A NOTE ON DIJOINS

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ABSTRACT. For every nonnegative integer arc weight function w, the minimum weight of a dicut is at least as large as the maximum number of dijoins such that no arc a is contained in more than $w(a)$ of these dijoins. We give two examples of digraphs with strict inequality for some weight w and discuss the possibility that, together with an example due to Schrijver, these are the only "minimal" such examples.

1. INTRODUCTION

Consider a digraph D with arc set A. A *dicut* is a non-empty set of arcs of the form $\{(u,v) \in A : u \in U, v \notin U\}$ such that there is no arc $(u,v) \in A$ with $v \in U$ and $u \notin U$. A *dijoin* is a set of arcs that intersects every dicut, or equivalently, a set of arcs whose contraction makes the digraph strongly connected. The following conjecture is still open.

Conjecture 1.1 (Woodall [8]). *The minimum cardinality of a dicut is equal to the maximum number of pairwise disjoint dijoins.*

Let D be a digraph and let $w : A \to Z_+$ be a weight function. The weight of a dicut is the sum of the weights of its arcs. The following conjecture is a weighted version of Woodall's conjecture.

Conjecture 1.2 (Edmonds and Giles [2]). *For every nonnegative integer arc weight function w, the minimum weight of a dicut is equal to the maximum number of dijoins such that no arc a is contained in more than $w(a)$ of these dijoins.*

Schrijver [6] exhibited an example showing that this conjecture is not true (see next section). However, it is known to be true for digraphs with the property that every source is connected to every sink (see Schrijver [7] and also Feofiloff and Younger [3]). It also holds when the digraph is obtained by choosing an arbitrary orientation of a tree and adding all transitive arcs [7]. We give two new counterexamples to the Edmonds–Giles conjecture and discuss the possibility that, together with Schrijver's example, these are the only "minimal" counterexamples.

2. EXAMPLES

Let D be a digraph and $w : A \to Z_+$ be a weight function. The weight of the smallest dicut is written $\tau(D,w)$. We denote $\nu(D,w)$ the cardinality of the largest collection of dijoin with the property that no arc a is in more than $w(a)$ of these dijoins. Digraphs D_1, D_2, and D_3 are defined in Table 1. With each D_i

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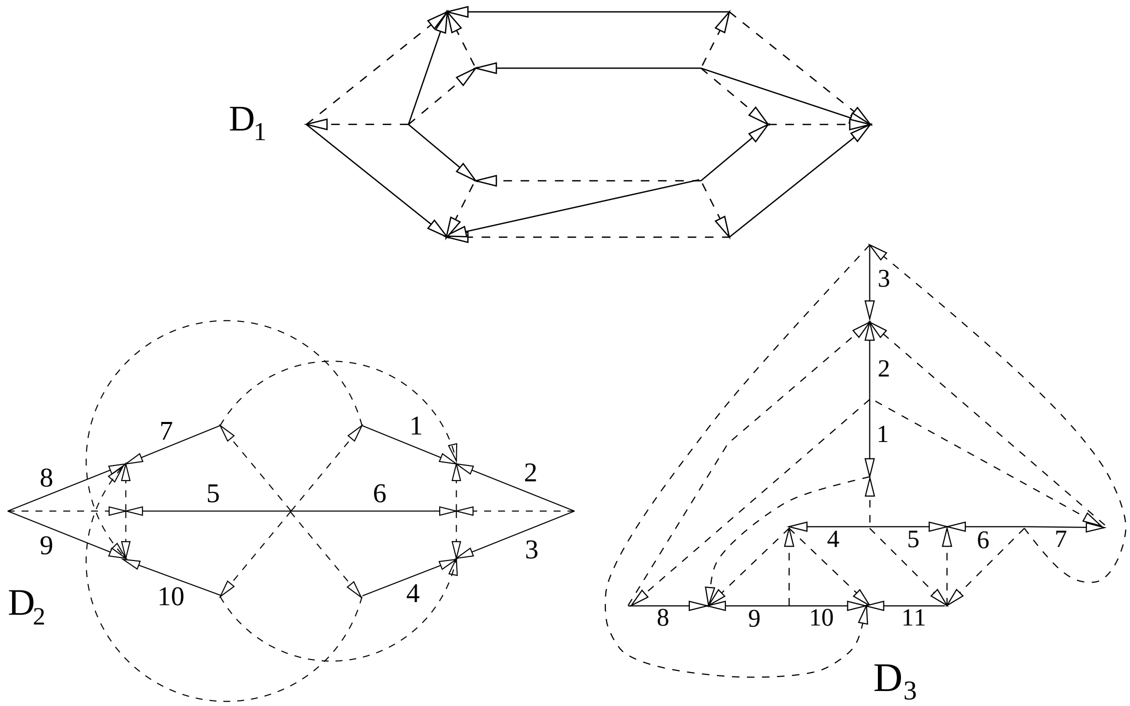

TABLE 1. Digraphs D_1, D_2, and D_3.

($i = 1, 2, 3$) there is an associated weight function w^i where $w^i(a) = 1$ (resp. $w^i(a) = 0$) if the arc a is represented by a solid (resp. dashed) line in the corresponding figure. It is easy to see that $\tau(D_i, w^i) = 2$ for each $i = 1, 2, 3$. We will also see that $\nu(D_i, w^i) = 1$ for each $i = 1, 2, 3$. Thus $\tau(D_i, w^i) > \nu(D_i, w^i)$ and each D_i, w^i contradicts Conjecture 1.2. The digraph D_1 with weights w^1 is due to Schrijver [7] who also showed that $2 = \tau(D_1, w^1) > \nu(D_1, w^1) = 1$. Digraphs D_2, D_3 are new.

Proposition 2.1. $\nu(D_2, w^2) = 1$ *and* $\nu(D_3, w^3) = 1$

Proof. For $i = 2, 3$, let $\mathcal{F}_i$ be the family of dicuts of D_i (viewed as arc sets), let A_i be the set of edges of D_i with weight one and let $\mathcal{F}'_i := \{S \cap A_i : S \in \mathcal{F}_i\}$.

Suppose for a contradiction there exists two arc-disjoint dijoins J_B and J_R of D_2 which are both included in A_2. Since $\{1, 2\} \in \mathcal{F}'_2$ we may assume arc $1 \in J_R$ and $2 \in J_B$. Since $\{2, 3\}$ and $\{3, 4\}$ are in $\mathcal{F}'_2$, it implies that $3 \in J_R$ and $4 \in J_B$. Consider first the case where $7 \in J_R$. Since $\{7, 8\}, \{8, 9\}, \{9, 10\}$ are in $\mathcal{F}'_2$, it implies $10 \in J_B$. But $\{3, 5, 7\} \in \mathcal{F}'_2$ implies $5 \in J_B$ and $\{2, 5, 10\} \in \mathcal{F}'_2$ implies $5 \in J_R$; a contradiction. Thus, $7 \in J_B$. Because $\{7, 8\}, \{8, 9\} \in \mathcal{F}'_2$ we have $8 \in J_R$ and $9 \in J_B$. But $\{1, 6, 8\} \in \mathcal{F}'_2$ implies $6 \in J_B$ and $\{4, 6, 9\} \in \mathcal{F}'_2$ implies $6 \in J_R$; a contradiction.

Suppose for a contradiction there exists two arc-disjoint dijoins J_B and J_R of D_3 which are both included in A_3. Since $\{1, 2\}, \{2, 3\} \in \mathcal{F}'_3$ we may assume $1, 3 \in J_B$ and $2 \in J_R$. Consider first the case where $8 \in J_B$. Since $\{8, 9\}, \{9, 10\}, \{10, 11\} \in \mathcal{F}'_3$ this implies $10 \in J_B$ and $9, 11 \in J_R$. Now $\{1, 6, 8\} \in \mathcal{F}'_3$ implies $6 \in J_R$ and $\{2, 4, 11\} \in \mathcal{F}'_3$ implies $4 \in J_B$. A contradiction as $\{5, 6\} \in \mathcal{F}'_3$ implies $5 \in J_B$ and $\{4, 5\} \in \mathcal{F}'_3$ implies $5 \in J_R$. Thus, $8 \in J_R$. Because $\{8, 9\}, \{9, 10\} \in \mathcal{F}'_3$ we have $9 \in J_B$ and $10 \in J_R$.

Since $\{3,7,9\} \in \mathcal{F}'_3$ we have $7 \in J_R$. Since $\{2,5,10\} \in \mathcal{F}'_3$ we have $5 \in J_B$. But $\{6,7\} \in \mathcal{F}'_3$ implies $6 \in J_B$ and $\{5,6\} \in \mathcal{F}'_3$ implies $6 \in J_R$; a contradiction. □

3. AN OPEN PROBLEM

A *transitive extension* of a digraph D is a digraph obtained from D by adding arcs of the form (u,v) where u and v correspond to the start and end of a directed path of D. Observe that contracting arc a in D is equivalent to setting $w(a)$ to some sufficiently large value M. Also note that adding transitive arcs of weight zero leaves the problem unchanged. Suppose D is contractible to a transitive extension $\tilde{D}$ of D_i (for some $i \in \{1,2,3\}$). Define $w : A \to Z_+$ as follows: $w(a) = M$ if a is contracted to obtain $\tilde{D}$, $w(a) = 0$ if a is an arc of $\tilde{D}$ but not of D_i, and $w(a) = w^i(a)$ for all remaining arcs. Then $2 = \tau(D,w) > \nu(D,w) = 1$. Thus,

Remark 3.1. If $\tau(D,w) = \nu(D,w)$ for all weight functions $w : A \to Z_+$ then D is not contractible to a transitive extension of D_1, D_2, or D_3.

A natural question is whether the converse also holds, i.e. are D_1, D_2, and D_3, the only obstructions to the property that $\tau(D,w) = \nu(D,w)$ for all weight functions $w : A \to Z_+$. To motivate this question let us consider a more general problem and let $\mathcal{H}$ be a finite family of sets over some finite ground set $E(\mathcal{H})$. Let $w : E(\mathcal{H}) \to Z_+$ be a weight function. Define $\tau(\mathcal{H},w)$ to be the weight of the minimum transversal of $\mathcal{H}$ and let $\nu(D,w)$ be the cardinality of the largest collection of sets with the property that no element $e \in E(\mathcal{H})$ is in more than $w(e)$ of these sets. Also let $\nu^*(\mathcal{H},w) = \max\{\sum_{S\in\mathcal{H}} y_S : \sum_{S\in\mathcal{H}, e\in S} y_S \le w(e), \forall e \in E(\mathcal{H}), y \ge \mathbf{0}\}$. Clearly, $\tau(\mathcal{H},w) \ge \nu^*(\mathcal{H},w) \ge \nu(\mathcal{H},w)$. If $\tau(\mathcal{H},w) = \nu(\mathcal{H},w)$ for all $w : E(\mathcal{H}) \to Z_+$ then $\mathcal{H}$ has the *Max-Flow Min-Cut property*. If $\tau(\mathcal{H},w) = \nu^*(\mathcal{H},w)$ for all $w : E(\mathcal{H}) \to Z_+$ then $\mathcal{H}$ is *ideal*. Suppose $\mathcal{H}$ is the family of dijoins of a digraph D. The Lucchesi-Younger directed cut theorem [5] together with Lehman's width-length theorem [4] imply that $\mathcal{H}$ is ideal. Ideal families of sets that do not have the Max-Flow Min-Cut property have been investigated in [1]. Several "minimal" examples are given in that paper. However, these families of sets are fairly constrained. The motivation for the aforementioned question is that the only examples in [1] arising from dijoins appear to be the families corresponding to D_1, D_2, and D_3.

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