On Padberg's conjecture about almost totally unimodular matrices

Gérard Cornuéjols* Luis F. Zuluaga†

July 6, 2000

Abstract

We consider Padberg's conjecture about almost totally unimodular matrices proposed in 1988 in *Operations Research Letters*. We show that it is not correct as stated and give a modified version of the conjecture.

1 Padberg's Conjecture

A matrix C is *totally unimodular* if $\det(C') \in \{0, \pm 1\}$ for every square submatrix C' of C. In particular, every entry of a totally unimodular matrix is $0, \pm 1$. A matrix C is *almost totally unimodular* if it is not totally unimodular but every proper submatrix of C is totally unimodular. Clearly, almost totally unimodular matrices are square and Gomory showed (cited in [1]) that if $C \in \{0, \pm 1\}^{n \times n}$ is almost totally unimodular then $\det(C) = \pm 2$.

A $0, \pm 1$ matrix with exactly two nonzero entries in each row and column, such that no proper submatrix has this property is called a *hole matrix*. It is an *unbalanced hole matrix* if the sum of the entries is $2 (mod\ 4)$. One can easily verify that unbalanced hole matrices are almost totally unimodular.

A matrix R such that RP is totally unimodular for every totally unimodular matrix P is called a *transformation that preserves total unimodularity*. Padberg's conjecture (p. 175 in [3]) on almost totally unimodular matrices is the following.

Conjecture 1 *Given any almost totally unimodular matrix A, there exists a transformation R that preserves total unimodularity such that RA is an unbalanced hole matrix.*

*Graduate School of Industrial Administration, Carnegie Mellon University, Schenley Park Pittsburgh, PA 15213 (gc0v@andrew.cmu.edu). Research supported by NSF grant DMI-9802773 and ONR grant N00014-97-1-0196

†Graduate School of Industrial Administration, Carnegie Mellon University, Schenley Park Pittsburgh, PA 15213 (lzuluaga@andrew.cmu.edu). Research supported by the William Larimer Mellon Fellowship.

2 Counterexample and a New Conjecture

We first show that Conjecture 1 is not correct as stated. Suppose A, R and an unbalanced hole matrix H satisfy Conjecture 1, i.e. $RA = H$. Since both A and H are square nonsingular matrices, then R is nonsingular. Also R is totally unimodular since R preserves total unimodularity. Now, R has at most one nonzero entry per row and column; as otherwise, if $R_{ij} \neq 0$ and $R_{ik} \neq 0$ for some row i and $j \neq k$, let P be the matrix containing only two nonzero entries $P_{j1} = R_{ij}$ and $P_{k1} = R_{ik}$. Then P is totally unimodular and $[RP]_{i1} = 2$. So we can assume (w.l.o.g.) that R is a permutation matrix. It follows that RA is obtained from A by permuting its rows. Therefore any almost totally unimodular matrix A containing a row with more than two nonzero entries contradicts Conjecture 1. For example, take the following almost totally unimodular matrix [2]

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Box \tag{1}$$

Now we state the following modified conjecture.

Conjecture 2 *Given any almost totally unimodular matrix A, there exists a totally unimodular matrix R such that RA is an unbalanced hole matrix.*

For example, a totally unimodular matrix R for the almost totally unimodular matrix A in (1) is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where} \quad RA = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Another example where Conjecture 2 holds is given by the following equation.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where the almost totally unimodular matrix is taken from [2]. In the examples above we have transformed an almost totally unimodular matrix into an unbalanced hole matrix of the form

$$\begin{bmatrix} \alpha_1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_3 & \alpha_4 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{2n-3} & \alpha_{2n-2} \\ \alpha_{2n} & 0 & \cdots & 0 & \alpha_{2n-1} \end{bmatrix} \tag{2}$$

where $\alpha_i \in \{+1, -1\}$ for $i = 1, \dots, 2n$ and $\sum_{i=1}^{2n} \alpha_i = 2(mod\ 4)$. It is worth noticing that for the almost totally unimodular matrix

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3}$$

there is no totally unimodular matrix R such that RA is an unbalanced hole matrix of the form (2). However,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} A' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

which shows that Conjecture 2 holds for the almost totally unimodular matrix in (3).We have verified Conjecture 2 on several other examples including infinite classes of almost totally unimodular matrices.

References

[1] P. Camion, *Characterization of totally unimodular matrices*, Proceedings of the American Mathematical Society, 16 (1965), pp. 1068–1073.

[2] G. Cornuéjols, *Combinatorial Optimization: Packing and Covering*, SIAM monograph, to be published.

[3] M. Padberg, *Total unimodularity and the Euler-subgraph problem*, Operations Research Letters, 7 (1988), pp. 173–179.