Even-Hole-Free Graphs
Part II: Recognition Algorithm

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Abstract

We present an algorithm that determines in polytime whether a graph contains an even hole. The algorithm is based on a decomposition theorem for even-hole-free graphs obtained in Part I of this paper. We also give a polytime algorithm to find an even hole in a graph when one exists.

1 Introduction

In a graph, a cycle is *even* if it contains an even number of nodes, and *odd* otherwise. A *hole* is a chordless cycle with at least four nodes. A graph that contains no even hole is called *even-hole-free*. (Graph G *contains* graph H means that H appears in G as an induced subgraph. Graph G is H*-free* means that G does not contain graph H.)

In this part, we present a polytime recognition algorithm for even-hole-free graphs. The algorithm builds on a structural theorem proved in [4]. The algorithm is not practical since the degree of the polynomial is high: our main contribution is in showing that this recognition problem is in the complexity class P. Previously, it was not even known whether this problem was in NP (it is trivially in co-NP, however). It was known (Bienstock [1]) that it is NP-complete to recognize whether a graph contains an even hole passing through a specified node. On the positive side, Porto [10] solved the even hole recognition problem in linear time for planar graphs and Markossian, Gasparian and Reed [9] solved it in polytime for diamond-and-cap-free graphs. A *diamond* is a cycle of length four with a single chord. A *cap* is a cycle of length greater than four with a single chord that forms a triangle with two edges of the cycle. In [5] we extended this last result to cap-free graphs. Here we give a solution for all graphs.

Finding an Even Hole

Note that our recognition algorithm for even-hole-free graphs can be used to find an even hole in graph G, if one exists: Let $v_1, \ldots, v_n$ denote the nodes of G and let $H = G$. In iteration i, test whether $H \setminus v_i$ contains an even hole. If the answer is yes, set $H = H \setminus v_i$ and otherwise keep H unchanged. Perform n iterations. At termination, the graph H is the desired even hole.

With 2 calls to the recognition algorithm, we can also check in polytime whether, given a graph G and a node v of G, all the even holes of G contain v. By contrast, as stated above [1], given a graph G and a node v of G, it is NP-complete to check whether there exists an even hole that contains v.

Cutsets

The decomposition theorem of [4] which we use here has two types of cutsets. We define these now.

For $S \subseteq V(G)$, we denote by $G \setminus S$ the subgraph obtained from the graph G by removing the nodes of S and all the edges with at least one node in S. The node set S is a *cutset* of the graph G if the graph $G \setminus S$ contains more connected components than G. For $S \subseteq V(G)$, $N(S)$ denotes the set of nodes in $V(G) \setminus S$ with at least one neighbor in S and $N[S]$ denotes $N(S) \cup S$. Node set S is a k*-star* if S is comprised of a clique C of size k and nodes with at least one neighbor in C, i.e. $S \subseteq N[C]$. We refer to C as the *clique center* of S. In this paper, we will use k-star cutsets, $k = 1, 2, 3$. We also refer to a 1-star as a *star*, to a 2-star as a *double star* and to a 3-star as a *triple star*. If S is comprised of a clique C and all nodes of G with at least one neighbor in C, it is called a *full k-star*.

A graph G has a *2-join* $V_1|V_2$, with special sets (A_1, A_2, B_1, B_2), if its nodes can be partitioned into sets V_1 and V_2 in such a way that, for $i = 1, 2$, V_i contains disjoint, nonempty node sets A_i and B_i, such that every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, and there are no other adjacencies between V_1 and V_2. Furthermore $|V_i| > 2$ for $i = 1, 2$, and if A_i and B_i are both of cardinality 1, then the graph induced by V_i is not a chordless path.

Star cutsets were introduced by Chvátal [2] and 2-joins by Cornuéjols and Cunningham [8]. In [6] and [3], 2-joins, star and double star cutsets were used to construct recognition algorithms for balanced $0, 1$ matrices and balanced $0, \pm 1$ matrices. Recently, they were used to decompose odd-hole-free graphs [7].

Base Classes

The decomposition theorem of [4] shows that every even-hole-free graph except those in two base classes contains a 2-join or a k-star cutset. These two base classes are the cap-free graphs and basic graphs. Cap-free graphs have been defined already. In [5], polytime algorithms are given for recognizing cap-free graphs and for recognizing even-hole-free cap-free graphs. The second base class of graphs used in the decomposition theorem of [4] is the class of *basic graphs*. We do not define basic graphs here. We just note that every basic graph is obtained from the line graph of a tree by adding two adjacent nodes x and y, and as a consequence we can check in polytime whether a graph is basic. Since there is a unique chordless path between any two nodes in the line graph of a tree, it also follows that we can check in polytime whether a basic graph is even-hole-free.

Decomposition Theorem

The following theorem follows from the main result proved in [4]. (In [4], the result is proved for odd-signable graphs, a class of graphs that contains even-hole-free graphs.)

Theorem 1.1 *A connected even-hole-free graph is cap-free or basic or contains a 2-join or a k-star cutset, $k = 1, 2, 3$.*

Idea of the Algorithm

The above decomposition theorem is the basis of our recognition algorithm for even-hole-free graphs. Whenever a 2-join or a k-star cutset is present in a graph G, we decompose G into two or more smaller or simpler graphs, called blocks. When G contains a k-star cutset, this is done as follows.

Definition 1.2 *Let S be a node cutset in a graph G and $C_1, \ldots, C_n$ the connected components of $G \setminus S$. We define the* blocks *of the decomposition to be graphs $G_1, \ldots, G_n$, where G_i is the subgraph of G induced by $V(C_i) \cup S$.*

When G contains a 2-join, the blocks are defined as follows.

Definition 1.3 *Let $V_1|V_2$ be a 2-join of G with special sets (A_1, A_2, B_1, B_2). If A_2 and B_2 are in different connected components of $G(V_2)$, define* block G_1 *to be the subgraph of G induced by node set $V_1 \cup \{a_2, b_2\}$, where $a_2 \in A_2$ and $b_2 \in B_2$. If $G(V_2)$ contains a path from A_2 to B_2, let Q be a shortest such path and define* block G_1 *to be the subgraph of G induced by node set V_1 plus a* marker path $P_2 = a_2, \ldots, b_2$ *that is chordless and satisfies the following properties. Node a_2 is adjacent to all the nodes in A_1, node b_2 is adjacent to all the nodes in B_1 and these are the only adjacencies between P_2 and V_1. Furthermore, the marker path P_2 has length 4 if Q has even length, and length 5 otherwise. Block G_2 is defined similarly. See Figure 1.*

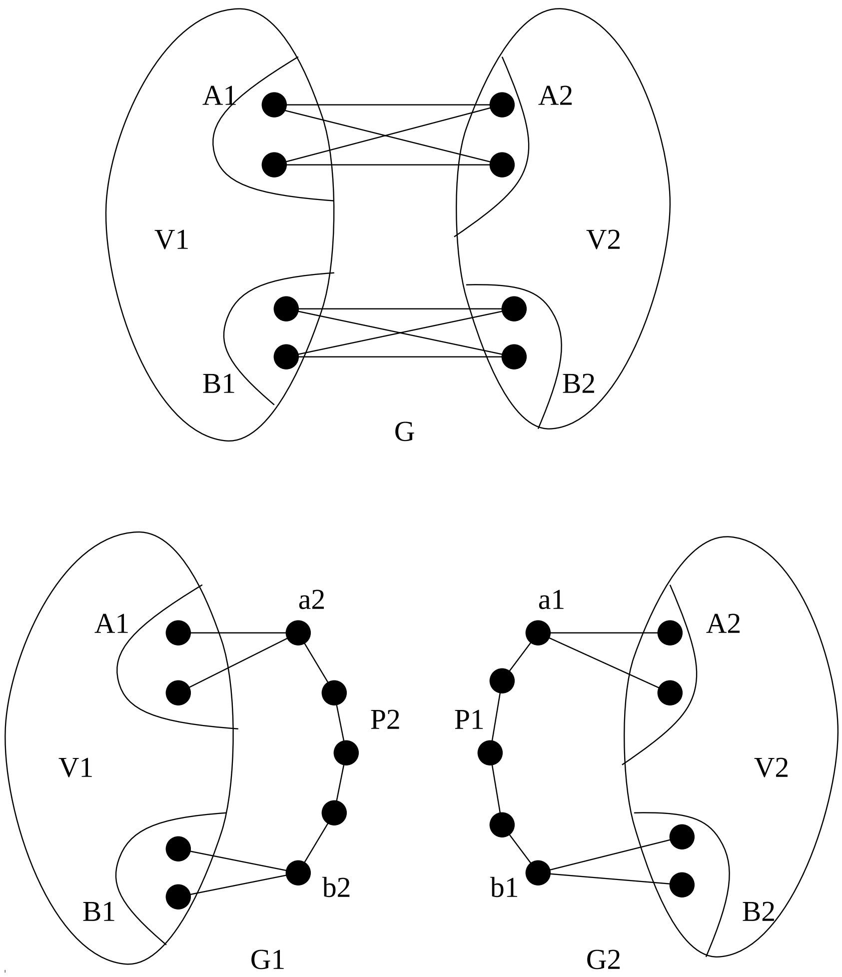

Figure 1: 2-Join Decomposition

If we were to follow the standard paradigm for creating an algorithm from a decomposition theorem, we would now show that

(a) we can find in polytime whether a decomposition exists in G;

(b) G is even-hole-free if and only if all the blocks are;

(c) when the decomposition is applied recursively to the blocks, the total number of blocks created is polynomial.

Unfortunately, although (a) is true for the two cutsets of Theorem 1.1, neither (b) nor (c) holds.

The problem with (c) is that, if we do not take care of dominated nodes properly, we can get an exponential number of blocks even decomposing just with star cutsets. (We say that u is *dominated* by v if u is adjacent to v and $N(u) \subseteq N[v]$.) Another problem is that we do not know how to bound the number of blocks if we mix k-star cutset and 2-join decompositions.

Our solution to (c) is to do k-star cutsets first, then 2-joins, and to deal with dominated nodes specially.

In Section 5, we discuss the 2-join decomposition of a graph G that has no k-star cutset, $k = 1, 2, 3$. We show that G is even-hole-free if and only if the two blocks G_1 and G_2 of the decomposition are even-hole-free. Furthermore, we show that the blocks G_1 and G_2 have no k-star cutsets, $k = 1, 2, 3$. Finally, if the 2-join decomposition is applied recursively, we show that only a linear number of blocks is created overall. By Theorem 1.1, G is even-hole-free if and only if all these blocks belong to a base class and are even-hole-free. This yields a polytime algorithm for checking whether a graph without k-star cutsets, $k = 1, 2, 3$, is even-hole-free.

A major difficulty that needs to be addressed when decomposing by a star, double star or triple star cutset is the fact that (b) above does not hold. Consider, for example, a graph G consisting of an even hole H and a node x with exactly two nonadjacent neighbors in H, say u, v, where both paths of H from u to v have an odd number of edges. If we decompose G by the star cutset $N[x]$ consisting of x and its two neighbors u, v, the two blocks of the decomposition are even-hole-free, whereas G contains the even hole H. Thus star cutset decomposition is not even-hole-free preserving.

To address this difficulty, we first apply a certain cleaning procedure to the input graph G. This procedure transforms G into a polynomial family of induced subgraphs of G with the property that, if G contains an even hole, then at least one graph in the family contains an even hole that will either not be broken by k-star cutset decomposition or will be detected while performing the decomposition.

Clean Graphs

Definition 1.4 *Let H be an even hole and $u \in V(G) \setminus V(H)$. We say that u is* good *w.r.t. H if it has at most three neighbors in H and the graph induced by $N(u) \cap V(H)$ is connected. Otherwise, u is called* bad.

Definition 1.5 *An even hole H of G is* clean *if there is no bad node w.r.t. H.*

Definition 1.6 *Let u be a good node w.r.t. an even hole H. We say that u is of Type gi w.r.t. H if $|N(u) \cap V(H)| = i$.*

Definition 1.7 *A* tent *w.r.t. an even hole H is either*

- *a Type g3 node w.r.t. to H, or*
- *an edge uv such that node u is a Type g1 node w.r.t. H, node v is a Type g2 node w.r.t. H, the neighbor x of u in H is distinct from the neighbors v_1, v_2 of v in H and x, v_1 have a common neighbor $y \neq v_2$ in H* (special tent).

Definition 1.8 *Let H be an even hole and u a Type g3 node w.r.t. H, with neighbors u_1, u_2 and u_3 in H such that u_1u_2 and u_2u_3 are edges. Let H' be the hole induced by $(V(H) \setminus \{u_2\}) \cup \{u\}$. We say that H' is obtained from H through a* Type g3 node substitution.

Consider a special tent uv w.r.t. an even hole H. Let H' be the hole induced by the node set $(V(H) \cup \{u, v\}) \setminus \{y, v_1\}$. We say that such a hole H' is obtained from H through a special tent substitution.

A tent substitution *is either a Type g3 node substitution or a special tent substitution. Note that holes H and H' are of the same length.*

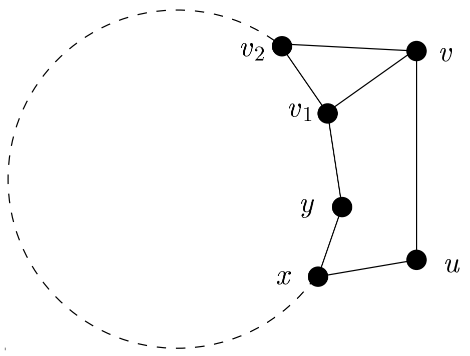

Figure 2: Special Tent

Definition 1.9 *Let G be a graph containing an even hole H. We define $\mathcal{C}_G(H)$ to be the family of all holes of G obtained from H through a sequence of tent substitutions.*

Definition 1.10 *An even hole H^* of G is* spotless *if all the holes in $\mathcal{C}_G(H^*)$ are clean.*

Definition 1.11 *A graph G is* clean *if it is either even-hole-free or it contains a spotless smallest even hole H^*.*

Given a graph G, Section 4 presents a cleaning procedure with the following property: it constructs in polytime a clean graph G' that is even-hole-free if and only if G is even-hole-free. The graph G' consists of a polynomial number of induced subgraphs of G, at least one of which is clean. The decomposition of clean graphs by k-star cutsets is presented in Section 3. The main result of that section is that a clean graph G can be decomposed recursively into a family of blocks that have no k-star cutsets and satisfy the following property: (i) either G is identified as containing an even hole during the decomposition process or (ii) when the decomposition process is completed, all blocks in the family are even-hole-free graphs if and only if G is even-hole-free.

Dominated Nodes

The other difficulty with k-star cutsets is that (c) does not hold. As mentioned earlier, our approach to (c) is to remove dominated nodes. We prove in Section 3 that the total number of blocks generated by recursive decomposition with k-star cutsets is polynomial if one first removes dominated nodes and uses full k-star cutsets. For this reason, in our recognition algorithm, we will actually use the following refinement of Theorem 1.1.

A *gem* is a graph on five nodes, such that four of the nodes induce a chordless path of length three and the fifth node is adjacent to all of the nodes of this path.

Theorem 1.12 *Let G be a connected even-hole-free graph. If G contains no gem or dominated node, then G is cap-free or basic or contains a 2-join or a full k-star cutset, $k = 1, 2, 3$.*

Proof: Follows from Theorem 1.1 and the next two lemmas. □

Lemma 1.13 *Assume G contains no gem and no 4-hole. Let C be a clique and $u \in V(G) \setminus C$. If $N[u] \subseteq N[C]$, then u is dominated by some node in C.*

Proof: Suppose $N[u] \subseteq N[C]$, but no node of C dominates u. Let $K \subseteq C$ be a minimal set such that $N[u] \subseteq N[K]$, i.e. for each $v \in K$, $N[u] \not\subseteq N[K \setminus \{v\}]$. Since $u \in N[K]$, u is adjacent to a node of K, say x. Since u is not dominated by x there exists $v \in N(u)$ such that v is not adjacent to x. Since $v \in N[K]$, v is adjacent to some node of $K \setminus \{x\}$, say y. Since x, y, v, u is not a 4-hole, u is adjacent to y. Since $N[u] \not\subseteq N[K \setminus x]$, there exists a node w adjacent to u and x but not y. Now either w, x, y, v, u induces a gem or w, x, y, v is a 4-hole. □

Lemma 1.14 *Assume G contains no dominated nodes, no gem and no 4-hole. If G contains a k-star cutset, $k = 1, 2, 3$, then G contains a full k-star cutset.*

Proof: Let C be the clique center of a k-star cutset S of G, where $k = 1, 2, 3$. Suppose $S' = C \cup N(C)$ is not a cutset of G. Then some component of $G \setminus S$, say C_1, must be entirely contained in $S' \setminus S$. Then $u \in C_1$ satisfies the conditions of Lemma 1.13 and therefore u is dominated by a node in C, contradicting the assumption. □

Dominated nodes can be identified in polytime and we will show in Section 3 that, in clean graphs, their removal is even-hole-preserving. In Section 3, we also show that, when G has a gem, there is a rather simple decomposition result. So Theorem 1.12 provides the basis for our recognition algorithm of even-hole-free graphs. The outline of the algorithm is as follows: check for 4-holes and a few other graphs that contain even holes and that can be identified in polytime (to simplify the analysis, later), then clean G, remove dominated nodes, decompose by full k-star cutsets, $k = 1, 2, 3$, then by 2-joins, and finally check that all the blocks are either basic or cap-free, and contain no even holes.

2 The Algorithm

A *wheel* (H, x) is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_n$. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A wheel is *even* if it contains an even number of sectors. It is easy to see that an even wheel always contains an even hole.

A $3PC(x,y)$ is a graph induced by three chordless paths from node x to y, having no common or adjacent intermediate nodes. Note that x and y are not adjacent. It is easy to see that a $3PC(x,y)$ always contains an even hole.

A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths, $P_1 = x_1, \ldots, y_1$, $P_2 = x_2, \ldots, y_2$ and $P_3 = x_3, \ldots, y_3$, having no common nodes and such that the only adjacencies between nodes of distinct paths are the edges of the two cliques of size three induced by the disjoint node sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. It is easy to see that a $3PC(x_1x_2x_3, y_1y_2y_3)$ always contains an even hole.

A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y$, $P_2 = x_2, \ldots, y$ and $P_3 = x_3, \ldots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P_i \setminus y$ and $P_j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by $\{x_1, x_2, x_3\}$.

We say that a graph G contains a $3PC(.,.)$ if it contains a $3PC(x,y)$ for some pair of nodes $x, y \in V(G)$. We say that a graph G contains a $3PC(\Delta, \Delta)$ if for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$ there exists a $3PC(x_1x_2x_3, y_1y_2y_3)$. Similarly we say that it contains a $3PC(\Delta, .)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$.

As mentioned above, an even-hole-free graph cannot contains an even wheel, a $3PC(.,.)$ nor a $3PC(\Delta, \Delta)$. Our recognition algorithm for even-hole-free graphs starts by checking whether the graph contains one of the two following structures (this can be done in polynomial time).

Definition 2.1 *A wheel (H, x) is a* short 4-wheel *if it contains four sectors and one of the following holds: the wheel has three short sectors, or it has two nonadjacent short sectors and a sector of length three.*

Definition 2.2 *A $3PC(.,.)$ is* short *if one path has length 2 and one has length 3. A $3PC(\Delta, \Delta)$ is* short *if one path has length one and one has length two. A* short 3PC *is either a short $3PC(.,.)$ or a short $3PC(\Delta, \Delta)$.*

RECOGNITION ALGORITHM FOR EVEN-HOLE-FREE GRAPHS

Input: A graph G.

Output: YES if G is even-hole-free, and NO otherwise.

Step 1: If G contains a 4-hole, a 6-hole, a short 4-wheel or a short $3PC$, output NO.

Step 2: Apply the Cleaning Algorithm of Section 4 to G and let $\mathcal{L}_1$ be the output family of graphs (so, if G has an even hole, then some graph in $\mathcal{L}_1$ has an even hole and is clean).

Step 3: Start with $\mathcal{L}_2 = \emptyset$. For each $L \in \mathcal{L}_1$, perform the Node Cutset Decomposition Algorithm of Section 3. If the algorithm identifies L as not being even-hole-free, output NO. Otherwise, union the output with $\mathcal{L}_2$ (so the graphs in $\mathcal{L}_2$ have no full k-star cutsets, $k = 1, 2, 3$).

Step 4: Start with $\mathcal{L}_3 = \emptyset$. For each $L \in \mathcal{L}_2$, perform the 2-Join Decomposition Algorithm of Section 5 and union the output with $\mathcal{L}_3$ (so the graphs in $\mathcal{L}_3$ have no 2-join).

Step 5: Start with $\mathcal{L}_4 = \mathcal{L}_5 = \emptyset$. For each $L \in \mathcal{L}_3$, check whether L contains a cap. If it does, add L to $\mathcal{L}_4$. Otherwise, add L to $\mathcal{L}_5$.

Step 6: For each $L \in \mathcal{L}_4$, check whether L is a basic graph. If some $L \in \mathcal{L}_4$ is not basic, output NO. Otherwise, for each $L \in \mathcal{L}_4$, check whether L contains an even hole. If some $L \in \mathcal{L}_4$ contains an even hole, output NO. Otherwise, go to Step 7.

Step 7: For each $L \in \mathcal{L}_5$, check whether L contains an even hole. If some $L \in \mathcal{L}_5$ contains an even hole, output NO. Otherwise, output YES.

The Cleaning Algorithm, the Node Cutset Decomposition Algorithm and the 2-Join Decomposition Algorithm will be shown to be polynomial in the next three sections. Steps 6 and 7 check cap-free and basic graphs. This can be performed in polytime, as pointed out already. So, the above recognition algorithm can be implemented to run in polynomial time.

In the next three sections, we will show that the following statements are equivalent.

(i) G is even-hole-free,

(ii) all the graphs in $\mathcal{L}_1$ are even-hole-free,

(iii) all the graphs in $\mathcal{L}_2$ are even-hole-free,

(iv) all the graphs in $\mathcal{L}_3$ are even-hole-free.

We will also show that the graphs in $\mathcal{L}_3$ do not contain a 4-hole, a dominated node, a gem, a full k-star cutset, $k = 1, 2, 3$, nor a 2-join. So, by Theorem 1.12, if G is even-hole-free, all the graphs in $\mathcal{L}_3$ must be either cap-free and even-hole-free, or basic and even-hole-free. The algorithm checks this in Steps 6 and 7. This establishes the validity of the algorithm (subject to being able to perform Steps 2, 3 and 4 as claimed).

3 k-Star Cutsets in Clean Graphs

Throughout this section, unless otherwise stated, we assume that G is a clean graph with spotless smallest even hole H^*. In addition, we assume that G contains no 4-hole, no short 4-wheel and no short $3PC$.

Lemma 3.1 *If node u is dominated by node v, then $G \setminus \{u\}$ contains a hole in $\mathcal{C}_G(H^*)$.*

Proof: Assume that H^* contains u. Let u_1 and u_2 be the neighbors of u in H^*. Since u is dominated by v, v is adjacent to u, u_1 and u_2. Since H^* is clean, v is of Type g3 w.r.t. H^*, and hence the hole induced by the node set $(V(H^*) \setminus \{u\}) \cup \{v\}$ is in $\mathcal{C}_G(H^*)$ and in $G \setminus \{u\}$. $\square$

Before proving the main results of this section, let us prove the following useful lemma.

Lemma 3.2 *Suppose C is a clique and $C \subseteq S \subseteq N[C]$ is a cutset breaking all the holes of $\mathcal{C}_G(H^*)$. Then, for each $H \in \mathcal{C}_G(H^*)$, $V(H) \cap C = \emptyset$.*

Proof: Suppose $H \in \mathcal{C}_G(H^*)$ is chosen such that the set $P = V(H) \cap C$ has maximum cardinality. As H is broken by S, there exists a node $x \in V(H) \cap S$ that has no neighbor in P. Let w be a neighbor of x in C. Now, if $P \neq \emptyset$, then w must be a Type g3 node w.r.t. H. After substituting w into H, we would get a hole in $\mathcal{C}_G(H^*)$ having more nodes from C than H, a contradiction. $\square$

This lemma, together with the definition of $\mathcal{C}_G(H^*)$, implies the following.

Corollary 3.3 *Suppose C is a clique and $C \subseteq S \subseteq N[C]$ is a cutset breaking all the holes of $\mathcal{C}_G(H^*)$. Then, for each $H \in \mathcal{C}_G(H^*)$, the tents w.r.t. H are disjoint from C.*

In the decomposition algorithm, we treat the decomposition of gems in a special way. Let us consider this case first, in the next two lemmas.

Lemma 3.4 *Let G be an even-hole-free graph and $\{x, y_0, y, z, z_0\}$ a node set that induces a gem, such that y_0, y, z, z_0 is a chordless path. Then $S = (N(x) \cup N(y) \cup N(z)) \setminus \{y_0, z_0\}$ is a triple star cutset breaking y_0 from z_0.*

Proof: Suppose not. Then, in $G \setminus S$, let P be a chordless path connecting y_0 to z_0. The nodes of P together with y and z induce a hole H. Node x has four neighbors on H, so (H, x) is an even wheel. $\square$

Remark 3.5 *If a triple star cutset S from Lemma 3.4 is such that the connected components of $G \setminus S$ that contain y_0 and z_0 respectively are both of size greater than 1, then $N(x) \cup N(y) \cup N(z)$ is a full triple star cutset.*

Lemma 3.6 *Let $\{x, y_0, y, z, z_0\}$ be a node set that induces a gem, such that y_0, y, z, z_0 is a chordless path. Let $S = N(x) \cup N(y) \cup N(z) \setminus \{y_0, z_0\}$ and C_1 (resp. C_2) be the connected component of $G \setminus S$ that contains y_0 (resp. z_0). If $|C_1| = 1$ (resp. $|C_2| = 1$), then $G \setminus \{y_0\}$ (resp. $G \setminus \{z_0\}$) contains a hole in $\mathcal{C}_G(H^*)$.*

Proof: Suppose that $|C_1| = 1$. If H^* does not contain y_0 then we are done, so suppose it does. Let $H^* = y_0, h_1, \ldots, h_n, y_0$. Then since $N(y_0) \subseteq S$, $h_1, h_n \in S$.
Case 1: h_1 or h_n is in $\{x, y\}$.

W.l.o.g. assume that $h_1 \in \{x, y\}$. Assume $h_1 = x$. Since H^* is a hole, h_n does not coincide with y and it cannot be a neighbor of x. Since $h_n \in S$, it must be a neighbor of y or z. If h_n is a neighbor of z then y_0, x, z, h_n, y_0 is a 4-hole. Hence h_n is a neighbor of y. But then y is of Type g3 w.r.t. H^* and so the hole induced by the node set $(V(H^*) \setminus \{y_0\}) \cup \{y\}$ is in $\mathcal{C}_G(H^*)$ and in $G \setminus \{y_0\}$.

When $h_1 = y$, the same argument holds by interchanging the roles of x and y.
Case 2: $h_1, h_n \in S \setminus \{x, y, z\}$

Assume first that one of the nodes x or y, is adjacent to both nodes h_1 and h_n. Assume w.l.o.g. that x is adjacent to both h_1 and h_n. Then x is of Type g3 w.r.t. H^* and the hole induced by the node set $(V(H^*) \setminus \{y_0\}) \cup \{x\}$ is in $\mathcal{C}_G(H^*)$ and in $G \setminus \{y_0\}$.

If x is adjacent to h_1 but not to h_n, and y is adjacent to h_n but not to h_1, then since the node set $V(H^*) \cup \{x, y\}$ cannot induce a short 4-wheel, x or y must have a neighbor in

$V(H^*) \setminus \{y_0, h_1, h_n\}$. W.l.o.g. assume that x has a neighbor in $V(H^*) \setminus \{y_0, h_1, h_n\}$. Since H^* is clean, x is adjacent to h_2. Since the hole induced by $(V(H^*) \setminus \{h_1\}) \cup \{x\}$ is clean, nodes $h_3, \ldots, h_{n-1}$ are not adjacent to y or x. But then the hole induced by the node set $(V(H^*) \setminus \{y_0, h_1\}) \cup \{x, y\}$ is in $\mathcal{C}_G(H^*)$ and in $G \setminus \{y_0\}$.

So we may assume that one of h_1 or h_n is adjacent to z. Assume w.l.o.g. that h_n is adjacent to z. Then y is adjacent to h_n, since otherwise y_0, y, z, h_n, y_0 is a 4-hole. Also x is adjacent to h_n, since otherwise y_0, x, z, h_n, y_0 is a 4-hole. Node z cannot be adjacent to h_1, since H^* is clean and of length greater than 4. Hence h_1 is adjacent to either x or y. But then one of x or y is adjacent to both h_1 and h_n, which is not possible. □

Definition 3.7 *A $3PC(x,y)$, with paths P_1, P_2 and P_3, is* decomposition detectable *w.r.t. the node cutset S if one of the following holds:*

(i) P_1 is of length 2 or 3, $V(P_1) \subseteq S$ and the intermediate nodes of P_2 and P_3 are in two different components of $G \setminus S$.

(ii) P_1 is of length 3, $V(P_1) \subseteq S$ and there are three distinct components of $G \setminus S$, C_1, C_2 and C_3, such that for some $z \in S \setminus \{x, y\}$, the intermediate nodes of P_2 are contained in $V(C_1) \cup V(C_2) \cup \{z\}$ and the intermediate nodes of P_3 are contained in $V(C_3)$.

A $3PC(x_1x_2x_3, y_1y_2y_3)$, with the three paths P_1, P_2 and P_3, is decomposition detectable *w.r.t. the node cutset S if $\{x_1, x_2, x_3, y_1, y_2, y_3\} \subseteq S$, P_1 is an edge and the intermediate nodes of P_2 and P_3 are contained in two different components of $G \setminus S$.*

A decomposition detectable 3PC *is either a decomposition detectable $3PC(.,.)$ or a decomposition detectable $3PC(\Delta, \Delta)$.*

In order to show that we end up with a polynomial number of pieces when we decompose a graph using our node cutsets, we need to refine the blocks. Let S be a k-star cutset, $k = 1, 2, 3$, with clique center C. Let $C_1, \ldots, C_n$ be the connected components of $G \setminus S$ and $G_1, \ldots, G_n$ the blocks of the decomposition. We define the *refined blocks* $G'_1, \ldots, G'_n$ as follows: for $i = 1, \ldots, n$, remove from G_i all nodes of $S \setminus C$ that do not have a neighbor in C_i.

Theorem 3.8 *Suppose that G contains no 4-hole, no short $3PC$, no short 4-wheel and that G is a clean graph with spotless smallest even hole H^*. When decomposing G with a full k-star cutset $S = N[C]$, $k = 1, 2, 3$, then either some hole in $\mathcal{C}_G(H^*)$ is entirely contained in one of the refined blocks of the decomposition or there exists a decomposition detectable $3PC$ w.r.t. S.*

Proof: Consider the following two cases.
Case 1: All the holes of $\mathcal{C}_G(H^*)$ are broken by S.

Then, by Lemma 3.2, for each $H \in \mathcal{C}_G(H^*)$, $V(H) \cap C = \emptyset$. Furthermore, by Corollary 3.3, no node of C is of Type g3. Let $C = \{v_1, \ldots, v_k\}$, where $k = |C|$. Denote by $P_1, \ldots, P_m$ the connected components of $G(V(H) \cap S)$. As H is broken by S, $m \geq 2$. On the other hand, as H is clean, each node of C is adjacent to at most one path $P_1, \ldots, P_m$. Hence $2 \leq m \leq k \leq 3$.
Case 1.1: $m = k = 3$.

Then we may assume that $V(P_i) = N(v_i) \cap V(H)$, $i = 1, 2, 3$.

If all the nodes of C are of Type g2 w.r.t. H, let u_i and w_i be the neighbors of v_i in H and assume w.l.o.g. that the nodes $u_1, w_1, u_2, w_2, u_3, w_3$ appear in this order when traversing H. Let Q_1 be the w_1u_2-subpath of H that does not contain u_1, w_2, u_3, w_3. Let Q_2 (respectively Q_3) be the w_2u_3-subpath (respectively w_3u_1-subpath) of H that does not contain nodes of Q_1. Since H is an even hole, at least one of the three paths Q_i is of odd length, say Q_1. But then the hole induced by $V(Q_1) \cup \{v_1, v_2\}$ is an even hole of length smaller than H, contradicting our choice of H.

If all the nodes of C are of Type g1 w.r.t. H, let u_i be the neighbor of v_i in H. Let Q_1 be the u_1u_2-subpath of H that does not contain u_3. Define Q_2 and Q_3 in a similar fashion. Since H is broken by S, some connected component of $G \setminus S$ contains the intermediate nodes of one of these paths, say Q_1, but not of the other two paths. So we get a decomposition detectable $3PC(u_1, u_2)$ satisfying (i) or (ii) of Definition 3.7.

If C has both Type g1 and Type g2 nodes w.r.t. H, assume w.l.o.g. that v_1 is of Type g1 and v_2 is of Type g2. Since H is a smallest even hole, v_1v_2 is a special tent w.r.t. H, contradicting Corollary 3.3.

Case 1.2: $m = 2$.

First, suppose that $k = 3$. Assume that $N[v_1] \cap V(H) = V(P_1)$ and $N[\{v_2, v_3\}] \cap V(H) = V(P_2)$, where $|N[v_2] \cap V(H)| \geq |N[v_3] \cap V(H)|$. If v_2 and v_3 both have a neighbor in H but do not have a common neighbor in H, then G contains a 4-hole. Hence, since v_2 and v_3 are of Type g1 or g2 or v_3 does not have a neighbor in H, $|V(P_2)| \leq 3$. If $|V(P_2)| = 3$, then $G(H \cup \{v_2, v_3\})$ is a short 4-wheel. It follows that $V(P_2) = N[v_2] \cap V(H)$.

Now, if v_1 and v_2 are of the same type, we get a decomposition detectable $3PC(\Delta, \Delta)$ or $3PC(.,.)$. If one is of Type g1 and the other of Type g2, v_1v_2 is a special tent. But this contradicts Corollary 3.3.

If $k = 2$, the arguments from the previous paragraph hold.

Case 2: A block G_i contains a hole of $\mathcal{C}_G(H^*)$.

Suppose $H \in \mathcal{C}_G(H^*)$ is a hole in G_i such that $V(H) \cap C$ has maximum cardinality. If $H \notin G'_i$, it follows from the definition of refined block that some node $x_2 \in V(H) \cap N(C)$ has no neighbor in $H \setminus N[C]$. So, there exists a chordless path $P' = x_1, x_2, x_3$ in H such that $x_1, x_2 \in N(C)$ and x_1 is adjacent to some $w_1 \in C \setminus V(H)$. If $V(H) \cap C \neq \emptyset$ or $w_1 \in N(x_3)$, then w_1 is of Type g3 and, after substituting w_1 into H, we would obtain a hole of $\mathcal{C}_G(H^*)$ in G_i with larger intersection with C than H, a contradiction. It follows that, for each $H \in \mathcal{C}_G(H^*)$, $V(H) \cap C = \emptyset$ and w_1x_3 is not an edge.

By the choice of x_2, this implies $x_3 \in N(C)$. So x_3 is adjacent to some node $w_3 \in C$. Then w_3 is not adjacent to x_1, else w_3 is of Type g3 and, after substituting w_3 into H, we have a contradiction to the choice of H. In fact, by the same argument, no node of C is of Type g3 w.r.t. H. As G is 4-hole-free and short 4-wheel-free, x_2 is adjacent to neither w_1 nor w_3. So x_2 is adjacent to some node $w_2 \in C$. Since G is 4-hole-free, w_2 is adjacent to both x_1 and x_3. Hence w_2 is of Type g3 w.r.t. H, a contradiction. $\square$

NODE CUTSET DECOMPOSITION ALGORITHM

Input: A graph G that does not contain a 4-hole, a short $3PC$ nor a short 4-wheel.

Output: Either G is identified as not being even-hole-free, or a list $\mathcal{L}$ of induced subgraphs of G with the following properties:

- The graphs in $\mathcal{L}$ do not contain a gem, a full k-star cutset, $k = 1, 2, 3$, nor any dominated nodes.
- If the input graph G contains an even hole and is clean, with spotless smallest even hole H^*, then one of the graphs in the list contains a hole in $\mathcal{C}_G(H^*)$.

Step 1: Initialize $\mathcal{M} = \{G\}$, $\mathcal{L} = \emptyset$.

Step 2: If $\mathcal{M}$ is empty, return $\mathcal{L}$ and stop. Otherwise, remove a graph F from $\mathcal{M}$. If F has no chordless path of length 4, go to Step 2. Otherwise, remove all dominated nodes from F and go to Step 3.

Step 3: If F contains a gem $\{x, y_0, y, z, z_0\}$, such that y_0, y, z, z_0 is a chordless path, go to Step 4. If F contains a full k-star cutset S, $k = 1, 2, 3$, go to Step 5. Otherwise, add F to $\mathcal{L}$ and go to Step 2.

Step 4: If $S = (N(x) \cup N(y) \cup N(z)) \setminus \{y_0, z_0\}$ is not a cutset breaking y_0 from z_0, go to Step 6. If the connected component of $F \setminus S$ that contains y_0 is of size 1, add graph $F \setminus \{y_0\}$ to $\mathcal{M}$ and go to Step 2. If the connected component of $F \setminus S$ that contains z_0 is of size 1, add graph $F \setminus \{z_0\}$ to $\mathcal{M}$ and go to Step 2. Otherwise, let $S = N(x) \cup N(y) \cup N(z)$ and go to Step 5.

Step 5: Check whether there exists a decomposition detectable $3PC(.,.)$ or $3PC(\Delta, \Delta)$ w.r.t. S. If yes, go to Step 6. Otherwise, construct the refined blocks of decomposition by S, add them to $\mathcal{M}$ and go to Step 2.

Step 6: Return that G is not even-hole-free and stop.

Lemma 3.9 *The Node Cutset Decomposition Algorithm produces the desired output.*

Proof: First suppose that the algorithm terminates in Step 6. Then by Lemma 3.4 and the fact that $3PC(.,.)$'s and $3PC(\Delta, \Delta)$'s contain even holes, the algorithm correctly identifies G as not being even-hole-free. Now suppose that the algorithm outputs the list $\mathcal{L}$, i.e. the algorithm does not terminate in Step 6. Then clearly, by Steps 2 and 3, the graphs in $\mathcal{L}$ do not contain any dominated node, gem or full k-star cutset, $k = 1, 2, 3$. Now further assume that the input graph G is clean and contains a spotless smallest even hole H^*. We want to show that some graph in list $\mathcal{L}$ contains a hole in $\mathcal{C}_G(H^*)$.

Let F be a graph taken off list $\mathcal{M}$ in Step 2. It is enough to show that if F contains a hole in $\mathcal{C}_G(H^*)$ then at least one of the graphs that gets put on list $\mathcal{M}$ or $\mathcal{L}$ in Steps 3, 4 and 5 also contains a hole in $\mathcal{C}_G(H^*)$. This follows from Lemma 3.1, Lemma 3.6 and Theorem 3.8. $\square$

Lemma 3.10 *The number of induced subgraphs in list $\mathcal{L}$ produced by the Node Cutset Decomposition Algorithm is bounded by $|V(G)|^5$.*

Proof: Let F be a graph taken off list $\mathcal{M}$ in Step 2. Suppose that F is decomposed in Step 5 by a full k-star cutset S, $k = 1, 2, 3$. Let $C_1, \ldots, C_n$ be the connected components of $F \setminus S$ and let $F_1, \ldots, F_n$ be the refined blocks of decomposition by S. Let C be the clique center of S.

Claim: *No two of the graphs $F_1, \ldots, F_n$ contain the same chordless path of length 4.*
Proof of Claim: Let P be a chordless path of length 4 and suppose that P appears in F_1 and F_2. Then $V(P) \subseteq V(F_1) \cap S$ and $V(P) \subseteq V(F_2) \cap S$. Since $V(P) \subseteq S$, it contains two nonadjacent nodes $a, b \in S \setminus C$, such that there exists a chordless path P' from a to b that uses only nodes in C as intermediate nodes. Since $a \in V(F_1) \cap V(F_2)$, by definition of the refined blocks, a has neighbors in both C_1 and C_2. Similarly b has neighbors in both C_1 and C_2. Note that by definition of S, nodes of C do not have neighbors in C_1 and C_2. But now there is a $3PC(a, b)$ that uses P' and paths in C_1 and C_2. This $3PC(., .)$ is decomposition detectable w.r.t. S and hence would have been detected in Step 5. This completes the proof of the claim.

By Step 2, the algorithm only adds to $\mathcal{L}$ subgraphs of G that have a chordless path of length 4. So, it follows from the claim that the number of graphs in $\mathcal{L}$ is at most $|V(G)|^5$. □

4 Cleaning

This section is devoted to the construction of the Cleaning Algorithm. We assume throughout this section that G contains no 4-hole, no 6-hole, no short 4-wheel and no short $3PC$ (recall Definitions 2.1 and 2.2). The Cleaning Algorithm will take as input the graph G and produce a polynomial family $\mathcal{L}$ of induced subgraphs of G such that, if G contains an even hole, then at least one of the graphs in $\mathcal{L}$ contains an even hole and is clean. Given a hole H, a node $v \notin H$ is *strongly adjacent* to H if v has at least two neighbors in H. Recall that an even hole H is clean if it has no bad strongly adjacent nodes (Definitions 1.4 and 1.5).

Lemma 4.1 *Let u be a bad node w.r.t. a smallest even hole H of G. Then either u has exactly two neighbors in H and these nodes are nonadjacent, or (H, u) is an even wheel and all the sectors of the wheel are odd.*

Proof: If u has two neighbors in H, then they are nonadjacent since u is bad. So assume that u has at least three neighbors in H. If u has an odd number of neighbors in H, then since H is an even hole, one of the sectors of the wheel (H, u) must be even. That sector together with u induces an even hole and since that hole cannot be smaller than H, u must be of Type g3, contradicting the assumption that u is bad. By a similar argument, if u has an even number of neighbors in H, then all the sectors of (H, u) must be odd. □

Definition 4.2 *Let v be a bad node w.r.t. a smallest even hole H of G. For $i = 1, 2, 3$, we say that v is of Type bi w.r.t. H if $V(H) \cap N(v)$ induces a graph G' with exactly two connected components, $|V(G')| \leq 4$ and the largest connected component of G' has exactly i nodes (see Figure 3). Otherwise, we call v a Type b4 node w.r.t. H.*

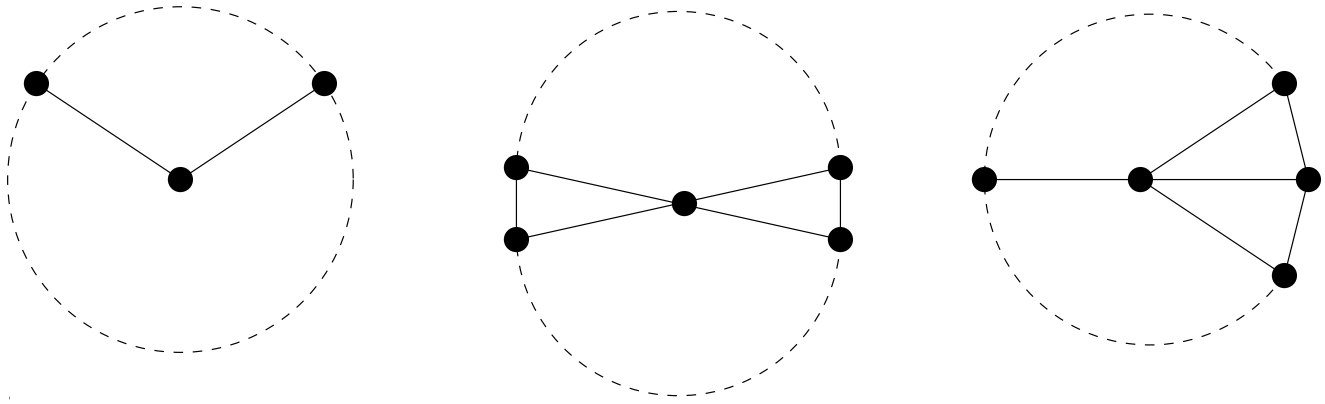

Figure 3: Bad nodes of Type b1, Type b2 and Type b3

Suppose H is a smallest even hole in G and v_1 and v_2 are two nonadjacent bad nodes w.r.t. H. Consider the following three types of subpaths of H.

e-path We call a subpath Q_i of H an edge-path (or *e-path*) if one of its endnodes is adjacent to v_1, the other is adjacent to v_2, at most one endnode is adjacent to both v_1, v_2, and no intermediate node of Q_i is adjacent to v_1 or v_2.

n-path We call a subpath P_i of H a node-path (or *n-path*) if it is a maximal path with the following property: the endnodes of P_i are adjacent to v_1 and no node of P_i is adjacent to v_2, or the endnodes of P_i are adjacent to v_2 and no node of P_i is adjacent to v_1. Note that an n-path can have length 0.

z-path We call a subpath P_0 of H a zero-path (or *z-path*) if it is a maximal path with all the nodes adjacent to both v_1 and v_2. As G is 4-hole-free, there is at most one z-path. Furthermore, if the z-path exists, it has at most two nodes.

We construct the graph H' from H defined as follows:
Contract each e-path Q_i of H to a single edge q_i.
Contract each n-path P_i of H to a single node p_i.
If H has a z-path P_0, contract it to a single node p_0 called the *z-node* of H'.

Since H has at least one node adjacent to v_1 but not v_2 and another adjacent to v_2 but not v_1, the graph H' has at least two nodes distinct from the z-node. Moreover, if H' has no z-node, it has at least four nodes. To see this, note that, since H has no z-path, it must have an even number of e-paths. If H has exactly two e-paths, then $V(H) \cup \{v_1, v_2\}$ contains an even hole smaller than H. So H has at least four e-paths and hence H' has at least four nodes.

We call an edge or a node of H' *even (odd)* if the corresponding path of H has even (odd) number of edges. We call an edge or a node of H' *real* if the corresponding path of H is an edge or a node respectively. Note that real edges are odd and real nodes are even.

Lemma 4.3 *Let q_i and q_{i+1} be two consecutive edges of H' such that their common endnode p_i is distinct from p_0. Then q_i and q_{i+1} have the same parity if and only if p_i is odd. Moreover, the edges of H' incident with p_0 are odd.*

Proof: Indeed, otherwise either (H, v_1) or (H, v_2) would have an even sector, contradicting Lemma 4.1. □

Lemma 4.4 *Suppose that H' has a z-node p_0 and that $q_i = p_i p_{i+1}$ is an even edge. Then q_i has a real endnode that is adjacent to p_0 by a real edge. Moreover, p_0 is a real node and H' has at least four edges.*

Proof: By Lemma 4.3, p_0 is not an endnode of q_i. If P_0 has a node u_0 that is adjacent to neither endnode of Q_i, then $V(Q_i) \cup \{u_0, v_1, v_2\}$ induces an even hole. Since H is a smallest even hole of G, $V(H) \setminus V(Q_i)$ contains three nodes. But now, since v_1 and v_2 are bad w.r.t. H, they are of Type b1. This implies that G contains a 4-hole, a contradiction. Hence, we may assume w.l.o.g. that p_i is a real node and is adjacent to p_0 by a real edge. As v_1 and v_2 are bad nodes w.r.t. H, it follows that p_{i+1} is not adjacent to p_0 in H'. Hence, since every node of P_0 must be adjacent to an endnode of Q_i, p_0 is a real node. Finally, since v_1 and v_2 are bad, H' has at least four edges. □

Lemma 4.5 *Let q_i and q_j be two nonconsecutive edges of H' with the same parity. Suppose that p_0 is not an endnode of q_i nor q_j. Then q_i and q_j have real endnodes that are adjacent by a real edge.*

Proof: Suppose not. Since $V(Q_i) \cup V(Q_j) \cup \{v_1, v_2\}$ does not induce a smaller even hole than H, it follows that H' has four edges, say $i = 1$ and $j = 3$, the paths Q_2 and Q_4 each have length 2, and v_1, v_2 are both of Type b1. Since G has no short $3PC(.,.)$, both Q_1 and Q_3 have length greater 1. It follows that $V(Q_2) \cup V(Q_4) \cup \{v_1, v_2\}$ is an 8-hole. Since H is a smallest hole, Q_1 and Q_3 both have length 2. But now $V(Q_1) \cup V(Q_2)$ forms a 6-hole with v_1 or v_2, contradicting the assumption that G contains no 6-hole. □

Lemma 4.6 *If p_i is a node of H' that is not adjacent to p_0, then either p_i is even or P_i is an edge.*

Proof: The result holds when $i = 0$, so we assume now $i \neq 0$. Suppose p_i is odd. Then, by Lemma 4.3, the two edges of H' that have p_i as a common endnode, say q_i and q_{i+1}, must have the same parity. So, if P_i is not an edge, $V(Q_i) \cup V(Q_{i+1}) \cup \{v_1, v_2\}$ induces a smaller even hole than H. □

Theorem 4.7 *Let v_1 and v_2 be nonadjacent bad nodes w.r.t. a smallest even hole H of G. Then either v_1 and v_2 have a common neighbor in H, or exactly one of v_1, v_2 is of Type b2 w.r.t. H.*

Proof: Let H' be defined from H as above. Assume v_1 and v_2 have no common neighbor in H. Then H' has no z-node. Let $p_1, \ldots, p_m$ be the nodes of H' appearing in this order when traversing H' and assume w.l.o.g. that v_1 is adjacent to p_1. Then p_k is adjacent to v_i if and only if $k \equiv i \pmod 2$. Furthermore, m is even since $p_1 p_m$ is an edge and p_1 is adjacent to v_1, which implies that p_m is adjacent to v_2.

Case 1: $m \geq 6$.

It follows from Lemma 4.5 that H' cannot have three consecutive even edges. Hence H' has two odd edges, the endnodes of which are not adjacent by a real edge. But this contradicts Lemma 4.5.

Case 2: $m = 4$.

Suppose v_1 is not a Type b2 node w.r.t. H. Then, by Lemmas 4.1 and 4.6, both p_1 and p_3 must be even. Now, if p_2 and p_4 are also even, then by Lemma 4.3, the edges of H' must be alternately odd and even. Thus H' has two odd edges whose endnodes are not adjacent by a real edge, contradicting Lemma 4.5. Hence v_2 is of Type b2.

If both v_1 and v_2 are of Type b2, then all the nodes of H' are odd and, by Lemma 4.3, all the edges of H' must have the same parity. But then, any two nonadjacent edges of H' contradict Lemma 4.5. $\square$

Lemma 4.8 *Let H be a Type b2 node free smallest even hole and let v_1 and v_2 be two nonadjacent bad nodes w.r.t. H. Then $H = u_0, u_1, \ldots, u_r$ where v_1 and v_2 are both adjacent to u_0. If v_1 and v_2 have exactly one common neighbor in H, then w.l.o.g. v_1 is adjacent to u_1 and the two sectors of (H, v_1) with common endnode u_1, contain all the neighbors of v_2 in H. Otherwise, v_1 and v_2 are both adjacent to u_1 and w.l.o.g. the two sectors of (H, v_1) with common endnode u_1, contains all the neighbors of v_2 in H.*

Proof: By Theorem 4.7, H has a z-path. Consider $H' = p_0, p_1, \ldots, p_m$ obtained from H as before, where p_0 is the z-node. Assume w.l.o.g. that $q_i = p_i p_{i+1}$ where $0 \leq i \leq m$ and $m + 1 \equiv 0$. Furthermore, assume w.l.o.g. that v_1 is adjacent to p_1, i.e. the endnodes of P_1 are adjacent to v_1. By Lemmas 4.3 and 4.4, all the edges of H' are odd, except maybe q_1 and q_{m-1}.

Case 1: H' has an even edge.

W.l.o.g. q_1 is even. By Lemma 4.4, q_0 is a real edge, both p_0 and p_1 are real nodes and $m \geq 3$. If $m = 3$, we are done. Assume $m = 4$. As p_0 and p_1 are real nodes, Lemma 4.1 implies that p_3 must be odd. But then, by Lemma 4.6, v_1 would be of Type b2. Hence $m \geq 5$. As both q_2 and q_3 are odd by Lemma 4.3, it follows that p_3 is odd. Hence, by Lemma 4.5 applied to q_2 and q_4, q_4 is even. But then q_1 and q_4 contradict Lemma 4.5.

Case 2: All the edges of H' are odd.

By Lemma 4.3, p_2 is odd. If $m \geq 4$, then the pair q_1 and q_3 contradicts Lemma 4.5. If $m = 3$, then, by Lemmas 4.1 and 4.6, v_2 would be of Type b2 w.r.t. H. Hence $m = 2$ and, by Lemma 4.1 applied to H and v_2, P_0 has two nodes u_0 and u_1. So we are done. $\square$

This lemma implies the next result.

Theorem 4.9 *Let H be a Type b2 node free smallest even hole. Let v_1 be a Type b3 node w.r.t. H and $N(v_1) \cap V(H) = \{u_1, u_2, u_3, u_4\}$, where u_2 is adjacent to u_1 and u_3. If v_2 is a bad node w.r.t. H, then $N(v_2) \cap \{u_2, u_4, v_1\} \neq \emptyset$.*

PROCEDURE BAD

Input: A graph G that does not contain a 4-hole, a 6-hole, a short 4-wheel nor a short 3PC.

Output: A family $\mathcal{L}$ of induced subgraphs of G that satisfies the following: If G contains a smallest even hole H, then, for some $G' \in \mathcal{L}$ containing H, the family $\mathcal{C}_{G'}(H)$ has no Type b2 nodes. Moreover, if there is a Type b1 or b3 node w.r.t. H but no Type b2 node w.r.t. a hole in $\mathcal{C}_G(H)$, then H is a spotless smallest even hole in some graph $G'' \in \mathcal{L}$.

Step 1: Set $\mathcal{L} = \{G\}$.

Step 2: For every (P_1, P_2, u), where $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ are disjoint chordless paths in G and $u \in N(x_1) \cap N(y_1)$, add to $\mathcal{L}$ the graphs obtained from G by removing the node set $N(\{x_1, x_2, y_1, y_2, u\}) \setminus (V(P_1) \cup V(P_2))$.

Theorem 4.10 *Procedure BAD produces the desired output.*

Proof: Let u be a Type bi node w.r.t. a smallest even hole H, where $i \leq 3$. Take $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ to be disjoint subgraphs of H such that $N(u) \cap \{x_1, x_2, y_1, y_2\}$ has maximum cardinality. Denote by G' the graph $G \setminus (N(\{x_1, x_2, y_1, y_2, u\}) \setminus (V(P_1) \cup V(P_2)))$. Then $G' \in \mathcal{L}$ and $H \subseteq G'$.

Claim: *In G, node u is a Type bi node w.r.t. all the holes in $\mathcal{C}_{G'}(H)$.*

Proof of Claim: Indeed, in G', the nodes x_1, x_2, y_1 and y_2 have degree 2. Since they belong to H, they also belong to all the holes in $\mathcal{C}_{G'}(H)$. It follows that P_1 and P_2 are subpaths in all the holes of $\mathcal{C}_{G'}(H)$. This completes the proof of the claim.

By Theorem 4.7, if $i = 2$, then every hole in $\mathcal{C}_{G'}(H)$ is Type b2 node free. By Theorem 4.9, if $i = 3$ and all holes in $\mathcal{C}_G(H)$ are Type b2 node free, then H is a spotless smallest even hole in G'. Finally, by Theorem 4.7, if $i = 1$ and all holes in $\mathcal{C}_G(H)$ are Type b2 node free, then H is a spotless smallest even hole in G'. □

Lemma 4.11 *Let H be a Type b2 node free smallest even hole and v_1, v_2 and v_3 be three pairwise nonadjacent bad nodes w.r.t. H. Then there exists a node $u \in V(H)$ that is adjacent to v_1, v_2 and v_3.*

Proof: By Theorem 4.7, there exists a node $u \in V(H)$ that is adjacent to v_1 and v_2. Suppose v_3 is not adjacent to u.

As v_1 and v_2 are two nonadjacent bad nodes w.r.t. H, by Lemma 4.8, we may let $H = u_0, u_1, \ldots, u_m$ where $u = u_0$, node u_1 is adjacent to v_1 (and possibly v_2) and the two sectors of (H, v_1) with common endnode u_1 contain all the neighbors of v_2 in H. Consider the following two cases.

Case 1: v_3 is adjacent to u_1.

As v_3 is not adjacent to u_0, and v_1 is adjacent to u_0 but not to u_2, it follows from Lemma 4.8 that the two sectors of v_1 sharing u_0 contain all the neighbors of v_3 in H. By Theorem 4.7, nodes v_2 and v_3 have a common neighbor in H. The only possibility is node u_1. So u_1 satisfies the lemma.

Case 2: v_3 is not adjacent to u_1.

Suppose that v_1, v_2 and v_3 do not have a common neighbor in H. Let u_i be adjacent to v_1 and v_3, and let u_j be adjacent to v_2 and v_3. Then $i > j$. First assume that $u_i = u_m$. It follows from Lemma 4.8 applied to v_1 and v_3 that $N(v_1) \cap V(H) = \{u_0, u_1, u_{m-1}, u_m\}$. But then (H, v_1) is a short 4-wheel, a contradiction.

It follows that $i < m$. Then $i = j+1$, otherwise the set $\{u_i, u_j, v_1, v_2, v_3, u_0\}$ would induce a 6-hole. If v_3 is not adjacent to u_{j-1}, then by Lemma 4.8, the two sectors of (H, v_3) sharing u_i must contain all the neighbors of v_2. But then v_3 is not adjacent to u_{i+1} and the two sectors of (H, v_3) sharing u_j must contain all the neighbors of v_1 in H, a contradiction. Hence v_3 is adjacent to both u_{j-1} and u_{i+1}. Now, by Lemma 4.8, the sectors of (H, v_3) sharing u_{i+1} (u_{j-1}) contain all the neighbors of v_1 (v_2). So (H, v_3) is a short 4-wheel, a contradiction. $\square$

Theorem 4.12 *Let H be a Type b2 node free smallest even hole. If there exist three nonadjacent bad nodes w.r.t. H, then there exists a node u in H such that all the bad nodes w.r.t. H are adjacent to node u or to one of the neighbors of u in H.*

Proof: Suppose v_1, v_2 and v_3 are three nonadjacent bad nodes w.r.t. H and u is a common neighbor in H (such a node exists by Lemma 4.11). Let u_1, u_2 denote the neighbors of u in H. Suppose v is a bad node w.r.t. H that is not adjacent to a node in $\{u, u_1, u_2\}$. Then, v is adjacent to at most one of the nodes v_1, v_2, v_3, else G contains a 4-hole. Say v is not adjacent to v_1 and v_2. Now, by Lemma 4.11, nodes v_1, v_2, v have a common neighbor in H, say w. But then w, v_1, u, v_2 is a 4-hole, a contradiction. $\square$

For a node set S, denote by $\alpha(S)$ the cardinality of a largest stable set in S.

Theorem 4.13 *Let H be a Type b2 node free smallest even hole and S be the set of all bad nodes w.r.t. H.*

a. *If $\alpha(S) = 1$, then there are two nonadjacent nodes u_1, u_2 in H such that either $S = N'$ where $N' = N(u_1) \cap N(u_2)$, or there exists $a \in S \setminus N'$ with the property that, if N denotes the set of nodes of $G \setminus (N' \cup \{a\})$ adjacent to all nodes in $N' \cup \{a\}$, then $|V(H) \cap N| \leq 3$ and $S \subseteq N \cup N' \cup \{a\}$.*

b. *If $\alpha(S) = 2$, then there are two nonadjacent nodes u_1, u_2 in H, and a third node w_1 in H (not necessarily distinct from u_1 or u_2) such that, if $A = S \setminus N(w_1)$ and $N'' = (N(u_1) \cap N(u_2)) \setminus N(w_1)$, then either $\alpha(A \setminus N'') \leq 1$, or there exists a node $a \in A \setminus N''$ and a node v_1 adjacent to u_1, u_2 and w_1 with the property that, if N is the set of nodes of $G \setminus (N'' \cup \{a, v_1\})$ that are adjacent to all the nodes in $N'' \cup \{a, v_1\}$, then $|V(H) \cap N| \leq 3$ and $\alpha(A \setminus (N \cup N'' \cup \{a\})) \leq 1$.*

Proof: **a.** Let u_1 and u_2 be two nodes of H such that

(i) the shortest path of H connecting u_1 and u_2 has at least three edges,

(ii) $N' = N(u_1) \cap N(u_2)$ has maximum cardinality.

By (i), $N' \subseteq S$. If $N' = S$, we are done. So, suppose $a \in S \setminus N'$. Denote by N the nodes of $G \setminus (N' \cup \{a\})$ adjacent to all nodes in $N' \cup \{a\}$. Then, since S is a clique containing $N' \cup \{a\}$, $S \subseteq N \cup N' \cup \{a\}$.

If $|V(H) \cap N| \geq 4$, then H would contain two nodes x_1 and x_2 satisfying (i) and having more common neighbors in S than u_1 and u_2, which contradicts (ii).

b. Suppose $v_1, v_2 \in S$ are nonadjacent.

By Theorem 4.7, nodes v_1 and v_2 have a common neighbor in H, say w_1. Let A be the set of bad nodes that are not adjacent to w_1. As G is 4-hole free, each node of A is adjacent to exactly one of v_1, v_2. For $i = 1, 2$, denote by A_i the set of nodes of A adjacent to v_i. Then $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$. As $\alpha(S) = 2$, it follows that both A_1 and A_2 are cliques (possibly empty). Now assume that u_1 and u_2 are two nodes of H such that

(i) v_1 is adjacent to both u_1 and u_2,

(ii) the shortest path in H connecting u_1 and u_2 has at least three edges,

(iii) $N'' = (N(u_1) \cap N(u_2)) \setminus N(w_1)$ has maximum cardinality.

As v_1 is a bad node w.r.t. H, such a pair of nodes u_1, u_2 always exists. (ii) and (iii) imply that $N'' \subseteq A$. As G is 4-hole free and $N(v_1) \cap A_2 = \emptyset$, it follows that $N'' \subseteq A_1$. If $A_1 = N''$, then $A \setminus N'' = A_2$, so $\alpha(A \setminus N'') \leq 1$ and we are done. So, suppose $a \in A_1 \setminus N''$. Denote by N the nodes of $G \setminus (N'' \cup \{a, v_1\})$ adjacent to all the nodes in $N'' \cup \{a, v_1\}$. Then, since A_1 is a clique containing $N'' \cup \{a\}$, it follows that $A_1 \subseteq N \cup N'' \cup \{a\}$, and hence $\alpha(A \setminus (N \cup N'' \cup \{a\})) \leq 1$.

If $|V(H) \cap N| \geq 4$, then N would contain two nodes x_1 and x_2 satisfying (i) and (ii) and having more common neighbors in A_1 than u_1 and u_2, which contradicts (iii). □

PROCEDURE b4

Input: A graph G that does not contain a 4-hole, a 6-hole, a short 4-wheel nor a short 3PC.

Output: A family $\mathcal{L}$ of induced subgraphs of G that satisfies the following: If G contains a smallest even hole H such that $\mathcal{C}_G(H)$ is Type bi node free for $i = 1, 2, 3$, then H is a spotless smallest even hole in some $G' \in \mathcal{L}$.

Step 1: Set $\mathcal{L} = \mathcal{L}_2 = \{G\}$ and $\mathcal{L}_1 = \mathcal{L}_3 = \emptyset$.

Step 2: For every chordless path $P = w_0, w_1, w_2, w_3, w_4$ in G, add to $\mathcal{L}$ the graph obtained from G by removing the node set $(\bigcup_{i=1}^{3} N(w_i)) \setminus V(P)$.

Step 3a: For every chordless path $P = w_0, w_1, w_2$ in G and $v_1 \neq w_0, w_2$ adjacent to w_1, add to $\mathcal{L}_1$ the graph obtained from G by removing the node set $N(w_1) \setminus \{w_0, w_2, v_1\}$.

For $k = 1$ **to** 2, **do**

begin

Step 3b: For every $L \in \mathcal{L}_k$ and for every nonadjacent $u_1, u_2 \in V(L)$, add to $\mathcal{L}_{k+1}$ the graph obtained from L by removing the node set $N(u_1) \cap N(u_2)$.

Step3c: For every $L \in \mathcal{L}_k$ and for every nonadjacent $u_1, u_2 \in V(L)$, let $N' = N(u_1) \cap N(u_2)$. For every $a \in V(L) \setminus N'$, let N denote the set of nodes of $L \setminus (N' \cup \{a\})$ that are adjacent to all the nodes in $N' \cup \{a\}$. For $i = 0, 1, 2, 3$, let $\mathcal{N}_i$ denote the family of all subsets of N with cardinality $|N| - i$. For every $M \in \mathcal{N}_i$, add to $\mathcal{L}_{k+1}$ the graph obtained from L by removing the node set $M \cup N' \cup \{a\}$.

end

Step 4: Add to $\mathcal{L}$ all the graphs in $\mathcal{L}_3$.

Theorem 4.14 *Procedure* **b4** *produces the desired output.*

Proof: Let H be a smallest even hole in G that is Type b2 node free, and S be the set of all bad nodes w.r.t. H.

If $\alpha(S) \geq 3$, then, by Theorem 4.12, Step 2 produces a graph G' in $\mathcal{L}$ where H is clean.

If $\alpha(S) = 1$, then, by Theorem 4.13a, Steps 3b and 3c applied to $G \in \mathcal{L}_2$ when $k = 2$, produces a graph $G' \in \mathcal{L}_3$ where H is clean.

Finally, if $\alpha(S) = 2$, then Step 3a produces a graph $L \in \mathcal{L}_1$ where the nodes of G in $N(w_1) \setminus \{w_0, w_2, v_1\}$ are removed. The bad nodes that remain are v_1 and $A = S \setminus N(w_1)$. By Theorem 4.13b, Steps 3b and 3c applied to L when $k = 1$ produce a graph in $\mathcal{L}_2$ that contains H and such that the set A_2 of remaining bad nodes w.r.t. H satisfies $\alpha(A_2) \leq 1$ (Note that N' in Step 3c (k=1) of the algorithm is equal to $N'' \cup \{v_1\}$ as defined in Theorem 4.13b whenever v_1 is adjacent to u_1 and u_2.) Now, by Theorem 4.13a, Steps 3b and 3c when $k = 2$ produce some graph $G' \in \mathcal{L}_3$ where H is clean.

So, in all cases, the algorithm produces a graph G' in $\mathcal{L}$ where H is clean. To complete the proof it remains to show that, if $\mathcal{C}_G(H)$ is Type bi node free for $i = 1, 2, 3$, then H is a spotless smallest even hole in G'. This follows from the next two claims.

Claim 1: If H^* is a clean smallest even hole and $\mathcal{C}_G(H^*)$ is Type bi node free, for $i = 1, 2, 3$, then any hole obtained from H^* through one special tent substitution is also clean.
Proof of Claim 1: Let xy be a special tent w.r.t. H^*, with intermediate paths P_1 and P_2, where P_1 is of length 2, and let H be the hole induced by the node set $V(P_2) \cup \{x, y\}$. W.l.o.g. assume that x is of Type g2 w.r.t. H^*, with neighbors x_1 and x_2 in H^*, and node y has a unique neighbor y_1 in H^*. Let p_1 be the intermediate node of P_1, and w.l.o.g. let x_2 and y_1 be the endnodes of P_1. We will show that the strongly adjacent nodes to H are of Type g2 or g3.

Suppose not and let u be a strongly adjacent node to H that is not of Type g2 or g3. Then u must have at least one neighbor in P_2. Let u_1 be the neighbor of u in P_2 that is closest to x_1, and let P' be the x_1u_1-subpath of P_2. Since H^* is clean, u is either not strongly adjacent to H^* or is of Type g2 or g3 w.r.t. H^*. Also u must be adjacent to a node in $\{x, y\}$, so we have the following three cases to consider.
Case 1: Node u is adjacent to both x and y.

First assume that u is adjacent to y_1. Then u must have at least two neighbors in P_2, since otherwise u is of Type g3 w.r.t. H. If u has two neighbors in P_2 then u_1 is adjacent to y_1 and (H, u) is a short 4-wheel. If u has three neighbors in P_2 then it is of Type g3 w.r.t. H^* and the hole induced by the node set $V(P') \cup \{x, u\}$ is even of length smaller than H^*, contradicting our choice of H^*. Hence u is not adjacent to y_1. By a similar argument u is not adjacent to x_1 either. Since u must have a neighbor in P_2 and since it is either not strongly adjacent to H^* or it is of Type g2 or g3 w.r.t. H^*, this implies that u does not have any neighbors in P_1. Node u_1 is not adjacent to y_1, since otherwise u, y, y_1, u_1, u is a 4-hole. Let H' be the hole induced by the node set $V(P') \cup V(P_1) \cup \{y, u\}$. But now (H', x) is a short 4-wheel.
Case 2: Node u is adjacent to x but not to y.

Node u is not adjacent to y_1, since otherwise u, x, y, y_1, u is a 4-hole. If u is adjacent to x_1 then u is of Type g3 w.r.t. H^*, with all neighbors in P_2. But then (H, u) is a short 4-wheel. Hence u is not adjacent to x_1 nor y_1, which implies that it cannot have any neighbors in P_1. But now there is a short $3PC(x, y_1)$, where two of the paths are x, P_1, y_1 and x, y, y_1 and the third path passes through u.

Case 3: Node u is adjacent to y but not to x.

Node u is not adjacent to x_1, since otherwise u, y, x, x_1, u is a 4-hole. If u is adjacent to y_1 then u is of Type g3 w.r.t. H^*, with all neighbors in P_2. But then (H, u) is a short 4-wheel. Hence u is not adjacent to x_1 nor y_1, which implies that it cannot have any neighbors in P_1. If u is of Type g1 or g3 w.r.t. H^*, then u is of Type b1 or b3 w.r.t. H, contradicting the assumption that $\mathcal{C}_G(H^*)$ is Type b1 and b3 node free. Since H^* is clean, u must be of Type g2 w.r.t. H^*, contradicting Lemma 4.1 applied to H and u.

Claim 2: If H^* is a clean smallest even hole and $\mathcal{C}_G(H^*)$ is Type bi node free, for $i = 1, 2, 3$, then any hole obtained from H^* through one Type g3 node substitution is also clean.

Proof of Claim 2: Let x be a Type g3 node w.r.t. H^*, with neighbors x_1, x_2 and x_3 in H^*. Assume that x_2 is the middle neighbor of x in H^* and let H be the hole obtained from H^* by substituting x for x_2. We will show that the strongly adjacent nodes to H are of Type g2 or g3. Let u be a strongly adjacent node to H. We consider the following two cases.

Case 1: Node u is not adjacent to x.

Then u cannot be adjacent to both x_1 and x_3, since otherwise x, x_1, u, x_3, x is a 4-hole. Since u is strongly adjacent to H, it is also strongly adjacent to H^*. Since H^* is clean, u is of Type g2 or g3 w.r.t. H^*. But then, since u is not adjacent to both x_1 and x_3, u is of Type g2 or g3 w.r.t. H as well.

Case 2: Node u is adjacent to x.

If u is not adjacent to x_1 nor x_3 then it is also not adjacent to x_2, since otherwise u would be a bad strongly adjacent node w.r.t. H^*. By Lemma 4.1 applied to H and u, node u cannot be of Type g2 w.r.t. H^*, and hence it is of Type g1 or g3 w.r.t. H^*. But then u is of Type b1 or b3 w.r.t. H, contradicting the assumption that $\mathcal{C}_G(H^*)$ is Type b1 and b3 node free. Therefore u must be adjacent to x_1 or x_3.

First assume that u is adjacent to both x_1 and x_3. Then u must also be adjacent to x_2, since otherwise u, x_1, x_2, x_3, u is a 4-hole. Since H^* is clean, u is of Type g3 w.r.t. H^* and hence w.r.t. H as well.

Now assume that u is adjacent to x_1 but not to x_3. Note that since H^* is clean, u can have at most three neighbors in $V(H^*) \setminus \{x_2\}$. If u has two neighbors in $V(H^*) \setminus \{x_2\}$, then u is of Type g2 or g3 w.r.t. H^* and hence of Type g3 w.r.t. H. If u has three neighbors in $V(H^*) \setminus \{x_2\}$, then (H, u) is a short 4-wheel. This completes the proof of Claim 2 and of the theorem. □

CLEANING ALGORITHM

Input: A graph G that does not contain a 4-hole, a 6-hole, a short 4-wheel nor a short 3PC.

Output: A family $\mathcal{L}$ of induced subgraphs of G such that, if G contains an even hole, then some $G' \in \mathcal{L}$ contains a spotless smallest even hole.

Step 1: Set $\mathcal{L} = \{G\}$.

Step 2: Apply Procedure BAD to G and let $\mathcal{L}'$ be the resulting output family.

Step 3: Apply Procedure BAD to each graph in $\mathcal{L}'$ and union the output with $\mathcal{L}$.

Step 4: Apply Procedure **b4** to each of the graphs in $\mathcal{L}'$ and union the output with $\mathcal{L}$.

If G contains an even hole then, after Step 2, $\mathcal{L}'$ contains a graph G' with a smallest even hole H such that $\mathcal{C}_{G'}(H)$ is Type b2 node free. Now, if H has a Type b1 or b3 node in G', we get the desired output in $\mathcal{L}$ after Step 3 and otherwise we get it after Step 4. So the Cleaning Algorithm produces the desired output. The size of the output can be estimated to be $O(n^{25})$.

5 2-Join Decompositions

In this section, we assume that G does not contain a 4-hole, a dominated node, a gem nor a full k-star cutset, $k = 1, 2, 3$. So, by Lemma 1.14, G contains no k-star cutset.

Let $V_1|V_2$ be a 2-join with special sets (A_1, A_2, B_1, B_2). For $i = 1, 2$, let $\mathcal{P}_i$ be the family of chordless paths $P = x_1, \ldots, x_n$ where $x_1 \in A_i$, $x_n \in B_i$ and $x_j \in V_i \setminus (A_i \cup B_i)$, $2 \leq j \leq n-1$.

Lemma 5.1 *The sets $\mathcal{P}_i$ are nonempty and contain no path of length 1, for $i = 1, 2$.*

Proof: Let $u \in A_1$ and $v \in B_1$.

First, suppose that there is no path in V_1 from A_1 to B_1. Then, since $|V_1| > 2$, either $\{u\} \cup A_2$ or $\{v\} \cup B_2$ is a star cutset. Hence $\mathcal{P}_1 \neq \emptyset$. Similarly, $\mathcal{P}_2 \neq \emptyset$.

Now, if uv is an edge, then no node of A_2 can be adjacent to a node of B_2 (since G is 4-hole-free). As $\mathcal{P}_2 \neq \emptyset$, it follows that $V_2 \setminus (A_2 \cup B_2) \neq \emptyset$. But then $\{u, v\} \cup A_2 \cup B_2$ would be a double star cutset. $\square$

The *blocks* of a 2-join decomposition are graphs G_1 and G_2 defined as follows. Block G_1 consists of the subgraph of G induced by node set V_1 plus a *marker path* $P_2 = a_2, \ldots, b_2$ that is chordless and satisfies the following properties. Node a_2 is adjacent to all the nodes in A_1, node b_2 is adjacent to all the nodes in B_1 and these are the only adjacencies between P_2 and the nodes of V_1. Furthermore, let $Q \in \mathcal{P}_2$. The marker path P_2 has length 4 if Q has even length, and length 5 otherwise. Block G_2 is defined similarly.

Theorem 5.2 *Let G_1 and G_2 be the blocks of a 2-join decomposition of G. Then, G is even-hole-free if and only if G_1 and G_2 are even-hole-free.*

Proof: First assume that G_1 or G_2 has an even hole, say G_1 does. Replacing in G_1 the marker path P_2 by a path $Q \in \mathcal{P}_2$ of the same parity yields a graph G_1' that contains an even hole. Since G_1' is a subgraph of G, this hole is also an even hole of G.

Conversely, suppose that G contains an even hole. If $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) has paths of different parities then, clearly, G_2 (resp. G_1) has an even hole. If all the paths of $\mathcal{P}_1 \cup \mathcal{P}_2$ have the same parity, then both G_1 and G_2 have even holes. So, we may assume that all the paths of $\mathcal{P}_1$ are odd and all the paths of $\mathcal{P}_2$ are even. But then each even hole H of G must be contained in $V_1 \cup A_2 \cup B_2$ or $V_2 \cup A_1 \cup B_1$. Hence H belongs either to G_1 or G_2. $\square$

Lemma 5.3 *If G does not contain a full k-star cutset, $k = 1, 2, 3$, then neither do the blocks of a 2-join decomposition of G.*

Proof: Let G_1 and G_2 be the blocks of a 2-join decomposition of G and suppose that one of them, say G_1, contains a full k-star cutset S, $k = 1, 2, 3$. We will obtain a contradiction by showing that this implies that G also contains a full k-star cutset. We consider the following three cases.

Case 1: $S = N[x]$

If x is not a node of the marker path P_2, then S is also a cutset in G. First assume that x coincides with a_2 or b_2, say $x = a_2$. Since P_2 is not an edge, the nodes of B_1 are all contained in the same component of $G_1 \setminus S$. Let u be a node of $G_1 \setminus S$ that is not in the same component as B_1. But then $N(a) \cup \{a\}$, where $a \in A_2$, is a full star cutset in G breaking u from B_1. Now assume that x is an intermediate node of P_2. Note that the graph induced by the node set $V_1 \cup \{a_2, b_2\}$ is connected since otherwise G would have a star cutset. Hence x is adjacent to a_2 or b_2, say a_2. Let $u \in A_1$ and $v \in B_1$ be the endnodes of a path in $\mathcal{P}_1$. Since P_2 is of length greater than 2, the nodes of $B_1 \cup \{u\}$ are all contained in the same component of $G_1 \setminus S$. Let y be a node of $G_1 \setminus S$ that is not in the same component as B_1. Then $N(u) \cup \{u\}$ is a full star cutset in G breaking y from v.

Case 2: $S = N(x) \cup N(y)$

If P_2 contains neither x nor y, then S is also a cutset in G. If P_2 contains both x and y, then since P_2 is of length greater than 3, either $N(x) \cup \{x\}$ or $N(y) \cup \{y\}$ is a full star cutset in G_1, and we are done by Case 1. So assume w.l.o.g. that $x = a_2$ and $y \in A_1$. Let u be a node of A_2. Then $N(u) \cup N(y)$ is a full double star cutset in G.

Case 3: $S = N(x) \cup N(y) \cup N(z)$

If P_2 does not contain a node in $\{x, y, z\}$, then S is also a cutset in G. So w.l.o.g. assume that $x = a_2$ and $y, z \in A_1$. But then $N(x) \cup N(y)$ is a full double star cutset in G. $\square$

We now present an algorithm that decomposes a graph using 2-joins.

Remark 5.4 *In [8], a set of forcing rules is given that decides in polytime whether a pair of edges a_1a_2 and b_1b_2 belong to a 2-join with special sets (A_1, A_2, B_1, B_2) such that for $i = 1, 2$ $a_i \in A_i$ and $b_i \in B_i$. The algorithm either outputs such a 2-join or it concludes that no such 2-join exists. We outline here this algorithm for the sake of completeness. As pointed out to us by Jim Geelen and Paul Seymour, these forcing rules can be formulated as a 2-SAT problem, thus providing an alternate, and elegant, proof that a 2-join can be found in polytime.*

Let a_1, a_2, b_1, b_2, u be five distinct nodes such that a_1a_2 and b_1b_2 are edges but neither a_1b_2 nor a_2b_1 is an edge and u is adjacent to at most one of the nodes a_2, b_2 (possibly none). The following rules yield a 2-join $V_1|V_2$ with $a_1, b_1, u \in V_1$ and $a_2, b_2 \in V_2$ or show that no such 2-join exists.

During the algorithm, the nodes h in V_1 are partitioned into three sets:

- Node h belongs to A_1 if it is adjacent to a_2 but not b_2,
- Node h belongs to B_1 if it is adjacent to b_2 but not a_2,
- Node h belongs to S_1 if it is adjacent to neither a_2 nor b_2.

The case where some node h in V_1 is adjacent to both a_2 and b_2 will not be permitted.

Initially, a_1, b_1, u are in V_1 and all the other nodes of G are in V_2. Forcing rules are used to move nodes from V_2 to V_1 as follows.

- If $v \in V_2$ is adjacent to at least one node in S_1, add v to V_1 and delete it from V_2,
- If $v \in V_2$ is adjacent to at least one node in $A_1 \cup B_1$ and $N(v) \cap (A_1 \cup B_1) \neq A_1$ or B_1, then add v to V_1 and delete it from V_2.

If some node v moved from V_2 to V_1 is adjacent to both a_2 and b_2, then the algorithm terminates since no 2-join with $a_1, b_1, u \in V_1$ and $a_2, b_2 \in V_2$ exists. If this situation never occurs, we continue moving nodes from V_2 to V_1 until no forcing rule applies. At this stage, denote by A_2 the nodes of V_2 adjacent to A_1, by B_2 those adjacent to B_2 and by S_2 the rest. The only adjacencies between nodes of V_1 and V_2 are between node sets A_1, A_2 and between B_1, B_2. There are three possibilities.

- If $|V_2| = 2$ or if $|A_2| = |B_2| = 1$ and V_2 induces a path, then no 2-join exists with $a_1, b_1, u \in V_1$ and $a_2, b_2 \in V_2$.
- If the first case does not occur and if $|A_1| \geq 2$ or $|B_1| \geq 2$ or $|A_1| = |B_1| = 1$ but V_1 does not induce a path, then $V_1|V_2$ is a 2-join with special sets (A_1, A_2, B_1, B_2).
- Finally, when neither of the above two cases occur, then $|A_1| = |B_1| = 1$ and V_1 induces a path. For each $h \in V_2$, move h from V_2 to V_1 and use the above forcing rules to find a 2-join with $a_1, b_1, u, h \in V_1$ and $a_2, b_2 \in V_2$. If this fails for all $h \in V_2$, then no 2-join exists with $a_1, b_1, u \in V_1$ and $a_2, b_2 \in V_2$.

Remark 5.5 *Constructing blocks of a 2-join decomposition can be done in polynomial time.*

By Remarks 5.4 and 5.5, one can see that every step of the following algorithm can be implemented to run in polynomial time.

2-JOIN DECOMPOSITION ALGORITHM

Input: A graph G that does not contain a 4-hole, a gem, a full k-star cutset, $k = 1, 2, 3$, nor any dominated nodes.

Output: A list $\mathcal{L}$ of graphs, with the following properties:

- The graphs in $\mathcal{L}$ do not contain a 4-hole, a gem, a full k-star cutset, $k = 1, 2, 3$, a 2-join nor any dominated nodes.
- G is even-hole-free if and only if all the graphs in $\mathcal{L}$ are even-hole-free.

Step 1: Let $\mathcal{L}' = \{G\}$ and $\mathcal{L} = \emptyset$.

Step 2: If $\mathcal{L}' = \emptyset$, stop. Otherwise, remove a graph F from $\mathcal{L}'$. Let $\mathcal{L}''$ be the set of all $\{\{a_1, b_1, u\}, \{a_2, b_2\}\}$ where a_1, b_1, a_2, b_2, u are five distinct nodes of F with the property that a_1b_1 and a_2b_2 are edges but not a_2b_1 nor a_1b_2, and node u is adjacent to at most one of the nodes a_2, b_2.

Step 3: If $\mathcal{L}'' = \emptyset$, add F to $\mathcal{L}$ and go to Step 2. Otherwise, remove $\{\{a_1, b_1, u\}, \{a_2, b_2\}\}$ from $\mathcal{L}''$.

Step 4: Check whether there is a 2-join $V_1|V_2$ with special sets (A_1, A_2, B_1, B_2) such that $u \in V_1$, for $i = 1, 2$, $a_i \in A_i$ and $b_i \in B_i$. If there is such a 2-join, go to Step 5. Otherwise, go to Step 3.

Step 5: Construct the blocks of the 2-join decomposition, add them to $\mathcal{L}'$ and go to Step 2.

Remark 5.6 *The number of graphs in list $\mathcal{L}$ produced by the 2-Join Decomposition Algorithm is $\mathcal{O}(|V(G)|)$. This is easily seen by observing that in each 2-join decomposition, the sum of the number of nodes in the two blocks is at most 12 more than the number of nodes in the original graph. If we stop doing 2-join decompositions when the size of the blocks is smaller than 24, then the number of blocks created is only linear in the number of nodes in the original graph.*

Lemma 5.7 *The 2-Join Decomposition Algorithm produces the desired output.*

Proof: By constructing blocks of a 2-join decomposition we do not create any gems, dominated nodes nor any 4-holes. So by Lemma 5.3, at every point in the algorithm the graphs in $\mathcal{L}'$ have the property that they do not contain a 4-hole, a gem, a full k-star cutset, $k = 1, 2, 3$, nor any dominated nodes. By the construction of $\mathcal{L}$, the graphs in $\mathcal{L}$ do not contain a 4-hole, a gem, a full k-star cutset, $k = 1, 2, 3$, a 2-join nor any dominated nodes. Furthermore, by Theorem 5.2, G is even-hole-free if and only if all the graphs in $\mathcal{L}$ are even-hole-free. □

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