^\top(x-f) \geq \alpha_0$. These two observations imply that π' is a cut-generating function because for any solution (x, y) feasible to (3), we have $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq \pi(x-f) \geq 1$. Furthermore, π' implies π by definition. It follows from Lemma 26 that π' is also restricted minimal. Then by Theorem 25, $0 \geq \pi'(-e^i) = \frac{\pi(-e^i)+\alpha_i}{1-\alpha_0} = \frac{\alpha_i}{1-\alpha_0}$ for all $i \in [p]$ and

$$0 \geq \limsup_{\epsilon\to 0^+} \frac{\pi'(-\epsilon e^i)}{\epsilon} = \frac{1}{1-\alpha_0}\left(\alpha_i + \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon}\right) = \frac{\alpha_i}{1-\alpha_0}$$

for all $i \in [n] \setminus [p]$. Together with $\alpha \in \mathbb{R}^n_+$ and $\alpha_0 \leq 0$, this implies $\alpha = 0$. ☐

Example 4. Theorem 8 implies in particular that the cut-generating functions π^1_α of Example 1 are strongly minimal. On the other hand, none of the minimal cut-generating functions π^2_α of Example 2 are strongly minimal. Indeed, for $f > 0$, the inequality $\alpha(x-f) \geq 1$ is valid for $S = \{0\}$ when $\alpha \leq -\frac{1}{f}$. Therefore, setting $\alpha_0 = 1$ and $\beta = 0$ in the definition of implication shows that π^2_α is implied by the trivial cut-generating function π_0 that takes the value 1 for all $r \in \mathbb{R}$. Note that π_0 is not minimal since it does not satisfy Lemma 10.

4.2 Existence of Strongly Minimal Cut-Generating Functions

We observe that Theorem 8 is stated for a rather special set S. One issue is the existence of strongly minimal cut-generating functions for a general set S. In particular, in Example 2, no strongly minimal cut-generating function exists despite the existence of minimal and restricted minimal cut-generating functions. We show this in the next proposition.

Proposition 28. *No strongly minimal cut-generating function exists for the model of Example 2.*

Proof. Let π be a cut-generating function for the model of Example 2. We will show that π cannot be strongly minimal. That is, we will show that there exists a cut-generating function $\pi' \neq \pi$ such that π' implies π.

Let $0 < \beta < 1$, $\alpha_0 = 1 - \beta > 0$, and $\alpha = \frac{\alpha_0}{-f} < 0$. Note that $\alpha(x-f) \geq \alpha_0$ is valid for $S = \{0\}$. Define the function π' by letting $\pi'(r) = \frac{\pi(r)-\alpha r}{\beta}$. Suppose $\pi' = \pi$. This implies $\pi(r) = \frac{r}{-f}$ for $r \in \mathbb{R}$, but we showed in Example 4 that such a linear function is implied by the trivial cut-generating function π_0. Therefore, π cannot be strongly minimal in this case. Hence, we may assume $\pi' \neq \pi$. We will show that π' is a cut-generating function. Since π' implies π, this will prove that π is not strongly minimal. For every feasible solution (x, y), we have $\sum_{r\in\mathbb{R}} \pi(r)y_r \geq 1$ and $\sum_{r\in\mathbb{R}} ry_r = -f$.

By the definition of π', we can write $\sum_{r\in\mathbb{R}} \pi'(r)y_r = \frac{1}{\beta}(\sum_{r\in\mathbb{R}} \pi(r)y_r - \alpha \sum_{r\in\mathbb{R}} ry_r) \geq \frac{1}{\beta}(1+\alpha f) = \frac{1}{\beta}(1-\alpha_0) = 1$. Thus, π' is a cut-generating function. $\square$

Next we prove Theorem 9 stated in the introduction.

Theorem 9. *Suppose $\overline{\operatorname{conv}}(S)$ is a full-dimensional polyhedron. Let $f \in \overline{\operatorname{conv}}(S)$. Then every cut-generating function for* (3) *is implied by a strongly minimal cut-generating function.*

Proof. Let π be a cut-generating function for (3). By Proposition 6, there exists a restricted minimal cut-generating function π^0 that implies π via scaling. By Proposition 5 and Theorem 2, π^0 is subadditive. Furthermore, $\pi^0(x-f) \geq 1$ for all $x \in S$. Consider an explicit description of $\overline{\operatorname{conv}}(S)$ with t linear inequalities: $\overline{\operatorname{conv}}(S) = \{x \in \mathbb{R}^n : \; {\alpha^i}^\top(x-f) \geq \alpha^i_0 \text{ for } i \in [t]\}$. Note that $\alpha^i_0 \leq 0$ for all $i \in [t]$ because $f \in \overline{\operatorname{conv}}(S)$. Let $\lambda^*_0 = 0$. We define a finite sequence of functions $\{\pi^i\}_{i=1}^t$ iteratively as follows:

- Given π^{i-1}, let λ^*_i be the largest value λ_i that satisfies $\frac{\pi^{i-1}(x-f)-\lambda_i {\alpha^i}^\top(x-f)}{1-\lambda_i\alpha^i_0} \geq 1$ for all $x \in S$.
- Define the function π^i by letting $\pi^i(r) = \frac{\pi^{i-1}(r)-\lambda^*_i {\alpha^i}^\top r}{1-\lambda^*_i\alpha^i_0}$.

Claim 1. *For all $i \in \{0,\dots,t\}$, $\lambda^*_i \geq 0$ and π^i is a restricted minimal cut-generating function.*

We prove the claim by induction. The claim holds for $i = 0$. Assume that it holds for $i = j-1$ where $j \in [t]$. Note that λ^*_j is well-defined because $\overline{\operatorname{conv}}(S)$ is full-dimensional and there exists $x^j \in S$ such that ${\alpha^j}^\top(x^j-f) > \alpha^j_0$. Furthermore, $\lambda^*_j \geq 0$ because $\pi^{j-1}(x-f) \geq 1$ for all $x \in S$. The function π^j is a subadditive cut-generating function because it satisfies $\pi^j(x-f) \geq 1$ for all $x \in S$ and π^{j-1} is subadditive by Proposition 5 and Theorem 2. Moreover, π^j is restricted minimal by Lemma 26 because it implies π^{j-1} by definition and π^{j-1} is restricted minimal. This concludes the proof of Claim 1.

Claim 2. *For all $i \in [t]$ and $x \in S$, $\pi^i(x-f) \leq \pi^{i-1}(x-f)$.*

Indeed, for all $i \in [t]$ and $x \in S$, we can write

$$\pi^i(x-f) = \frac{\pi^{i-1}(x-f) - \lambda^*_i {\alpha^i}^\top(x-f)}{1-\lambda^*_i\alpha^i_0} \leq \frac{\pi^{i-1}(x-f) - \lambda^*_i\alpha^i_0}{1-\lambda^*_i\alpha^i_0} \leq \pi^{i-1}(x-f).$$

The first inequality above follows from the validity of ${\alpha^i}^\top(x-f) \geq \alpha^i_0$ for S, the second inequality follows from $\alpha^i_0 \leq 0$ and the fact that $\pi^{i-1}(x-f) \geq 1$ for all $x \in S$. This concludes the proof of Claim 2.

Claim 3. *For all $i \in [t]$ and $\lambda > 0$, there exists $x \in S$ such that $\frac{\pi^i(x-f)-\lambda{\alpha^i}^\top(x-f)}{1-\lambda\alpha^i_0} < 1$.*

To see this, fix $i \in [t]$ and suppose that the claim is not true. Then there exists $\lambda > 0$ such that

$$\begin{aligned} 1 \leq \frac{\pi^i(x-f) - \lambda{\alpha^i}^\top(x-f)}{1-\lambda\alpha^i_0} &= \frac{\frac{\pi^{i-1}(x-f)-\lambda^*_i{\alpha^i}^\top(x-f)}{1-\lambda^*_i\alpha^i_0} - \lambda{\alpha^i}^\top(x-f)}{1-\lambda\alpha^i_0} \\ &= \frac{\pi^{i-1}(x-f) - (\lambda^*_i + \lambda(1-\lambda^*_i\alpha^i_0)){\alpha^i}^\top(x-f)}{1-(\lambda^*_i + \lambda(1-\lambda^*_i\alpha^i_0))\alpha^i_0} \end{aligned}$$

for all $x \in S$. Because $\lambda(1-\lambda_i^*\alpha_0^i) > 0$, we get $\lambda_i^* + \lambda(1-\lambda_i^*\alpha_0^i) > \lambda_i^*$ which contradicts the maximality of λ_i^*. This concludes the proof of Claim 3.

Claim 4. *For all $i \in [t]$ and $\lambda \in \mathbb{R}_+^i \setminus \{0\}$, there exists $x \in S$ such that* $\frac{\pi^i(x-f)-\sum_{\ell=1}^i \lambda_\ell {\alpha^\ell}^\top (x-f)}{1-\sum_{\ell=1}^i \lambda_\ell \alpha_0^\ell} < 1$.

We have already proved this for $i=1$ in Claim 3. Assume now that the claim holds for $i = j-1 \in [t-1]$. Let $\lambda \in \mathbb{R}_+^j \setminus \{0\}$. If $\lambda_j = 0$, we can write

$$\frac{\pi^j(x-f)-\sum_{\ell=1}^j \lambda_\ell {\alpha^\ell}^\top (x-f)}{1-\sum_{\ell=1}^j \lambda_\ell \alpha_0^\ell} \le \frac{\pi^{j-1}(x-f)-\sum_{\ell=1}^{j-1} \lambda_\ell {\alpha^\ell}^\top (x-f)}{1-\sum_{\ell=1}^{j-1} \lambda_\ell \alpha_0^\ell} < 1.$$

Here we have used Claim 2 to obtain the first inequality and the induction hypothesis to obtain the second inequality. If $\lambda_j > 0$, we get

$$\frac{\pi^j(x-f)-\sum_{\ell=1}^j \lambda_\ell {\alpha^\ell}^\top (x-f)}{1-\sum_{\ell=1}^j \lambda_\ell \alpha_0^\ell} \le \frac{\pi^j(x-f)-\sum_{\ell=1}^{j-1} \lambda_\ell \alpha_0^\ell - \lambda_j {\alpha^j}^\top (x-f)}{1-\sum_{\ell=1}^{j-1} \lambda_\ell \alpha_0^\ell - \lambda_j \alpha_0^j} < 1$$

by using Claim 3 to obtain the second inequality. This concludes the proof of Claim 4.

By Claim 1, π^t is a restricted minimal cut-generating function. Furthermore, π^t implies π^0. By Proposition 27, to prove that π^t is strongly minimal, it is enough to show that for any valid inequality $\alpha^\top(x-f) \ge \alpha_0$ for S such that $\alpha \ne 0$, there exists $x \in S$ such that $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0} < 1$. Let $\alpha^\top(x-f) \ge \alpha_0$ be any valid inequality for S such that $\alpha \ne 0$. We have $\alpha_0 \le 0$ because $f \in \overline{\text{conv}}(S)$. By Farkas' Lemma, there exists $\lambda \in \mathbb{R}_+^t \setminus \{0\}$ such that $\alpha = \sum_{\ell=1}^t \lambda_\ell \alpha^\ell$ and $\sum_{\ell=1}^t \lambda_\ell \alpha_0^\ell \ge \alpha_0$. By Claim 4 above, there exists $x \in S$ such that $\pi^t(x-f) - \sum_{\ell=1}^t \lambda_\ell \alpha^\ell < 1 - \sum_{\ell=1}^t \lambda_\ell \alpha_0^\ell \le 1-\alpha_0$. Proposition 27 now implies that π^t is strongly minimal.

□

5 Extreme Cut-Generating Functions

A cut-generating function π is *extreme* if, whenever cut-generating functions π_1, π_2 satisfy $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, we have $\pi = \pi_1 = \pi_2$. It follows from this definition that extreme cut-generating functions are minimal. The following result shows that extreme cut-generating functions must in fact be strongly minimal; see Gomory and Johnson [GJ72a], Johnson [Joh74], and Kılınç-Karzan [KK] for analogous results.

Lemma 29. *Suppose $f \in \overline{\text{conv}}(S)$. Any extreme cut-generating function for* (3) *is strongly minimal.*

Proof. We prove the contrapositive, namely, any cut-generating function that is not strongly minimal cannot be extreme. Let π be a cut-generating function for (3) that is not strongly minimal. Then there exist a cut-generating function $\pi' \ne \pi$, a valid inequality $\alpha^\top(x-f) \ge \alpha_0$ for S, and $\beta \ge 0$ such that $\alpha_0 + \beta \ge 1$ and $\pi(r) \ge \alpha^\top r + \beta\pi'(r)$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\text{conv}}(S)$, we must have $\alpha_0 \le 0$, and $\beta \ge 1$. We divide the rest of the proof into two cases. In each case, we exhibit cut-generating functions π_1, π_2 that are distinct from π and satisfy $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$.
Case i: $\alpha_0 + \beta > 1$. Let $\delta > 0$ be such that $\alpha_0 + \beta = 1 + \delta$. Let π_1 and π_2 be defined as $\pi_1 = \frac{1}{1+\delta}\pi$ and $\pi_2 = \frac{1+2\delta}{1+\delta}\pi$. It is easy to check that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Furthermore, π_1 and π_2 are distinct from π

since for any $x \in S$, $\pi_1(x-f) \neq \pi(x-f)$ and $\pi_2(x-f) \neq \pi(x-f)$. We show that π_1 and π_2 are indeed cut-generating functions. Let (x, y) be a feasible solution to (3) so that $f+\sum_{r\in\mathbb{R}^n} ry_r = x \in S$. Then $\sum_{r\in\mathbb{R}^n} \pi_1(r)y_r \geq \frac{1}{1+\delta}(\sum_{r\in\mathbb{R}^n} \alpha^\top ry_r + \beta\sum_{r\in\mathbb{R}^n} \pi'(r)y_r) \geq \frac{1}{1+\delta}(\alpha^\top(x-f)+\beta) \geq \frac{\alpha_0+\beta}{1+\delta} = 1$. Similarly, $\sum_{r\in\mathbb{R}^n} \pi_2(r)y_r = \frac{1+2\delta}{1+\delta}\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq \frac{1+2\delta}{1+\delta} > 1$. Thus, π_1 and π_2 are cut-generating functions.
Case ii: $\alpha_0 + \beta = 1$. Let π_1 and π_2 be defined as $\pi_1 = \pi'$ and $\pi_2 = \pi + (\pi - \pi')$. It is again easy to see that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. The function π_1 is a cut-generating function that is distinct from π by hypothesis. Furthermore, π_2 is distinct from π because π_1 is distinct from π. We show that π_2 is a cut-generating function. Note that $\alpha_0 + \beta = 1$; hence, $\pi_2(r) = \pi(r) + (\pi(r) - (\alpha_0+\beta)\pi'(r)) = \pi(r) + ((\pi(r) - \beta\pi'(r)) - \alpha_0\pi'(r)) \geq \pi(r) + (\alpha^\top r - \alpha_0\pi'(r))$ for all $r \in \mathbb{R}^n$. For any feasible solution (x, y) to (3), we can write $\sum_{r\in\mathbb{R}^n} \pi_2(r)y_r \geq \sum_{r\in\mathbb{R}^n} \pi(r)y_r + \sum_{r\in\mathbb{R}^n} \alpha^\top ry_r - \alpha_0\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq \sum_{r\in\mathbb{R}^n} \pi(r)y_r + \alpha^\top(x-f) - \alpha_0 \geq 1$ where the second inequality is obtained by using $\alpha_0 \leq 0$. Thus, π_2 is a cut-generating function. □

The main purpose of this section is to prove a 2-Slope Theorem for extreme cut-generating functions for (3) when $S = \mathbb{Z}_+$, in the spirit of the Gomory-Johnson 2-Slope Theorem for $S = \mathbb{Z}$. We assume that $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$.

When $S = \mathbb{Z}_+$, any cut-generating function for (3) must take nonnegative values at nonnegative rationals because minimal cut-generating functions are subadditive and take nonnegative values at nonnegative integers. In the remainder, we restrict our attention to cut-generating functions for (3) that take nonnegative values $\pi(r)$ for *all* $r \in \mathbb{R}_+$. This is satisfied in particular by cut-generating functions that are left or right-continuous on the nonnegative halfline. Therefore, we make the following assumption.

Assumption 1. *When $S = \mathbb{Z}_+$, a cut-generating function π is assumed to satisfy $\pi(r) \geq 0$ for all $r \geq 0$.*

This assumption means, in particular, that a cut-generating function π is extreme if and only if it cannot be written as $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ where π_1 and π_2 are distinct cut-generating functions satisfying Assumption 1. We now state the main result of this section.

Theorem 30 (2-Slope Theorem). *Let $S = \mathbb{Z}_+$ and $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. Suppose Assumption 1 holds. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a continuous function whose restriction to any compact interval is a piecewise linear function with at most two slopes. If π is a strongly minimal cut-generating function for (3), then π is extreme.*

Theorem 30 implies, for example, that the cut-generating functions π^1_α of Example 1 are extreme.

We are going to need several lemmas in the proof of Theorem 30. Recall that any minimal cut-generating function π for (3) is subadditive by Theorem 2. Thus, $\pi(r^1)+\pi(r^2) \geq \pi(r^1+r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$. We denote by $E(\pi)$ the set of all pairs (r^1, r^2) for which this inequality is satisfied as an equality.

Lemma 31. *Suppose S is full-dimensional and $f \in \overline{\text{conv}}(S)$. Let π be a strongly minimal cut-generating function for (3). Suppose there exist cut-generating functions π_1 and π_2 such that $\pi = \frac{1}{2}\pi_1+\frac{1}{2}\pi_2$. Then π_1 and π_2 are strongly minimal cut-generating functions and $E(\pi) \subset E(\pi_1)\cap E(\pi_2)$.*

Proof. We first prove that π_1 and π_2 are strongly minimal cut-generating functions. Suppose π_1 is not strongly minimal. Then there exists a cut-generating function $\pi'_1 \neq \pi_1$, a valid inequality

$\alpha^\top(x-f) \geq \alpha_0$ for S, and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi_1(r) \geq \beta\pi_1'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\mathrm{conv}}(S)$, α_0 and $\beta \geq 1$. Define the function $\pi' : \mathbb{R}^n \to \mathbb{R}$ as $\pi' = \frac{\beta}{\beta+1}\pi_1' + \frac{1}{\beta+1}\pi_2$. The function π' is a cut-generating function because it is a convex combination of two cut-generating functions. Furthermore, $\pi(r) = \frac{1}{2}\pi_1(r) + \frac{1}{2}\pi_2(r) \geq \frac{\beta}{2}\pi_1'(r) + \frac{1}{2}\pi_2(r) + \frac{1}{2}\alpha^\top r = \frac{\beta+1}{2}\pi_1'(r) + \frac{1}{2}\alpha^\top r$ for all $r \in \mathbb{R}^n$. Because the linear inequality $\frac{1}{2}\alpha^\top(x-f) \geq \frac{\alpha_0}{2}$ is valid for S, $\frac{\beta+1}{2} \geq 0$, and $\frac{\beta+1}{2} + \frac{\alpha_0}{2} \geq 1$, the function π' implies π. If $\alpha = 0$ and $\beta = 1$, then $\pi' = \frac{1}{2}\pi_1' + \frac{1}{2}\pi_2$ and $\pi' \neq \pi$ because $\pi_1' \neq \pi_1$. If $\alpha = 0$ and $\beta > 1$, then $\pi \geq \frac{\beta+1}{2}\pi'$. For any $x \in S$, we have $\pi(x-f) > \pi'(x-f)$ because π' is a cut-generating function and $\pi'(x-f) \geq 1$. If $\alpha \neq 0$, then there exists $\bar{x} \in S$ such that $\alpha^\top(\bar{x}-f) > \alpha_0$. Such a point $\bar{x}$ exists because S is full-dimensional. Then we can write $\pi(\bar{x}-f) \geq \frac{\beta+1}{2}\pi'(\bar{x}-f) + \frac{1}{2}\alpha^\top(\bar{x}-f) > \frac{\beta+1}{2}\pi'(\bar{x}-f) + \frac{\alpha_0}{2} \geq \pi'(\bar{x}-f) + \frac{\alpha_0+\beta-1}{2} \geq \pi'(\bar{x}-f)$ by using $\pi'(\bar{x}-f) \geq 1$ and $\alpha_0 + \beta \geq 1$ to obtain the third and fourth inequality, respectively. In all three cases, $\pi' \neq \pi$ which contradicts the strong minimality of π.

Now let $(r^1, r^2) \in E(\pi)$. Because π_1 and π_2 are minimal cut-generating functions, they are subadditive by Theorem 2. Then

$$\begin{aligned}\pi(r^1+r^2) = \pi(r^1) + \pi(r^2) &= \frac{1}{2}(\pi_1(r^1) + \pi_1(r^2)) + \frac{1}{2}(\pi_2(r^1) + \pi_2(r^2)) \\ &\geq \frac{1}{2}\pi_1(r^1+r^2) + \frac{1}{2}\pi_2(r^1+r^2) = \pi(r^1+r^2).\end{aligned}$$

This shows that the inequality above must in fact be satisfied as an equality and $\pi_j(r^1) + \pi_j(r^2) = \pi_j(r^1+r^2)$ for $j \in [2]$. Equivalently, $(r^1, r^2) \in E(\pi_1) \cap E(\pi_2)$. Hence, $E(\pi) \subset E(\pi_1) \cap E(\pi_2)$. ☐

Lemma 32. *Let $S = \mathbb{Z}_+$ and $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a continuous function whose restriction to any compact interval is a piecewise linear function. If π is a minimal cut-generating function for* (3)*, then there exist $\epsilon > 0$ and $s^- < 0 < s^+$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$.*

Proof. Suppose π is a minimal cut-generating function for (3). By Theorem 20, $\pi(0) = 0$ and π is subadditive. Together with $\pi(0) = 0$, the continuity and piecewise linearity of π imply that there exist $\epsilon > 0$ and $s^-, s^+ \in \mathbb{R}$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$. Because π is a cut-generating function for (3), it must satisfy $\pi(-f) \geq 1$ and $\pi(\lceil f \rceil - f) \geq 1$. The subadditivity of π then implies $k\pi(\frac{-f}{k}) \geq \pi(-f) \geq 1$ and $k\pi(\frac{\lceil f \rceil - f}{k}) \geq \pi(\lceil f \rceil - f) \geq 1$ for all $k \in \mathbb{Z}_{++}$. For k large enough, $\frac{-f}{k} \in [-\epsilon, 0]$ and $\frac{\lceil f \rceil - f}{k} \in [0, \epsilon]$. This proves $s^- < 0 < s^+$. ☐

A fundamental tool in the proof of Theorem 30 will be the Interval Lemma, as was already the case in the proof of Gomory and Johnson's 2-Slope Theorem [GJ03]. The Interval Lemma has numerous variants (see, for example, Aczél [Acz66], Kannappan [Kan09], Dey et al. [DRLM10], and Basu et al. [BCCZ12]). Below we give another variant which is well-suited to our needs in proving Theorem 30 because it only assumes a function that is bounded from below on a finite interval. This condition is known to be equivalent to the classical continuity assumption in the literature on Cauchy's additive equation; see Kannappan [Kan09, Theorem 1.2]. We include a proof of our Interval Lemma here for the sake of completeness. Our proof follows the approach of [BCCZ12]. Interval lemmas are usually stated in terms of a single function, but they can also be worded using three functions; this variant is known as Pexider's additive equation (see, for example, Aczél [Acz66] or Basu, Hildebrand, and Köppe [BHK14]). We state and prove our lemma in this more general form.

Lemma 33 (Interval Lemma)**.** *Let $a_1 < a_2$ and $b_1 < b_2$. Consider the intervals $A = [a_1, a_2]$, $B = [b_1, b_2]$, and $A+B = [a_1+b_1, a_2+b_2]$. Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and $h : A+B \to \mathbb{R}$. Assume that f is bounded from below on A. If $f(a) + g(b) = h(a + b)$ for all $a \in A$ and $b \in B$, then f, g, and h are affine functions with identical slopes in the intervals A, B, and $A + B$, respectively.*

Proof. The lemma will follow from several claims about the functions f, g, h.

Claim 1. *Let $a \in A$, and let $b \in B$, $\epsilon > 0$ be such that $b + \epsilon \in B$. For all $k \in \mathbb{Z}_{++}$ such that $a + k\epsilon \in A$, we have $f(a + k\epsilon) - f(a) = k[g(b + \epsilon) - g(b)]$.*

For $\ell \in [k]$, we have $f(a+\ell\epsilon)+g(b) = h(a+b+\ell\epsilon) = f(a+(\ell-1)\epsilon)+g(b+\epsilon)$ by the hypothesis of the lemma. This implies $f(a + \ell\epsilon) - f(a + (\ell - 1)\epsilon) = g(b + \epsilon) - g(b)$ for $\ell \in [k]$. Summing all k equations, we obtain $f(a + k\epsilon) - f(a) = k[g(b + \epsilon) - g(b)]$. This concludes the proof of Claim 1.

Let $\bar{a}, \bar{a}' \in A$ be such that $\bar{a}' - \bar{a} \in \mathbb{Q}$ and $\bar{a}' > \bar{a}$. Define $c = \frac{f(\bar{a}')-f(\bar{a})}{\bar{a}'-\bar{a}}$.

Claim 2. *For every $a, a' \in A$ such that $a' - a \in \mathbb{Q}$, we have $f(a') - f(a) = c(a' - a)$.*

Assume without any loss of generality that $a' > a$. Choose a positive rational ϵ such that $\bar{a}' - \bar{a} = \bar{p}\epsilon$ for some integer $\bar{p}$, $a' - a = p\epsilon$ for some integer p, and $b_1 + \epsilon \in B$. By Claim 1,

$$f(\bar{a}') - f(\bar{a}) = \bar{p}[g(b_1 + \epsilon) - g(b_1)] \text{ and } f(a') - f(a) = p[g(b_1 + \epsilon) - g(b_1)].$$

Dividing the first equality by $\bar{a}' - \bar{a} = \bar{p}\epsilon$ and the second by $a' - a = p\epsilon$, we obtain

$$\frac{f(a') - f(a)}{a' - a} = \frac{g(b_1 + \epsilon) - g(b_1)}{\epsilon} = \frac{f(\bar{a}') - f(\bar{a})}{\bar{a}' - \bar{a}} = c.$$

Thus, $f(a') - f(a) = c(a' - a)$. This concludes the proof of Claim 2.

Claim 3. *For every $a \in A$, $f(a) = f(a_1) + c(a - a_1)$.*

Let $\delta(x) = f(x) - cx$. We show that $\delta(a) = \delta(a_1)$ for all $a \in A$ to prove the claim. Because f is bounded from below on A, δ is bounded from below on A as well. Let M be a number such that $\delta(a) \geq M$ for all $a \in A$.

Suppose for a contradiction that there exists some $a^* \in A$ such that $\delta(a^*) \neq \delta(a_1)$. The lower bound on δ implies $\delta(a_1), \delta(a^*) \geq M$. Let $D = \max\{\delta(a_1), \delta(a^*)\}$. Let $N \in \mathbb{Z}_{++}$ be such that $N|\delta(a^*) - \delta(a_1)| > D - M$. By Claim 2, $\delta(a_1) = \delta(a)$ and $\delta(a^*) = \delta(a')$ for every $a, a' \in A$ such that $a_1 - a$ and $a^* - a'$ are rational. If $\delta(a^*) < \delta(a_1)$, choose $\bar{a}, \bar{a}' \in A$ such that $\bar{a} < \bar{a}'$, $\delta(a_1) = \delta(\bar{a})$, $\delta(a^*) = \delta(\bar{a}')$, $\bar{a} + N(\bar{a}' - \bar{a}) \in A$, and $b_1 + (\bar{a}' - \bar{a}) \in B$. Otherwise, choose $\bar{a}, \bar{a}' \in A$ such that $\bar{a} < \bar{a}'$, $\delta(a_1) = \delta(\bar{a}')$, $\delta(a^*) = \delta(\bar{a})$, $\bar{a} + N(\bar{a}' - \bar{a}) \in A$, and $b_1 + (\bar{a}' - \bar{a}) \in B$. In either case we have $\bar{a} < \bar{a}'$ and $\delta(\bar{a}) > \delta(\bar{a}')$. Furthermore, the choices of $\bar{a}$, $\bar{a}'$, and N imply

$$N[\delta(\bar{a}') - \delta(\bar{a})] = -N\left|\delta(\bar{a}') - \delta(\bar{a})\right| = -N\left|\delta(a^*) - \delta(a_1)\right| < M - D.$$

Let $\epsilon = \bar{a}' - \bar{a}$. By Claim 1,

$$\delta(\bar{a} + N\epsilon) - \delta(\bar{a}) = N[\delta(b_1 + \epsilon) - \delta(b_1)] = N[\delta(\bar{a} + \epsilon) - \delta(\bar{a})] = N[\delta(\bar{a}') - \delta(\bar{a})].$$

Combining this with the previous inequality, we obtain

$$\delta(\bar{a} + N\epsilon) - \delta(\bar{a}) = N[\delta(\bar{a}') - \delta(\bar{a})] < M - D.$$

Because $\delta(\bar{a}) \leq \max\{\delta(a_1), \delta(a^*)\} = D$, this yields $\delta(\bar{a}+N\epsilon) < M-D+\delta(\bar{a}) < M$ which contradicts the choice of M. This concludes the proof of Claim 3.

Claim 4. *For every $b \in B$, $g(b) = g(b_1) + c(b - b_1)$.*

Let k be the smallest positive integer such that $k(a_2 - a_1) \geq b - b_1$, and let $\epsilon = b - (b_1 + (k-1)(a_2 - a_1))$. For every $\ell \in [k-1]$, we have $g(b_1 + \ell(a_2 - a_1)) - g(b_1 + (\ell - 1)(a_2 - a_1)) = f(a_1 + (a_2 - a_1)) - f(a_1) = c(a_2 - a_1)$ by Claim 1. Similarly, $g(b) - g(b_1 + (k-1)(a_2 - a_1)) = g(b_1 + (k-1)(a_2 - a_1) + \epsilon) - g(b_1 + (k-1)(a_2 - a_1)) = f(a_1 + \epsilon) - f(a_1) = c\epsilon$ by Claim 1. Summing all k equations, we obtain $g(b) - g(b_1) = c\epsilon + c(k-1)(a_2 - a_1) = c(b - b_1)$. This concludes the proof of Claim 4.

Finally, let $w \in A + B$, and let $a \in A$, $b \in B$ be such that $w = a + b$. By the hypothesis of the lemma and by Claims 3 and 4, we have $h(w) = f(a) + g(b) = f(a_1) + c(a - a_1) + g(b_1) + c(b - b_1) = h(a_1 + b_1) + c(w - (a_1 + b_1))$. □

We are now ready to prove Theorem 30. Our proof follows the proof outline of the original Gomory-Johnson 2-Slope Theorem for $S = \mathbb{Z}$ [GJ03].

Proof of Theorem 30. Let I be a compact interval of $\mathbb{R}$ containing $[\lfloor -f \rfloor, 1]$. By Lemma 32, there exist $0 < \epsilon \leq 1$ and $s^- < 0 < s^+$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$. Thus, s^- and s^+ are the two slopes of π. Assume without any loss of generality that the slopes of π are distinct in the consecutive intervals delimited by the points $\min I = r_{-q} < \ldots < r_{-1} < r_0 = 0 < r_1 < \ldots < r_t = \max I$. It follows that π has slope s^+ in interval $[r_i, r_{i+1}]$ if i is even and slope s^- if i is odd.

Consider cut-generating functions π_1, π_2 such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. By Lemma 31, π_1 and π_2 are strongly minimal cut-generating functions. By Theorem 8, π, π_1, and π_2 are symmetric and satisfy $\pi(0) = \pi_1(0) = \pi_2(0) = 0$ and $\pi(-1) = \pi_1(-1) = \pi_2(-1) = 0$. The symmetry condition implies in particular that $\pi(-f) = \pi_1(-f) = \pi_2(-f) = 1$.

We are going to obtain the theorem as a consequence of several claims.

Claim 1. *In intervals $[r_i, r_{i+1}]$ with i even, π_1 and π_2 are affine functions with positive slopes s_1^+ and s_2^+, respectively.*

Let $i \in \{-q, \ldots, t-1\}$ even. Let $0 < \epsilon \leq r_1$ be such that $r_i + \epsilon < r_{i+1}$. Define $A = [0, \epsilon]$, $B = [r_i, r_{i+1} - \epsilon]$. Then $A + B = [r_i, r_{i+1}]$. Note that the slope of π is s^+ in all three intervals and $\pi(a) + \pi(b) = \pi(a + b)$ for every $a \in A$ and $b \in B$. By Lemma 31, $\pi_1(a) + \pi_1(b) = \pi_1(a + b)$ and $\pi_2(a) + \pi_2(b) = \pi_2(a + b)$ for every $a \in A$ and $b \in B$. Consider either $j \in [2]$. The function π_j is a cut-generating function, so $\pi_j(a) \geq 0$ for every $a \in A$ by Assumption 1. Lemma 33 implies that π_j is an affine function with common slope s_j^+ in all three intervals A, B, and $A + B$. Because π_j is a minimal cut-generating function, it is subadditive and satisfies $k\pi_j(\frac{\lceil f \rceil - f}{k}) \geq \pi_j(\lceil f \rceil - f) \geq 1$ for all $k \in \mathbb{Z}_{++}$. Choosing k large enough ensures $\frac{\lceil f \rceil - f}{k} \in A$ and $k\pi_j(\frac{\lceil f \rceil - f}{k}) = s_j^+(\lceil f \rceil - f) \geq 1$. This shows $s_j^+ > 0$ and concludes the proof of Claim 1.

Claim 2. *In intervals $[r_i, r_{i+1}]$ with i odd, π_1 and π_2 are affine functions with negative slopes s_1^- and s_2^-, respectively.*

The proof of the claim is similar to the proof of Claim 1. One only needs to choose the intervals A, B, and $A + B$ slightly more carefully while using Lemma 33. Let $i \in \{-q, \ldots, t-1\}$ odd. Let $0 < \epsilon \leq -r_{-1}$ be such that $r_i + \epsilon < r_{i+1}$ and $\epsilon \leq r_1$. Define $A = [-\epsilon, 0]$, $B = [r_i + \epsilon, r_{i+1}]$. Then $A + B = [r_i, r_{i+1}]$. Consider either $j \in [2]$. Because π_j is a minimal cut-generating function, it is subadditive and satisfies $\pi_j(a) \geq -\pi_j(-a) = s_j^+ a$ for all $a \in A$. Thus, π_j is minorized by a linear function and bounded from below on A. Now using Lemmas 31 and 33, we see that π_j is an affine function with common slope s_j^- in all three intervals A, B, and $A+B$. The negativity of s_j^- then follows from this, the subadditivity of π_j, $\pi_j(0)=0$, and $\pi_j(-f)=1$. This concludes the proof of Claim 2.

Claims 1 and 2 show that π_1 and π_2 are continuous functions whose restrictions to the interval I are piecewise linear functions with two slopes.

Claim 3. $s^+ = s_1^+ = s_2^+$, $s^- = s_1^- = s_2^-$.

Define L_{-1}^+ and L_{-f}^+ as the sum of the lengths of intervals with positive slope contained in $[-1,0]$ and $[-f,0]$, respectively. Define L_{-1}^- and L_{-f}^- as the sum of the lengths of intervals with negative slope contained in $[-1,0]$ and $[-f,0]$, respectively. Note that $L_{-f}^+, L_{-f}^-, L_{-1}^+, L_{-1}^-$ are all nonnegative, $L_{-1}^+ + L_{-1}^- = 1$, and $L_{-f}^+ + L_{-f}^- = f$. Since $\pi(0)=\pi_1(0)=\pi_2(0)=0$, $\pi(-f)=\pi_1(-f)=\pi_2(-f)=1$, and $\pi(-1)=\pi_1(-1)=\pi_2(-1)=0$, the vectors (s^+,s^-), (s_1^+,s_1^-), (s_2^+,s_2^-) all satisfy the system

$$\begin{aligned} L_{-1}^+\sigma^+ + L_{-1}^-\sigma^- &= 0,\\ L_{-f}^+\sigma^+ + L_{-f}^-\sigma^- &= -1. \end{aligned}$$

Note that $(L_{-1}^+, L_{-1}^-) \neq 0$ because $L_{-1}^+ + L_{-1}^- = 1$. Suppose the constraint matrix of the system above is singular. Then the vector (L_{-f}^+, L_{-f}^-) must be a multiple λ of (L_{-1}^+, L_{-1}^-). However, this is impossible because the system has a solution (s^+,s^-) and the right-hand sides of the two equations would have to satisfy $0\lambda = -1$. Therefore, the constraint matrix is nonsingular and the system must have a unique solution. This implies $s^+ = s_1^+ = s_2^+$ and $s^- = s_1^- = s_2^-$, concluding the proof of Claim 3.

The functions π, π_1, and π_2 are continuous piecewise linear functions which have the same slope in each interval $[r_i, r_{i+1}]$ of I. Therefore, $\pi(r)=\pi_1(r)=\pi_2(r)$ for all $r\in I$. Because I can be chosen to be any compact interval that contains $[\lfloor -f\rfloor, 1]$, we have $\pi=\pi_1=\pi_2$. □

Example 5. In Theorem 30, the cut-generating function π is assumed to be "strongly minimal". This assumption cannot be weakened to "minimal" or "restricted minimal" as illustrated by the following example. For $0<f<1$ and $\alpha\geq 1$, consider the function $\pi_\alpha^4 : \mathbb{R}\to\mathbb{R}$ defined by

$$\pi_\alpha^4(r) = \begin{cases} \frac{\alpha r}{1-f}, & \text{for } r \geq 0,\\ \frac{-r}{f}, & \text{for } -f < r < 0,\\ 1 + \frac{\alpha(r+f)}{1-f}, & \text{for } r \leq -f. \end{cases}$$

The above function π_α^4 is a continuous piecewise linear function with only two slopes (see Figure 2). Furthermore, $\frac{\alpha r}{1-f} \leq \pi_\alpha^4(r) \leq 1 + \frac{\alpha(r+f)}{1-f}$ for all $r\in\mathbb{R}$. We claim that

(i) π_α^4 is a restricted minimal cut-generating function for (3),
(ii) π_α^4 is neither strongly minimal nor extreme when $\alpha > 1$.

As a consequence of Theorem 25, to prove (i), it suffices to show that $\pi_\alpha^4(0)=0$, $\pi_\alpha^4(-1)\leq 0$, π_α^4 is subadditive and symmetric. The first two properties are straightforward to verify. We prove that π_α^4 is subadditive, that is, $\pi_\alpha^4(r^1)+\pi_\alpha^4(r^2) \geq \pi_\alpha^4(r^1+r^2)$ for all $r^1, r^2\in\mathbb{R}$. We may assume $r^1 \leq r^2$.

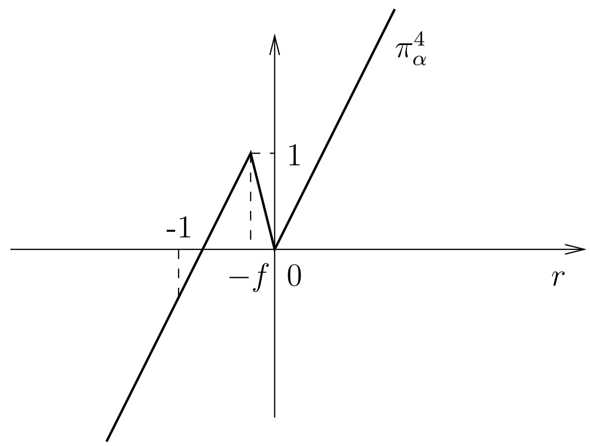

Figure 2: The restricted minimal cut-generating function π_α^4 has only two slopes, but it is not extreme.

- If $r^1 \le -f$, then $\pi_\alpha^4(r^1) + \pi_\alpha^4(r^2) \ge 1 + \frac{\alpha(r^1+f)}{1-f} + \frac{\alpha r^2}{1-f} = 1 + \frac{\alpha(r^1+r^2+f)}{1-f} \ge \pi_\alpha^4(r^1 + r^2)$.
- If $r^1 > -f$ and $r^1 + r^2 < 0$, then $\pi_\alpha^4(r^1) + \pi_\alpha^4(r^2) \ge \frac{-r^1}{f} + \frac{-r^2}{f} = \frac{-(r^1+r^2)}{f} \ge \pi_\alpha^4(r^1 + r^2)$.
- If $r^1 + r^2 \ge 0$, then $\pi_\alpha^4(r^1) + \pi_\alpha^4(r^2) \ge \frac{\alpha r^1}{1-f} + \frac{\alpha r^2}{1-f} = \pi_\alpha^4(r^1 + r^2)$.

Thus π_α^4 is subadditive. Furthermore, π_α^4 is symmetric since the point $(-f/2, 1/2)$ is a point of symmetry in the graph of the function.

To prove (ii), note that for any $\alpha > 1$, $\pi_\alpha^4(-1) < 0$. It follows from Theorem 8 that π_α^4 is not strongly minimal and from Lemma 29 that π_α^4 is not extreme. Indeed, for any $\alpha > 1$, π_α^4 can be written as $\pi_\alpha^4 = \frac{1}{2}\pi_{\alpha-\epsilon}^4 + \frac{1}{2}\pi_{\alpha+\epsilon}^4$, where both functions $\pi_{\alpha-\epsilon}^4$ and $\pi_{\alpha+\epsilon}^4$ are restricted minimal cut-generating functions if we choose $0 < \epsilon \le \alpha - 1$.

Finally, we observe that when $\alpha = 1$, the conditions of Theorem 8 are satisfied. This implies that π_α^4 is strongly minimal in this case and therefore extreme by Theorem 30.

Proposition 21 states that if $K_2 \subset K_1$ are two closed convex cones, then any minimal cut-generating function for (3) when $S = K_1 \cap \mathbb{Z}$ is also a minimal cut-generating function when $S = K_2 \cap \mathbb{Z}$. This statement is not true for extreme cut-generating functions, nor for strongly minimal cut-generating functions. Indeed, for $0 < \alpha \le 1$, the function π_α^1 is strongly minimal for $S = \mathbb{Z}_+$ (Lemma 29) and extreme for $S = \mathbb{Z}_+$ (Theorem 30), but it is neither strongly minimal for $S = \{0\}$ (Proposition 28) nor extreme for $S = \{0\}$ (Lemma 29).

6 Minimal Cut-Generating Functions for mixed-integer Linear Programs

We now turn to mixed-integer linear programming. As before, it is convenient to work with an infinite model.

$$\begin{aligned} x =& f + \sum_{r\in\mathbb{R}^n} r s_r + \sum_{r\in\mathbb{R}^n} r y_r, \\ & x \in S, \\ & s_r \in \mathbb{R}_+, \forall r \in \mathbb{R}^n, \\ & y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ & s, y \text{ have finite support.} \end{aligned} \tag{11}$$

As earlier, we assume that $S \neq \emptyset$. We will also need to assume that f is not in the closure of S, that is, $f \in \mathbb{R}^n \setminus \operatorname{cl} S$.

Two functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ are said to form a *cut-generating function pair* if the inequality $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ holds for every feasible solution (x, s, y) of (11) [GJ72a, Joh74, CCZ11a]. Cut-generating function pairs can be used to generate cutting-planes in mixed-integer linear programming by simply restricting the above inequality to the vectors r that appear as nonbasic columns.

Note that the assumption $f \in \mathbb{R}^n \setminus \operatorname{cl} S$ is needed for the existence of ψ in cut-generating function pairs (ψ, π). Suppose for example that $S = \mathbb{R} \setminus \{f\}$. Let $\bar{r} \in \mathbb{R} \setminus \{0\}$ and $\epsilon > 0$. Then the solution $x = f + \epsilon\bar{r}$, $y = 0$, $s_{\bar{r}} = \epsilon$, and $s_r = 0$ for all $r \neq \bar{r}$ is feasible to (11). Therefore, in any cut-generating function pair (ψ, π) for (11), the function $\psi : \mathbb{R} \to \mathbb{R}$ would have to satisfy $\sum_{r\in\mathbb{R}} \pi(r)y_r + \sum_{r\in\mathbb{R}} \psi(r)s_r = \epsilon\psi(\bar{r}) \geq 1$. This, however, implies $\psi(\bar{r}) \geq \frac{1}{\epsilon}$ for all $\epsilon > 0$, contradicting $\psi(\bar{r}) \in \mathbb{R}$.

The definitions of minimality, restricted minimality, and strong minimality extend readily to cut-generating function pairs for the model (11). A cut-generating function pair (ψ', π') for (11) *dominates* another cut-generating function pair (ψ, π) if $\psi \geq \psi'$ and $\pi \geq \pi'$, *implies* (ψ, π) *via scaling* if there exists $\beta \geq 1$ such that $\psi \geq \beta\psi'$ and $\pi \geq \beta\pi'$, and *implies* (ψ, π) if there exists $\beta \geq 0$ and a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for S such that $\alpha_0 + \beta \geq 1$, $\psi(r) \geq \beta\psi'(r) + \alpha^\top r$, and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. A cut-generating function pair (ψ, π) is *minimal* (resp. *restricted minimal*, *strongly minimal*) if it is not *dominated* (resp. *implied via scaling*, *implied*) by a cut-generating function pair other than itself. As for the model (3), strongly minimal cut-generating function pairs for (11) are restricted minimal, and restricted minimal cut-generating function pairs for (11) are minimal.

The following theorem extends Theorem 1, Proposition 6, and Theorem 9 to the model (11). The proof of each claim is similar to the proof of its aforementioned counterpart for the model (3) and is therefore omitted.

Theorem 34.

(i) Every cut-generating function pair for (11) *is dominated by a minimal cut-generating function pair.*

(ii) Every cut-generating function pair for (11) *is implied via scaling by a restricted minimal cut-generating function pair.*

(iii) Suppose $\overline{\operatorname{conv}}(S)$ is a full-dimensional polyhedron. Let $f \in \overline{\operatorname{conv}}(S)$. Then every cut-generating function pair for (11) *is implied by a strongly minimal cut-generating function pair.*

Next we state two simple lemmas which will be used in the proof of Theorem 37. We omit a complete proof of Lemma 35. Its first claim follows from the observation that for any cut-generating function pair (ψ, π), the related pair (ψ, π') where π' is the pointwise minimum of ψ and π is a cut-generating function pair that dominates (ψ, π). Its second claim has a similar proof to that of Lemma 11. The reader is referred to [CCZ11a] for the proof of Lemma 35 in the case $S = \mathbb{Z}^n$, which remains valid for general S.

Lemma 35. *Let (ψ, π) be a minimal cut-generating function pair for* (11). *Then*
(i) $\pi \leq \psi$,
(ii) ψ is sublinear, that is, subadditive and positively homogeneous.

Lemma 36. *Let $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$. If π is a cut-generating function for* (3)*, ψ is sublinear, and $\psi \geq \pi$, then (ψ, π) is a cut-generating function pair for* (11).

Proof. Let $(\bar{x}, \bar{s}, \bar{y})$ be a feasible solution of (11), and let $\bar{r} = \sum_{r \in \mathbb{R}^n} r \bar{s}_r$. Note that $(\bar{x}, \tilde{y})$, where $\tilde{y}_{\bar{r}} = \bar{y}_{\bar{r}} + 1$ and $\tilde{y}_r = \bar{y}_r$ for $r \neq \bar{r}$, is a feasible solution to (3). Then $\pi(\bar{r}) + \sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r) \tilde{y}_r \geq 1$ because π is a cut-generating function for (3). Using the sublinearity of ψ and $\psi \geq \pi$, we can write $\sum_{r \in \mathbb{R}^n} \psi(r) \bar{s}_r + \sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r \geq \psi(\bar{r}) + \sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r \geq \pi(\bar{r}) + \sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r \geq 1$. This shows that (ψ, π) is a cut-generating function pair for (11). □

Gomory and Johnson [GJ72a, GJ72b] characterized minimal cut-generating function pairs for (11) when $S = \mathbb{Z}$. Johnson [Joh74] generalized this result as follows: Consider $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$. The pair (ψ, π) is a minimal cut-generating function pair for (11) when $S = \mathbb{Z}^n$ if and only if π is a minimal cut-generating function for (3) when $S = \mathbb{Z}^n$ and ψ satisfies

$$\psi(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \text{ for all } r \in \mathbb{R}^n. \tag{12}$$

In the next result, we give similar characterizations of minimal, restricted minimal, and strongly minimal cut-generating function pairs for (11). Our proof follows the proofs in [Joh74, CCZ11a] for similar results on minimal cut-generating function pairs in the case $S = \mathbb{Z}^n$.

Theorem 37. *Let $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$.*
(i) The pair (ψ, π) is a (restricted) minimal cut-generating function pair for (11) *if and only if π is a (restricted) minimal cut-generating function for* (3) *and ψ satisfies* (12).
(ii) Suppose S is full-dimensional and $f \in \overline{\operatorname{conv}}(S)$. The pair (ψ, π) is a strongly minimal cut-generating function pair for (11) *if and only if π is a strongly minimal cut-generating function for* (3) *and ψ satisfies* (12).

Proof. We will prove the claim (ii) only. The proof of claim (i) is similar.

We first prove the "only if" part. Suppose (ψ, π) is a strongly minimal cut-generating function pair for (11). Because (ψ, π) is minimal, we have that $\psi \geq \pi$ and ψ is sublinear by Lemma 35. Furthermore, π is a cut-generating function for (3) since for any feasible solution $(\bar{x}, \bar{y})$ to (3), there exists a feasible solution $(\bar{x}, \bar{s}, \bar{y})$ to (11) such that $\bar{s}_r = 0$ for all $r \in \mathbb{R}^n$, and $\sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r = \sum_{r \in \mathbb{R}^n} \psi(r) \bar{s}_r + \sum_{r \in \mathbb{R}^n} \pi(r) \bar{y}_r \geq 1$. We claim that π is a strongly minimal cut-generating function for (3). Suppose not. Then there exists a cut-generating function $\pi' \neq \pi$, a valid inequality

$\alpha^\top(x-f) \geq \alpha_0$ for S, and $\beta \geq 0$ such that $\alpha_0+\beta \geq 1$ and $\pi(r) \geq \beta\pi'(r)+\alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\text{conv}}(S)$, $\alpha_0 \leq 0$ and $\beta \geq 1$. Define the function $\psi' : \mathbb{R}^n \to \mathbb{R}$ by letting $\psi'(r) = \frac{\psi(r)-\alpha^\top r}{\beta}$. The pair (ψ', π') is a cut-generating function pair for (11). To see this, first note that ψ' is sublinear because ψ is. Furthermore, $\psi' \geq \pi'$ because $\psi'(r) = \frac{\psi(r)-\alpha^\top r}{\beta} \geq \frac{\pi(r)-\alpha^\top r}{\beta} \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. It then follows from Lemma 36 that (ψ', π') is a cut-generating function pair. Because $\pi' \neq \pi$ and (ψ', π') implies (ψ, π), this contradicts the strong minimality of (ψ, π). Thus, π is a strongly minimal cut-generating function for (3). In particular, π is minimal, and subadditive by Theorem 2.

Define the function $\psi'' : \mathbb{R}^n \to \mathbb{R}$ by letting $\psi''(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$. We first show that ψ'' is well-defined, that is, it is finite everywhere, and that $\psi'' \leq \psi$. By Lemma 35, $\pi \leq \psi$ and ψ is sublinear. Thus, for all $\epsilon > 0$ and $r \in \mathbb{R}^n$, we have

$$-\psi(-r) = \frac{-\psi(-\epsilon r)}{\epsilon} \leq \frac{-\pi(-\epsilon r)}{\epsilon} \leq \frac{\pi(\epsilon r)}{\epsilon} \leq \frac{\psi(\epsilon r)}{\epsilon} = \psi(r).$$

The second inequality above holds because $\pi(r) + \pi(-r) \geq \pi(0) = 0$ for all $r \in \mathbb{R}^n$ by the subadditivity of π. This implies

$$-\psi(-r) \leq \psi''(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \leq \psi(r),$$

which proves both claims since ψ is real-valued.

It is easy to verify from the definition of ψ'' that it is sublinear. Furthermore, $\pi \leq \psi''$ by Lemma 23. It then follows from Lemma 36 that (ψ'', π) is a cut-generating function pair for (11). Because the cut-generating function pair (ψ, π) is minimal and $\psi'' \leq \psi$, we get $\psi = \psi''$, proving that ψ satisfies (12).

We now prove the "if" part. Suppose π is a strongly minimal cut-generating function for (3) and ψ satisfies (12). Note that ψ is sublinear by definition and $\psi \geq \pi$ by Lemma 23. It follows from Lemma 36 that (ψ, π) is a cut-generating function pair for (11). Let (ψ', π') be a cut-generating function pair that implies (ψ, π). We will show $\psi' = \psi$ and $\pi' = \pi$, proving that (ψ, π) is strongly minimal. Let (ψ'', π'') be a minimal cut-generating function pair that dominates (ψ', π'). By the choice of (ψ', π'), there exist a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\psi(r) \geq \beta\psi'(r) + \alpha^\top r \geq \beta\psi''(r) + \alpha^\top r$, $\pi(r) \geq \beta\pi'(r) + \alpha^\top r \geq \beta\pi''(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Furthermore, $\alpha_0 \leq 0$ and $\beta \geq 1$ because $f \in \overline{\text{conv}}(S)$. By the "only if" part of claim (i), π'' is a minimal cut-generating function for (3) and $\psi''(r) = \limsup_{\epsilon \to 0^+} \frac{\pi''(\epsilon r)}{\epsilon}$ for all $r \in \mathbb{R}^n$. The function π'' implies π. By the strong minimality of π, we have $\pi'' = \pi$. Then $\pi(r) \geq \beta\pi(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Let $\bar{x} \in S$ be such that $\alpha^\top(\bar{x} - f) > \alpha_0$; such a point exists because S is full-dimensional. If $\beta > 1$, then $\pi(\bar{x} - f) \leq \frac{-\alpha^\top(\bar{x}-f)}{\beta-1} < \frac{-\alpha_0}{\beta-1} \leq 1$ which contradicts the fact that π is a cut-generating function. Hence, we can assume $\beta = 1$. Then $\alpha^\top r \leq 0$ for all $r \in \mathbb{R}^n$; therefore, $\alpha = 0$. Using $\alpha = 0$ and $\beta = 1$, we get $\pi = \pi'' \leq \pi' \leq \pi$ and $\psi'' \leq \psi' \leq \psi$. Finally, note that $\psi''(r) = \limsup_{\epsilon \to 0^+} \frac{\pi''(\epsilon r)}{\epsilon} = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} = \psi(r)$ for all $r \in \mathbb{R}^n$. This shows $\psi'' = \psi' = \psi$ and concludes the proof. $\square$

Example 6. Let $n = 1$, $S = \mathbb{Z}_+$, and $0 < f < 1$. We consider the classical Gomory function $\psi(r) = \max\{\frac{-r}{f}, \frac{r}{1-f}\}$ for the continuous nonbasic variables. In the spirit of [DW10b], the *trivial lifting* of ψ can be defined as

$$\pi^5(r) = \inf_{x \in \mathbb{Z}_+} \{\psi(r + x)\}.$$

Note that π^5 coincides with the Gomory function $\pi_1^1(r) = \min\{\frac{r-\lfloor r\rfloor}{1-f}, \frac{\lceil r\rceil - r}{f}\}$ of Example 1 on the negative points and with ψ on the nonnegative points. Using standard techniques, one can verify that (ψ, π^5) is a cut-generating function pair for (11). Nevertheless, (ψ, π^5) is not a *minimal* pair. To prove this, it is enough by Theorem 37 and Proposition 16 to show that π^5 does not satisfy (8) and hence is not a minimal cut-generating function for (3). Indeed, note that $\pi^5(1) = \frac{1}{1-f}$, whereas $\pi^5(-f-k) = 1$ for all $k \in \mathbb{Z}_{++}$. Therefore, $\pi^5(1) = \frac{1}{1-f} \neq 0 = \sup\{\frac{1}{k}(1-\pi^5(-f-k)) : k \in \mathbb{Z}_{++}\}$ which violates (8).

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