The Strong Perfect Graph Conjecture

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June 2002

survey prepared for the
International Congress of Mathematicians, Beijing, China 2002

Abstract

A graph is perfect if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. Examples of perfect graphs include bipartite graphs, line graphs of bipartite graphs and the complements of such graphs. These four classes of perfect graphs will be called basic. In 1960, Berge formulated two conjectures about perfect graphs, one stronger than the other. The weak perfect graph conjecture, which states that a graph is perfect if and only if its complement is perfect, was proved in 1972 by Lovász. This result is now known as the perfect graph theorem. The strong perfect graph conjecture (SPGC) states that a graph is perfect if and only if it does not contain an odd hole or its complement. The SPGC has attracted a lot of attention. It was proved recently (May 2002) in a remarkable sequence of results by Chudnovsky, Robertson, Seymour and Thomas. The proof is difficult and, as of this writing, they are still checking the details. Here we give a flavor of the proof. Let us call Berge graph a graph that does not contain an odd hole or its complement. Conforti, Cornuéjols, Robertson, Seymour, Thomas and Vušković (2001) conjectured a structural property of Berge graphs that implies the SPGC: Every Berge graph $G$ is basic or has a skew partition or a homogeneous pair, or $G$ or its complement has a 2-join. A skew partition is a partition of the vertices into nonempty sets $A, B, C, D$ such that every vertex of $A$ is adjacent to every vertex of $B$ and there is no edge between $C$ and $D$. Chvátal introduced this concept in 1985 and conjectured that no minimally imperfect graph has a skew partition. This conjecture was proved recently by Chudnovsky and Seymour (May 2002). Cornuéjols and Cunningham introduced 2-joins in 1985 and showed that they cannot occur in a minimally imperfect graph different from an odd hole. Homogeneous pairs were introduced in 1987 by Chvátal and Sbihi, who proved that they cannot occur in minimally imperfect graphs. Since skew partitions, 2-joins and homogeneous pairs cannot occur in minimally imperfect Berge graphs, the structural property of Berge graphs stated above implies the SPGC. This structural property was proved: (i) When $G$ contains the line graph of a bipartite subdivision of a 3-connected graph (Chudnovsky, Robertson, Seymour and Thomas (September 2001)); (ii) When $G$ contains a stretcher (Chudnovsky and Seymour (January 2002)); (iii) When $G$ contains no proper wheels, stretchers or their complements (Conforti, Cornuéjols and Zambelli (May 2002)); (iv) When $G$ contains a proper wheel, but no stretchers or their complements (Chudnovsky and Seymour (May 2002)). (ii), (iii) and (iv) prove the SPGC.

2000 Mathematics Subject Classification: 05C17

Keywords: perfect graph, odd hole, strong perfect graph conjecture, strong perfect graph theorem, Berge graph, decomposition, 2-join, skew partition, homogeneous pair.

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This work was supported in part by NSF grant DMI-0098427 and ONR grant N00014-97-1-0196.
1 Introduction

In this paper, all graphs are simple (no loops or multiple edges) and finite. The vertex set of graph $G$ is denoted by $V(G)$ and its edge set by $E(G)$. A stable set is a set of vertices no two of which are adjacent. A clique is a set of vertices every pair of which are adjacent. The cardinality of a largest clique in graph $G$ is denoted by $\omega(G)$. The cardinality of a largest stable set is denoted by $\alpha(G)$. A $k$-coloring is a partition of the vertices into $k$ stable sets (these stable sets are called color classes). The chromatic number $\chi(G)$ is the smallest value of $k$ for which there exists a $k$-coloring. Obviously, $\omega(G) \leq \chi(G)$ since the vertices of a clique must be in distinct color classes of the $k$-coloring. An induced subgraph of $G$ is a graph with vertex set $S \subseteq V(G)$ and edge set comprising all the edges of $G$ with both ends in $S$. It is denoted by $G(S)$. The graph $G(V(G) - S)$ is denoted by $G \setminus S$. A graph $G$ is perfect if $\omega(H) = \chi(H)$ for every induced subgraphs $H$ of $G$. A graph is minimally imperfect if it is not perfect but all its proper induced subgraphs are.

A hole is a graph induced by a chordless cycle of length at least 4. A hole is odd if it contains an odd number of vertices. Odd holes are not perfect since their chromatic number is 3 whereas the size of their largest clique is 2. It is easy to check that odd holes are minimally imperfect. The complement of a graph $G$ is the graph with the same vertex set as $G$, and $\bar{G}$ is an edge of $G$ if and only if it is not an edge of $G$. The odd holes and their complements are the only known minimally imperfect graphs. In 1960 Berge [3] proposed the following conjecture, known as the Strong Perfect Graph Conjecture.

Conjecture 1.1 (Strong Perfect Graph Conjecture) (Berge [3]) The only minimally imperfect graphs are the odd holes and their complements.

At the same time, Berge also made a weaker conjecture, which states that a graph $G$ is perfect if and only if its complement $\bar{G}$ is perfect. This conjecture was proved by Lovász [29] in 1972 and is known as the Perfect Graph Theorem.

Theorem 1.2 (Perfect Graph Theorem) (Lovász [29]) Graph $G$ is perfect if and only if graph $G$ is perfect.

Proof: Lovász [30] proved the following stronger result.

Claim 1: A graph $G$ is perfect if and only if, for every induced subgraph $H$, the number of vertices of $H$ is at most $\alpha(H) \omega(H)$.

Since $\alpha(H) = \omega(H)$ and $\omega(H) = \alpha(H)$, Claim 1 implies Theorem 1.2.

Proof of Claim 1: We give a proof of this result due to Gasparian [25]. First assume that $G$ is perfect. Then, for every induced subgraph $H$, $\omega(H) = \chi(H)$. Since the number of vertices of $H$ is at most $\alpha(H) \chi(H)$, the inequality follows.

Conversely, assume that $G$ is not perfect. Let $H$ be a minimally imperfect subgraph of $G$ and let $n$ be the number of vertices of $H$. Let $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then $H$ satisfies

$$\omega = \chi(H \setminus v)$$

for every vertex $v \in V(H)$

and $\omega = \omega(H \setminus S)$ for every stable set $S \subseteq V(H)$.

Let $A_0$ be an $\alpha$-stable set of $H$. Fix an $\omega$-coloring of each of the $\alpha$ graphs $H \setminus s$ for $s \in A_0$, let $A_1, \ldots, A_{\omega}$ be the stable sets occurring as a color class in one of these colorings and let $A := \{A_0, A_1, \ldots, A_{\omega}\}$. Let $A$ be the corresponding stable set versus vertex incidence matrix. Define $B := (B_0, B_1, \ldots, B_{\omega})$ where $B_i$ is an $\omega$-clique of $H \setminus A_i$. Let $B$ be the corresponding clique versus vertex incidence matrix.

Claim 2: Every $\omega$-clique of $H$ intersects all but one of the stable sets in $A$.

Proof of Claim 2: Let $S_1, \ldots, S_\omega$ be any $\omega$-coloring of $H \setminus v$. Since any $\omega$-clique $C$ of $H$ has at most one vertex in each $S_i$, $C$ intersects all $S_i$’s if $v \not\in C$ and all but one if $v \in C$. Since $C$ has at most one vertex in $A_0$, Claim 9 follows.

In particular, it follows that $AB^T - J - I$. Since $J - I$ is nonsingular, $A$ and $B$ have at least as many columns as rows, that is $n \geq \omega + 1$. This completes the proof of Claim 1. \qed
2 Four Basic Classes of Perfect Graphs

Bipartite graphs are perfect since, for any induced subgraph \( H \), the bipartition implies that \( \chi(H) \leq 2 \) and therefore \( \omega(H) = \chi(H) \).

A graph \( L \) is the line graph of a graph \( G \) if \( V(L) = E(G) \) and two vertices of \( L \) are adjacent if and only if the corresponding edges of \( G \) are adjacent.

**Proposition 2.1** Line graphs of bipartite graphs are perfect.

**Proof:** If \( G \) is bipartite, \( \chi'(G) = \Delta(G) \) by a theorem of König [28], where \( \chi' \) denotes the edge-chromatic number and \( \Delta \) the largest vertex degree.

If \( L \) is the line graph of a bipartite graph \( G \), then \( \chi(L) = \chi'(G) \) and \( \omega(L) = \Delta(G) \). Therefore \( \chi(L) = \omega(L) \). Since induced subgraphs of \( L \) are also line graphs of bipartite graphs, the result follows. □

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász’s perfect graph theorem (Theorem 1.2) that the complements of bipartite graphs and of line graphs of bipartite graphs are perfect. This can also be verified directly, without using the perfect graph theorem. To summarize, in this section we have introduced four basic classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

3 2-Join

A graph \( G \) has a 2-join if its vertices can be partitioned into sets \( V_1 \) and \( V_2 \), each of cardinality at least three, with nonempty disjoint subsets \( A_1, B_1 \subseteq V_1 \) and \( A_2, B_2 \subseteq V_2 \), such that all the vertices of \( A_1 \) are adjacent to all the vertices of \( A_2 \), all the vertices of \( B_1 \) are adjacent to all the vertices of \( B_2 \) and these are the only adjacencies between \( V_1 \) and \( V_2 \). There is an \( O(|V(G)|^3|E(G)|^2) \) algorithm to find whether a graph \( G \) has a 2-join [23].

When \( G \) contains a 2-join, we can decompose \( G \) into two blocks \( G_1 \) and \( G_2 \) defined as follows.

**Definition 3.1** If \( A_2 \) and \( B_2 \) are in different connected components of \( G(V_2) \), define block \( G_1 \) to be \( G(V_1 \cup \{p_1, q_1\}) \), where \( p_1 \in A_2 \) and \( q_1 \in B_2 \). Otherwise, let \( P_1 \) be a shortest path from \( A_2 \) to \( B_2 \) and define block \( G_1 \) to be \( G(V_1 \cup V(P_1)) \). Block \( G_2 \) is defined similarly.

Next we show that the 2-join decomposition preserves perfection (Cornuérjols and Cunningham [23]; see also Kapoor [27] Chapter 8). Earlier, Bixby [4] had shown that the simpler join decomposition preserves perfection.

**Theorem 3.2** Graph \( G \) is perfect if and only if its blocks \( G_1 \) and \( G_2 \) are perfect.

**Proof:** By definition, \( G_1 \) and \( G_2 \) are induced subgraphs of \( G \). It follows that, if \( G \) is perfect, so are \( G_1 \) and \( G_2 \). Now we prove the converse: If \( G_1 \) and \( G_2 \) are perfect, then so is \( G \). Let \( G^* \) be an induced subgraph of \( G \). We must show

\[
(*) \quad \omega(G^*) = \chi(G^*).
\]

For \( i = 1, 2 \), let \( V_i^* = V_i \cap V(G^*) \). The proof of (*) is based on a coloring argument, combining \( \omega(G^*) \)-colorings of the perfect graphs \( G(V_1^*) \) and \( G(V_2^*) \) (Claim 3) into an \( \omega(G^*) \)-coloring of \( G^* \) (Claim 4). To prove Claim 3, we will use the following results.

**Claim 1:** (Lovász’s Replication Lemma [29]) Let \( \Gamma \) be a perfect graph and \( v \in V(\Gamma) \). Create a new vertex \( v' \) adjacent to \( v \) and to all the neighbors of \( v \). Then the resulting graph \( \Gamma' \) is perfect.
Proof of Claim 1: It suffices to show that \( \omega(\Gamma') = \chi(\Gamma')\) since, for induced subgraphs, the proof follows similarly. We distinguish two cases. Suppose first that \( v \) is contained in some \( \omega(\Gamma')\)-clique of \( \Gamma \). Then \( \omega(\Gamma') = \omega(\Gamma) + 1 \). Since at most one new color is needed in \( \Gamma' \), \( \omega(\Gamma') = \chi(\Gamma') \) follows.

Now suppose that \( v \) is not contained in any \( \omega(\Gamma')\)-clique of \( \Gamma \). Consider any \( \omega(\Gamma')\)-coloring of \( \Gamma \) and let \( A \) be the color class containing \( v \). Then, \( \omega(\Gamma' \setminus (A - \{ v \})) = \omega(\Gamma) - 1 \), since every \( \omega(\Gamma')\)-clique of \( \Gamma \) meets \( A - \{ v \} \). By the perfection of \( \Gamma \), the graph \( \Gamma' \setminus (A - \{ v \}) \) can be colored with \( \omega(\Gamma) - 1 \) colors. Using one additional color for the vertices \( (A - \{ v \}) \cup \{ v' \} \), we obtain an \( \omega(\Gamma')\)-coloring of \( \Gamma' \). This proves Claim 1.

We say that \( \Gamma' \) is obtained from \( \Gamma \) by replicating \( v \). Replication can be applied recursively. We say that \( v \) is replicated \( k \) times if \( k \) copies of \( v \) are made, including \( v \).

Claim 2: Let \( \Gamma \) be a graph and \( uv \) an edge of \( \Gamma \) such that the vertices \( u \) and \( v \) have no common neighbor. Let \( \Gamma' \) be the graph obtained from \( \Gamma \) by replicating vertex \( v \) into \( v' \). Let \( H \) be the graph obtained from \( \Gamma' \) by deleting edge \( uv' \). Then \( \Gamma \) is perfect if and only if \( H \) is perfect.

Proof of Claim 2: If \( H \) is perfect, then so is \( \Gamma \) since \( \Gamma \) is an induced subgraph of \( H \).

Conversely, suppose that \( \Gamma \) is perfect and \( H \) is not. Let \( \Gamma^* \) be a minimally imperfect subgraph of \( H \). Let \( \Gamma'^* \) be the subgraph of \( \Gamma' \) induced by the vertices of \( \Gamma^* \). Since \( \Gamma^* \) is perfect but \( H^* \) is not, \( V(H^*) \) must contain vertices \( u \) and \( v' \). Also \( \chi(\Gamma^*) = \chi(\Gamma') \) and \( \omega(\Gamma^*) = \omega(\Gamma') + 1 \). Therefore \( uv' \) is the unique maximum clique in \( \Gamma^* \) and \( \omega(H^*) = 2 \). The only neighbor of \( v \) in \( H^* \) is \( u \) since otherwise \( v, v' \) would be in a clique of cardinality three in \( H^* \). Now \( v' \) is a vertex of degree 1 in \( H^* \), a contradiction to the assumption that \( H^* \) is minimally imperfect. This proves Claim 2.

For \( i = 1, 2 \), let \( A_i^* = A_i \cap V(G^*) \), \( B_i^* = B_i \cap V(G^*) \), \( a_i = \omega(A_i^*) \) and \( b_i = \omega(B_i^*) \). Let \( G^*_i = G_i \setminus (V_i - V^*_i) \) and \( \omega \geq \omega(G^*_i) \). In an \( \omega \)-coloring of \( G_i^* \), let \( C(A_i^*) \) and \( C(B_i^*) \) denote the sets of colors in \( A_i^* \) and \( B_i^* \) respectively.

Claim 3: There exists an \( \omega \)-coloring of \( V_i^* \) such that \( |C(A_i^*)| = a_i \) and \( |C(B_i^*)| = b_i \). Furthermore, if \( G_i \) contains path \( P_i \) and

(i) if \( P_i \) has an odd number of edges, then \( |C(A_i^*) \cap C(B_i^*)| = \max(0, a_i + b_i - \omega) \),

(ii) if \( P_i \) has an even number of edges, then \( |C(A_i^*) \cap C(B_i^*)| = \min(a_i, b_i) \).

Proof of Claim 3: First assume that block \( G_i \) is induced by \( V_i \cup \{ p_i, q_i \} \). In \( G_i^* \), replicate \( p_i \) \( \omega \)-\( a_i \) times and \( q_i \) \( \omega - b_i \) times. By Claim 1, this new graph \( H \) is perfect and \( \omega(H) = \omega \). Therefore an \( \omega \)-coloring of \( H \) exists. This coloring induces an \( \omega \)-coloring of \( V_i^* \) with \( |C(A_i^*)| = a_i \) and \( |C(B_i^*)| = b_i \). Now assume that \( G_i \) contains path \( P_i \). We consider two cases.

(i) \( P_i \) has an odd number of edges.

Let \( P_i = x_1, \ldots, x_{2k} \). In \( G_i^* \), replicate vertex \( x_{2k} \) into \( x'_{2k} \) and remove edge \( x_{2k-1}x'_{2k} \). By Claim 2, the new graph is perfect. For \( i \) odd, \( 1 \leq i < 2k \), replicate vertex \( x_i \) \( \omega - a_i \) times. For \( i \) even, \( 1 < i \leq 2k - 2 \), replicate vertex \( x_i \) \( a_i \) times. If \( a_i + b_i < \omega \), replicate \( x_{2k} \) \( a_i \) times and replicate \( x'_{2k} \) \( \omega - a_i \) times. By Claim 1, this new graph \( H \) is perfect. Since \( \omega(H) = \omega \), \( H \) has an \( \omega \)-coloring. Note that \( |C(A_i^*)| = a_i \) and \( |C(B_i^*)| = b_i \) and every vertex of \( P_i \) belongs to two cliques of size \( \omega \). So the colors that appear in the replicates of \( x_{2k} \) are precisely \( C(A_i^*) \). Therefore \( B_i^* \) is colored with colors that do not appear in \( C(A_i^*) \). Thus \( |C(A_i^*) \cap C(B_i^*)| = 0 \).

If \( a_i + b_i \geq \omega \), replicate \( x_{2k} \) \( \omega - b_i \) times and remove \( x'_{2k} \). The new graph \( H \) is perfect and \( \omega(H) = \omega \). Therefore \( H \) has an \( \omega \)-coloring. Again \( |C(A_i^*)| = a_i \) and \( |C(B_i^*)| = b_i \), and the \( \omega - b_i \) colors that appear in the replicates of \( x_{2k} \) belong to \( C(A_i^*) \). Since these colors cannot appear in \( C(B_i^*) \), the number of common colors in \( C(A_i^*) \) and \( C(B_i^*) \) is \( a_i + b_i - \omega \).

(ii) \( P_i \) has an even number of edges.

Assume w.l.o.g. that \( a_i \leq b_i \). Let \( P_i = x_1, \ldots, x_{2k+1} \). In \( G_i^* \), replicate vertex \( x_i \) \( \omega - a_{i+1} \) times for \( i \) odd, \( 1 \leq i \leq 2k - 1 \), and replicate vertex \( x_i \) \( a_{i+1} \) times for \( i \) even, \( 1 < i \leq 2k \). Finally, replicate \( x_{2k+1} \) \( \omega - b_{i+1} \) times. By Claim 1, the new graph \( H \) is perfect and \( \omega(H) = \omega \). In an \( \omega \)-coloring of \( H \), \( |C(A_i^*)| = a_i \) and \( |C(B_i^*)| = b_i \) and the colors that appear in the replicates of \( x_{2k} \) are precisely \( C(A_i^*) \). But then these colors do not appear in the replicates of \( x_{2k+1} \) and consequently they must appear in \( C(B_i) \). Thus \( |C(A_i) \cap C(B_i)| = \min(a_i, b_i) \). This proves Claim 3.
Claim 4: $G^*$ has an $\omega(G^*)$-coloring.

Proof of Claim 4:
Let $\omega = \omega(G^*)$. Clearly, $\omega \geq a_1 + a_2$ and $\omega \geq b_1 + b_2$. To prove the claim, we will combine $\omega$-colorings of $V^*_1$ and $V^*_2$.

If at least one of the sets $A^*_1, A^*_2, B^*_1, B^*_2$ is empty, one can easily construct the desired $\omega$-coloring of $G^*$. So we assume now that these sets are nonempty. This implies that $\omega \geq \omega(A^*_1)$ and $\omega \geq \omega(A^*_2)$. By Claim 3, there exist $\omega$-colorings of $V^*_1$ such that $|C(A^*_1)| = a_1$ and $|C(B^*_1)| = b_1$. Thus, if $A^*_2$ and $B^*_2$ are in different connected components of $G(V^*_2)$, an $\omega$-coloring of $V^*_1$ can be combined with $\omega$-colorings of the components of $G(V^*_2)$ into an $\omega$-coloring of $G^*$. So we can assume that both $P_1$ and $P_2$ exist. Since $G_1$ contains no odd hole, every chordless path from $A_1$ to $B_1$ has the same parity as $P_1$. It follows from the definition of 2-join decomposition that $P_1$ and $P_2$ have the same parity.

(i) $P_1$ and $P_2$ both have an odd number of edges.
Then by Claim 3 (i), there exists an $\omega$-coloring of $V^*_1$ with $|C(A^*_1) \cap C(B^*_1)| = \max(0, a_1 + b_1 - \omega)$. In the coloring of $V^*_1$, label by 1 through $a_1$ the colors that occur in $A^*_1$ and by $\omega$ through $\omega - b_1 + 1$ the colors that occur in $B^*_1$. In the coloring of $V^*_2$, label by $\omega$ through $\omega - a_2 + 1$ the colors that occur in $A^*_2$ and by 1 through $b_2$ the colors that occur in $B^*_2$. If this is not an $\omega$-coloring of $G^*$, there must exist a common color in $A^*_1$ and $A^*_2$ or in $B^*_1$ and $B^*_2$. Then either $a_1 \geq \omega - a_2 + 1$ or $b_2 \geq \omega - b_1 + 1$, a contradiction.

(ii) $P_1$ and $P_2$ both have an even number of edges.
Then by Claim 3 (ii), there exists an $\omega$-coloring of $V^*_1$ with $|C(A^*_1) \cap C(B^*_1)| = \min(a_1, b_1)$. In the coloring of $V^*_1$, label by 1 through $a_1$ the colors that occur in $A^*_1$ and by 1 through $b_1$ the colors that occur in $B^*_1$. In the coloring of $V^*_2$, label by $\omega$ through $\omega - a_2 + 1$ the colors that occur in $A^*_2$ and by $\omega$ through $\omega - b_2 + 1$ the colors that occur in $B^*_2$. If this is not an $\omega$-coloring of $G^*$, there must exist a common color in $A^*_1$ and $A^*_2$ or in $B^*_1$ and $B^*_2$. Then either $a_1 \geq \omega - a_2 + 1$ or $b_1 \geq \omega - b_2 + 1$, a contradiction. \qed

Corollary 3.3 If a minimally imperfect graph $G$ has a 2-join, then $G$ is an odd hole.

Proof: Since $G$ is not perfect, Theorem 3.2 implies that block $G_1$ or $G_2$ is not perfect, say $G_1$. Since $G_1$ is an induced subgraph of $G$ and $G$ is minimally imperfect, it follows that $G = G_1$. Since $|V_2| \geq 3$, $V_2$ induces a chordless path. Thus $G$ is a minimally imperfect graph with a vertex of degree 2. This implies that $G$ is an odd hole [32]. \qed

We end this section with another decomposition that preserves perfection. A graph $G$ has a 6-join if $V(G)$ can be partitioned into eight nonempty sets $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$ with the property that, for any $x_i \in X_i$ $(i = 1, 2, 3)$ and $y_j \in Y_j$ $(j = 1, 2, 3)$, the graph induced by $x_1, y_1, x_2, y_2, x_3, y_3$ is a 6-hole and these kinds of edges are the only adjacencies between $X = X_1 \cup X_2 \cup X_3 \cup X_4$ and $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$.

Theorem 3.4 (Assev and Vuskovic [2]) No minimally imperfect graph contains a 6-join.

If $G$ contains a 6-join, define blocks $G_X$ and $G_Y$ as follows. $G_X$ is the graph induced by $X \cup \{y_1, y_2, y_3\}$ where $y_j \in Y_j$ $(j = 1, 2, 3)$. Similarly $G_Y$ is the graph induced by $Y \cup \{x_1, x_2, x_3\}$ where $x_i \in X_i$ $(i = 1, 2, 3)$. It can be shown [1] that $G$ is perfect if and only if its blocks $G_X$ and $G_Y$ are perfect.

4 Skew Partition and Homogeneous Pair

A graph has a skew partition if its vertices can be partitioned into four nonempty sets $A, B, C, D$ such that there are all the possible edges between $A$ and $B$ and no edges from $C$ to $D$. It is easy to verify that the odd holes and their complements do not have a skew partition. Chvátal [6] conjectured that no minimally imperfect graph has a skew partition.

Theorem 4.1 (Skew Partition Theorem) (Chudnovsky and Seymour [13]) No minimally imperfect graph has a skew partition.
Chudnovsky and Seymour obtained this result as a consequence of their proof of the SPGC. In order to prove the SPGC, they first proved the following weaker result.

**Theorem 4.2** (Chudnovsky and Seymour [12]) A minimally imperfect Berge graph with smallest number of vertices does not have a skew partition.

We do not give the proof of this difficult theorem here. Instead, we prove results due to Hoàng [26] on two special skew partitions called $T$-cutset and $U$-cutset respectively.

Assume that $G$ is a minimally imperfect graph with skew partition $A, B, C, D$. Let $a = \omega(A)$, $b = \omega(B)$, $\omega = \omega(G)$ and $\alpha = \alpha(G)$. The vertex sets $A \cup B \cup C$ and $A \cup B \cup D$ induce perfect graphs $G_1$ and $G_2$ respectively and both of these graphs contain an $\omega$-clique. Indeed, each vertex of a minimally imperfect graph belongs to $\omega$ $\omega$-cliques [32] and, for $u \in C$, these $\omega$-cliques are contained in $G_1$. For $u \in D$, they are contained in $G_2$.

**Lemma 4.3** (Hoàng [26]) Let $C_i$ be an $\omega$-coloring of $G_i$, for $i = 1, 2$. Then $C_1$ and $C_2$ cannot have the same number of colors in $A$.

**Proof:** Suppose $C_1$ and $C_2$ have the same number of colors in $A$ and assume w.l.o.g. that these colors are $1, 2, \ldots, k$. Let $K$ be the subgraph of $G$ induced by the vertices with colors $1, 2, \ldots, k$ and let $H = G \setminus K$. Since every $\omega$-clique of $G$ is in $G_1$ or $G_2$, the largest clique in $K$ has size $k$ and the largest clique in $H$ has size $\omega - k$. The graphs $H$ and $K$ are perfect since they are proper subgraphs of $G$. Color $K$ with $k$ colors and $H$ with $\omega - k$ colors. Now $G$ is colored with $\omega$ color, a contradiction to the assumption that $G$ is minimally imperfect. \qed

**Lemma 4.4** No $\omega$-clique is contained in $A \cup B$.

**Proof:** Suppose that a $\omega$-clique were contained in $A \cup B$. Then any $\omega$-coloring of $G_i$, for $i = 1, 2$, would contain $a$ colors in $A$ and $b = \omega - a$ colors in $B$, contradicting Lemma 4.3. \qed

**Lemma 4.5** Every $\alpha$-stable set intersects $A \cup B$.

**Proof:** By Lemma 4.4 applied to the complement graph, no $\alpha$-stable set is contained in $C \cup D$. \qed

**Lemma 4.6** If some $u \in A$ has no neighbor in $C$, then there exists an $\omega$-coloring of $G_1$ with $b$ colors in $D$.

**Proof:** Let $C_1$ be an $\omega$-coloring of $G_1$ with minimum number $k$ of colors in $B$ and suppose that this number is strictly greater than $b$. Consider the subgraph $H$ of $G_1$ induced by the vertices colored with the colors of $C_1$ that appear in $B$. The graph $H \cup u$ can be colored with $k$ colors since it is perfect and has no clique of size greater than $k$. Keeping the other colors of $C_1 \setminus (H \cup u)$, we get an $\omega$-coloring of $G_1$ with fewer colors on $B$ than $C_1$, a contradiction. \qed

**Lemma 4.7** If some $u \in A$ has no neighbor in $C$, then every vertex of $A$ has a neighbor in $D$ and every vertex of $B$ has a neighbor in $C$.

**Proof:** By Lemma 4.6, there exists an $\omega$-coloring of $G_1$ with $b$ colors in $B$. Thus, by Lemma 4.3, there exists no $\omega$-coloring of $G_2$ with $b$ colors in $B$. By Lemma 4.6, this implies that every vertex of $A$ has a neighbor in $D$.

Suppose that $v \in B$ has no neighbor in $C$. In the complement graph, $u$ and $v$ are adjacent to all the vertices of $C$. By Lemma 4.3, $|A| \geq 2$ and $|B| \geq 2$. So $A' = A - u$, $B' = B - v$, $C' = C$, $D' = D \cup \{u, v\}$ form a skew partition. But $u$ has no neighbor in $B$ and $v$ has no neighbor in $A$, contradicting the first part of the lemma. So every $v \in B$ has a neighbor in $C$. \qed

A $T$-cutset is a skew partition with $u \in C$ and $v \in D$ such that every vertex of $A$ is adjacent to both $u$ and $v$. 


Lemma 4.8 (Hoàng [26]) No minimally imperfect graph contains a T-cutset.

Proof: In the complement, u and v contradict Lemma 4.7.

A T-cutset is a skew partition with $u, v \in C$ such that every vertex of $A$ is adjacent to $u$ and every vertex of $B$ is adjacent to $v$.

Lemma 4.9 (Hoàng [26]) No minimally imperfect graph contains a U-cutset.

Proof: In the complement, $u$ and $v$ contradict Lemma 4.7.

We conclude this section with the notion of homogeneous pair introduced by Chvátal and Sbihi [8]. A graph $G$ has a homogeneous pair if $V(G)$ can be partitioned into subsets $A_1, A_2$ and $B$, such that:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$.
- If a node of $B$ is adjacent to a node of $A_1$ ($A_2$) then it is adjacent to all the nodes of $A_1$ ($A_2$).

Theorem 4.10 (Chvátal and Sbihi [8]) No minimally imperfect graph contains a homogeneous pair.

5 Decomposition of Berge Graphs

A graph is a Berge graph if it does not contain an odd hole or its complement. Clearly, all perfect graphs are Berge graphs. The SPGC states that the converse is also true.

Conjecture 5.1 (Conforti, Cornuéjols, Robertson, Seymour, Thomas and Vušković (2001)) (Decomposition Conjecture) Every Berge graph $G$ is basic or has a skew partition or a homogeneous pair, or $G$ or $\overline{G}$ has a 2-join.

This conjecture implies the SPGC. Indeed, suppose that the Decomposition Conjecture holds but not the SPGC. Then there exists a minimally imperfect graph $G$ distinct from an odd hole or its complement. Choose $G$ with the smallest number of vertices. $G$ is a Berge graph and it cannot have a skew partition by Theorem 4.2. $G$ cannot have an homogeneous pair by Theorem 4.10. Neither $G$ nor $\overline{G}$ can have a 2-join by Corollary 3.3. So $G$ must be basic by the Decomposition Conjecture. Therefore $G$ is perfect, a contradiction.

Note that there are other decompositions that cannot occur in minimally imperfect Berge graphs, such as 6-joins (Theorem 3.4) or universal 2-amalgams [15] (universal 2-amalgams generalize both 2-joins and homogeneous pairs). These decompositions could be added to the statement of Conjecture 5.1 while still implying the SPGC. However they do not appear to be needed. Paul Seymour commented that homogeneous pairs might not be necessary either. In fact, we had initially formulated Conjecture 5.1 without homogeneous pairs. I added them to the statement to be on the safe side since they currently come up in the proof of the SPGC (see below).

Several special cases of Conjecture 5.1 are known. For example, it holds when $G$ is a Meyniel graph (Burllet and Fonlupt [5] in 1984), when $G$ is claw-free (Chvátal and Sbihi [9] in 1988 and Maffray and Reed [31] in 1999), diamond-free (Fonlupt and Zemirline [24] in 1987), bull-free (Chvátal and Sbihi [8] in 1987), or dart-free (Chvátal, Fonlupt, Sun and Zemirline [7] in 2000). All these results involve special types of skew partitions (such as star cutsets) and, in some cases, homogeneous pairs [8]. A special case of 2-join called augmentation of a flat edge appears in [31]. In 1999, Conforti and Cornuéjols [14] used more general 2-joins to prove Conjecture 5.1 for WP-free Berge graphs, a class of graphs that contains all bipartite graphs and all line graphs of bipartite graphs. This paper was the precursor of a sequence of decomposition results involving 2-joins:

Theorem 5.2 (Conforti, Cornuéjols and Vušković [18]) A square-free Berge graph is bipartite, the line graph of a bipartite graph, or has a 2-join or a star cutset.
Theorem 5.3 (Chudnovsky, Robertson, Seymour and Thomas [10]) If $G$ is a Berge graph that contains the line graph of a bipartite subdivision of a 3-connected graph, then $G$ has a skew partition, or $G$ or $G$ has a 2-join or is the line graph of a bipartite graph.

Given two vertex disjoint triangles $a_1, a_2, a_3$ and $b_1, b_2, b_3$, a stretcher is a graph induced by three chordless paths, $P^1 = a_1, \ldots, b_1$, $P^2 = a_2, \ldots, b_2$ and $P^3 = a_3, \ldots, b_3$, at least one of which has length greater than one, such that $P^1, P^2, P^3$ have no common vertices and the only adjacencies between the vertices of distinct paths are the edges of the two triangles. The next result is a real tour-de-force and a key step in the proof of the SPGC.

Theorem 5.4 (Chudnovsky and Seymour [12]) If $G$ is a Berge graph that contains a stretcher, then $G$ is the line graph of a bipartite graph or $G$ has a skew partition or a homogeneous pair, or $G$ or $G$ has a 2-join.

A wheel $(H, v)$ consists of a hole $H$ together with a vertex $v$, called the center, with at least three neighbors in $H$. If $v$ has $k$ neighbors in $H$, the wheel is called a $k$-wheel. A line wheel is a 4-wheel $(H, v)$ that contains exactly two triangles and these two triangles have only the center $v$ in common. A twin wheel is a 3-wheel containing exactly two triangles. A universal wheel is a wheel $(H, v)$ where the center $v$ is adjacent to all the vertices of $H$. A triangle-free wheel is a wheel containing no triangle. A proper wheel is a wheel that is not any of the above four types. These concepts were first introduced in [14]. The following theorem generalizes an earlier result by Conforti, Cornuèjols and Zambelli [21] and Thomas [35].

Theorem 5.5 (Conforti, Cornuèjols and Zambelli [22]) If $G$ is a Berge graph that contains no proper wheels, stretchers or their complements, then $G$ is basic or has a skew partition.

The last step in proving the SPGC is the following difficult theorem.

Theorem 5.6 (Chudnovsky and Seymour [13]) If $G$ is a Berge graph that contains a proper wheel, but no stretchers or their complements, then $G$ has a skew partition, or $G$ or $G$ has a 2-join.

A monumental paper containing these results is forthcoming [11]. Independently, Conforti, Cornuèjols, Vušković and Zambelli [20] proved that the Decomposition Conjecture holds for Berge graphs containing a large class of proper wheels but, as of May 2002, they could not prove it for all proper wheels. Theorems 5.4, 5.5 and 5.6 imply that Conjecture 5.1 holds, and therefore the SPGC is true.

Corollary 5.7 (Strong Perfect Graph Theorem) The only minimally imperfect graphs are the odd holes and their complements.

Conforti, Cornuèjols and Vušković [19] proved a weaker version of the Decomposition Conjecture where “skew partition” is replaced by “double star cutset”. A double star is a vertex set $S$ that contains two adjacent vertices $u, v$ and a subset of the vertices adjacent to $u$ or $v$. Clearly, if $G$ has a skew partition, then $G$ has a double star cutset: Take $S = A \cup B$, $u \in A$ and $v \in B$. Although the decomposition result in [19] is weaker than Conjecture 5.1 for Berge graphs, it holds for a larger class of graphs than Berge graphs: By changing the decomposition from “skew partition” to “double star cutset”, the result can be obtained for all odd-hole-free graphs instead of just Berge graphs.

Theorem 5.8 (Conforti, Cornuèjols and Vušković [19]) If $G$ is an odd-hole-free graph, then $G$ is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or $G$ has a double star cutset or a 2-join.

One might try to use Theorem 5.8 to construct a polynomial time recognition algorithm for odd-hole-free graphs. Conforti, Cornuèjols, Kapoor and Vušković [17] obtained a polynomial time recognition algorithm for the class of even-hole-free graphs. This algorithm is based on the decomposition of even-hole-free graphs by 2-joins, double star and triple star cutsets obtained in [16].

A useful tool for studying Berge graphs is due to Roussel and Rubio [34]. This lemma was proved independently by Robertson, Seymour and Thomas [33], who popularized it and named it The Wonderful Lemma. It is used repeatedly in the proofs of Theorems 5.3-5.6.
Lemma 5.9 (The Wonderful Lemma) (Roussel and Rubio [34]) Let $G$ be a Berge graph and assume that $V(G)$ can be partitioned into a set $S$ and an odd chordless path $P = u, u', \ldots, v', v$ of length at least 3 such that $u, v$ are both adjacent to all the vertices in $S$ and $G(S)$ is connected. Then one of the following holds:

(i) An odd number of edges of $P$ have both ends adjacent to all the vertices in $S$.

(ii) $P$ has length 3 and $G(S \cup \{u', v'\})$ contains an odd chordless path between $u'$ and $v'$.

(iii) $P$ has length at least 5 and there exist two nonadjacent vertices $x, x'$ in $S$ such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a path.

References


