A Connection Between Cutting Plane Theory and the Geometry of Numbers *

Gérard Cornuéjols Yanjun Li

Graduate School of Industrial Administration

Carnegie Mellon University, Pittsburgh, USA

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Abstract

In this paper, we relate several questions about cutting planes to a fundamental problem in the geometry of numbers, namely, the closest vector problem. Using this connection we show that the dominance, membership and validity problems are NP-complete for Chvátal and split cuts.

Key words: Closest vector problem, dominance problem, membership problem, validity problem, Chvátal cut, split cut, disjunctive cut, Gomory cut.

1. Introduction

Given a rational polyhedron $P = \{x \in R^n | Ax \leq b\}$, where $A \in Z^{m \times n}$ and $b \in Z^m$, a *Chvátal cut* [4] $ax \leq d$ is obtained from a valid inequality $ax \leq c$ for P, where $a \in Z^n$ and $d = \lfloor c \rfloor$. A *split cut* [5] $ax \leq d$ is a valid inequality for $\text{Conv}(P \cap \{x \in R^n | \pi x \leq \pi_0\}, P \cap \{x \in R^n | \pi x \geq \pi_0 + 1\})$ for some $(\pi, \pi_0) \in Z^{n+1}$, where $\text{Conv}(P_1, P_2)$ denotes the convex hull of $P_1 \bigcup P_2$. Split cuts are an important special case of *disjunctive cuts* [2]. The equivalence between split cuts, Gomory mixed integer cuts and mixed integer rounding cuts was proved by Nemhauser and Wolsey [13, 14]. See also [6, 7]. Chvátal cuts are special split cuts where $P \cap \{x \in R^n | \pi x \geq \pi_0 + 1\}$ is empty, $\pi = a$ and $\pi_0 = d$. The intersection of all the cuts in a given family is called the *elementary closure* of P for this family. The elementary closure for Chvátal cuts is called *Chvátal closure* and for split cuts it is called *split closure*.

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A cut $ax \leq d$ for a rational polyhedron P *dominates* $\tilde{a}x \leq \tilde{d}$ if $P \cap \{x \in R^n | ax \leq d\} \subseteq P \cap \{x \in R^n | \tilde{a}x \leq \tilde{d}\}$. The *dominance problem* for a family of cuts is, given a rational polyhedron P and an inequality $\tilde{a}x \leq \tilde{d}$ with $(\tilde{a}, \tilde{d}) \in Z^{n+1}$, to decide whether there exists a cut $ax \leq d$ for P in the family such that $ax \leq d$ dominates $\tilde{a}x \leq \tilde{d}$.

The *membership problem* [11, 15] for the elementary closure of a family of cuts is, given a rational polyhedron P and a rational point $x^* \in P$, to decide whether x^* belongs to the elementary closure of P for this family of cuts.

The *validity problem* for the elementary closure of a family of cuts is, given P and an inequality, to decide whether this inequality is valid for the elementary closure of P for the family of cuts. The validity problem for the Chvátal closure was considered in [9]. It is well known [11] that the membership and validity problems have the same computational complexity since optimizing a linear function over the elementary closure can be done using binary search to solve a sequence of membership problems.

One can verify that these three decision problems are in NP both for Chvátal cuts and split cuts. The complexity of the membership problem for the Chvátal closure was raised by Schrijver [15] and settled by Eisenbrand [8]. Using a result of Caprara and Fischetti, Eisenbrand reduces the weighted binary clutter problem [10] in a clever way to the membership problem to prove its NP-completeness. The complexity of the membership problem for the split closure was raised by Cornuéjols and Li [6] and recently settled by Caprara and Letchford [3]. They reduce the max cut problem to the membership problems for split cuts, balanced split cuts and binary split cuts, and thereby prove their NP-completeness. The dominance problem has not been studied previously in the literature. Note that it involves a single cut whereas, by contrast, the validity problem involves the whole elementary closure.

In this paper, we reduce the closest vector problem, a fundamental problem in the geometry of numbers, to the dominance and membership problems for Chvátal and split cuts, and concisely prove their NP-completeness even when P is a simplicial cone.

2. Connecting the closest vector problem to the dominance and membership problems

Given a vector $b = (b_1, b_2, \cdots, b_n) \in Q^n$ and n linearly independent vectors $a_i = (a_{1i}, a_{2i}, \cdots, a_{ni}) \in Q^n$ $(1 \leq i \leq n)$, the *Closest Vector Problem* (CVP) is to find a vector closest to b in the lattice $L = \{x \in Q^n : x = \sum_{i=1}^{n} k_i a_i,\ k_i \in Z,\ 1 \leq i \leq n\}$. CVP was shown to be NP-hard for any l_p-norm $(p \geq 1)$ [16]. The corresponding decision problem is NP-complete [12, 16]:

Lemma 1. Given $b \in Q^n$, a lattice L generated by n basis vectors $a_i \in Q^n$ $(1 \leq i \leq$

n), and $u \in Q_+$, the feasibility problem for

$$(1)\ \begin{cases} \|x-b\|_p < u, \\ x \in L \end{cases}$$

is NP-complete.

We are going to reduce the above feasibility problem for the l_∞-norm and l_1-norm to the dominance and membership problems for Chvátal and split cuts on a simplicial cone. Let $\tilde{a}_i = (\bar{a}_i, b_i)$ $(1 \le i \le n)$ and $\tilde{a}_{n+1} = (0, 0, \cdots, 0, -1)$, where $\bar{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in}) \in Q^n$ $(1 \le i \le n)$ are n arbitrary linearly independent vectors, and let $b_i \in Q$ $(1 \le i \le n)$. The simplicial cone that we will work on is $C = \{x \in R^{n+1} | x = \beta + \sum_{i=1}^{n+1} \lambda_i \tilde{a}_i,\ \lambda_i \ge 0 \text{ for } 1 \le i \le n+1\}$, where $\beta = (0, 0, \cdots, 0, \frac{1}{2})$.

Theorem 2. The dominance problem for Chvátal cuts is NP-complete.

Proof. Consider the simplicial cone C defined above and an inequality $ax \le d$ with $(a, d) \in Z^{n+2}$ such that the hyperplane $ax = d$ intersects the extreme rays of C at the points $\beta + \alpha_i \tilde{a}_i$ $(1 \le i \le n+1)$, where $0 < \alpha_i < +\infty$ $(1 \le i \le n)$ and $\alpha_{n+1} = \frac{1}{2}$.
An inequality $\pi x \ge \pi_0 + 1$ is a Chvátal cut of C, where $(\pi, \pi_0) \in Z^{n+2}$, if and only if $0 < \pi\beta - \pi_0 < 1$ and $\pi \tilde{a}_i \ge 0$ $(1 \le i \le n+1)$. Let $\epsilon = \pi\beta - \pi_0$. It is easy to check that the hyperplane $\pi x = \pi_0 + 1$ intersects the extreme rays of C at the points $\beta + \lambda_i \tilde{a}_i$, where $\lambda_i = \frac{1-\epsilon}{\pi \tilde{a}_i}$ $(1 \le i \le n+1)$. Note that when $\pi \tilde{a}_i = 0$, the hyperplane $\pi x = \pi_0 + 1$ is parallel to the ray $\tilde{a}_i$, and by convention we let $\lambda_i = +\infty$.
The inequality $ax \le d$ is dominated by $\pi x \le \pi_0 + 1$ if and only if $\alpha_i \le \lambda_i$ $(1 \le i \le n+1)$. Therefore the inequality $ax \le d$ is dominated by a Chvátal cut of C if and only if there exists $(\pi, \pi_0) \in Z^{n+2}$ such that $0 \le \pi \tilde{a}_i \le \frac{1-\epsilon}{\alpha_i}$ $(1 \le i \le n+1)$ and $0 < \epsilon < 1$, where $\epsilon = \pi\beta - \pi_0$.
By the choice of β, we have $0 < \epsilon = \frac{\pi_{n+1}}{2} - \pi_0 < 1$. Since π_0 and π_{n+1} are integer, it follows that $\epsilon = \frac{1}{2}$ and π_{n+1} is an odd number. By the choice of $\tilde{a}_{n+1}$ and α_{n+1}, it follows that $0 \le -\pi_{n+1} \le 1$. Thus $\pi_{n+1} = -1$.
Let $\pi = (\bar{\pi}, \pi_{n+1})$. Then $ax \le d$ is dominated by a Chvátal cut on C if and only if there exists a solution $\bar{\pi}$ to

$$(2)\ \begin{cases} 0 \le \bar{\pi}\bar{a}_i - b_i \le \frac{1}{2\alpha_i},\ 1 \le i \le n, \\ \bar{\pi} \in Z^n \end{cases}$$

Let L be the lattice with basis vectors $\{a_i = (a_{1i}, a_{2i}, \cdots, a_{ni})\}_{i=1}^n$. Because the feasibility problem for (1) (when $p = \infty$) is NP-complete, by Lemma 1, and (2) has a more general form than (1), the feasibility problem for (2) is NP-complete. □

Theorem 3. The dominance problem for split cuts is NP-complete.

Proof. A split cut from a disjunction $(\pi x \le \pi_0) \vee (\pi x \ge \pi_0 + 1)$ that is not valid for C, must be violated by β. Thus $0 < \pi\beta - \pi_0 < 1$, *i.e.*, $0 < \frac{\pi_{n+1}}{2} - \pi_0 < 1$. Let $\epsilon = \pi\beta - \pi_0$. Then $\epsilon = \frac{1}{2}$ and π_{n+1} is an odd number.

Let $\beta + \lambda_i \tilde{a}_i$ $(1 \leq i \leq n+1)$, where $\lambda_i > 0$, be the $n+1$ intersection points of the extreme rays of C with the split cut. Since these points are either on the hyperplane $\pi x = \pi_0$ or $\pi x = \pi_0 + 1$, we have

$$\lambda_i = \begin{cases} \frac{1-\epsilon}{\pi \tilde{a}_i}, & \text{if } \pi \tilde{a}_i \geq 0, \\ \frac{\epsilon}{-\pi \tilde{a}_i}, & \text{if } \pi \tilde{a}_i < 0. \end{cases}$$

Since $\epsilon = \frac{1}{2}$, $\lambda_i = \frac{\frac{1}{2}}{|\pi \tilde{a}_i|}$ $(1 \leq i \leq n+1)$. Let $ax \leq d$ with $(a, d) \in Z^{n+2}$ be an inequality such that the hyperplane $ax = d$ intersects the extreme rays of C at the points $\beta + \alpha_i \tilde{a}_i$ $(1 \leq i \leq n+1)$, where $0 < \alpha_i < +\infty$ $(1 \leq i \leq n)$ and $\alpha_{n+1} = \frac{1}{2}$. Let $\pi = (\bar{\pi}, \pi_{n+1})$. Now the inequality $ax \leq d$ is dominated by a split cut if and only if

$$(3) \begin{cases} |\bar{\pi} \bar{a}_i + \pi_{n+1} b_i| \leq \frac{1}{2\alpha_i}, \ 1 \leq i \leq n, \\ |\pi_{n+1}| \leq 1, \ \pi_{n+1} \text{ is odd}, \\ \bar{\pi} \in Z^n \end{cases}$$

is feasible. Since π_{n+1} can only take the value -1 or 1 and there is no sign restriction on $\bar{\pi}$, (3) is feasible if and only if

$$(4) \begin{cases} |\bar{\pi} \bar{a}_i - b_i| \leq \frac{1}{2\alpha_i}, \ 1 \leq i \leq n, \\ \bar{\pi} \in Z^n \end{cases}$$

is feasible.

Let L be the lattice with basis vectors $\{a_i = (a_{1i}, a_{2i}, \cdots, a_{ni})\}_{i=1}^n$. Since $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an arbitrary positive rational vector, the NP-completeness of the feasibility problem for (4) follows from Lemma 1 (when $p = \infty$). □

Theorem 4. The membership problem for the split closure is NP-complete.

Proof. Let $x^* = \beta + \frac{1}{4} \sum_{i=1}^{n+1} \tilde{a}_i \in C$ be an interior point of C. Using results in the proof of Theorem 3, a split cut from the disjunction $(\pi x \leq \pi_0) \vee (\pi x \geq \pi_0 + 1)$ is violated by x^* if and only if $\sum_{i=1}^{n+1} \frac{\frac{1}{4}}{\lambda_i} < 1$, where $\lambda_i = \frac{\frac{1}{2}}{|\pi \tilde{a}_i|}$ $(1 \leq i \leq n+1)$ and π_{n+1} is an odd number. Therefore, the point x^* is cut off by a split cut if and only if $\sum_{i=1}^{n+1} |\pi \tilde{a}_i| = \sum_{i=1}^{n} |\bar{\pi} \bar{a}_i + \pi_{n+1} b_i| + |\pi_{n+1}| < 2$ has a solution $\pi = (\bar{\pi}, \pi_{n+1}) \in Z^{n+1}$, where π_{n+1} is an odd number.

Being odd and satisfying $\sum_{i=1}^{n} |\bar{\pi} \bar{a}_i + \pi_{n+1} b_i| + |\pi_{n+1}| < 2$ enforces π_{n+1} to be either -1 or 1. Because $\bar{\pi}$ has no sign restriction, x^* is cut off by a split cut if and only if $\sum_{i=1}^{n} |\bar{\pi} \bar{a}_i - b_i| < 1$ has a solution $\bar{\pi} \in Z^n$. Now, the NP-completeness follows from Lemma 1 (when $p = 1$ and $u = 1$). □

Kannan's proof [12] of Lemma 1 above (see also Theorem 5 in [1]) actually implies the NP-completeness of the following variant of CVP:

Lemma 5. Given $b \in Q^n$, a lattice L generated by n basis vectors $a_i \in Q^n$ ($1 \leq i \leq n$), and $u \in Q_+$, the feasibility problem for

$$(5) \quad \begin{cases} \|x - b\|_1 < u, \\ x \in L, \ x \geq b \end{cases}$$

is NP-complete.

Theorem 6. The membership problem for the Chvátal closure is NP-complete.

Proof. Taking the same point x^* as in the proof of Theorem 4 and using the argument in the proof of Theorem 4 and the fact that $\pi \tilde{a}_i \geq 0$ for a Chvátal cut, we conclude that x^* is cut off by a Chvátal cut if and only if

$$(6) \quad \begin{cases} \sum_{i=1}^{n} |\bar{\pi} \bar{a}_i - b_i| < 1, \\ \bar{\pi} \in Z^n, \quad \bar{\pi} \bar{a}_i \geq b_i, \ 1 \leq i \leq n \end{cases}$$

has a solution. The NP-completeness now follows from Lemma 5 (when $u = 1$). $\square$

Using binary search to optimize a linear function over the elementary closure, it follows that the validity and membership problems have the same computational complexity. This implies the following result.

Corollary 7. The validity problems for the Chvátal and split closures are NP-complete.

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