Minimal inequalities for an infinite relaxation of integer programs

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Abstract

We show that maximal S-free convex sets are polyhedra when S is the set of integral points in some rational polyhedron of $\mathbb{R}^n$. This result extends a theorem of Lovász characterizing maximal lattice-free convex sets. Our theorem has implications in integer programming. In particular, we show that maximal S-free convex sets are in one-to-one correspondance with minimal inequalities.

1 Introduction

Consider a mixed integer linear program, and the optimal tableau of the linear programming relaxation. We select n rows of the tableau, relative to n basic integer variables $x_1, \ldots, x_n$. Let $s_1, \ldots, s_m$ denote the nonbasic variables. Let $f_i \geq 0$ be the value of x_i in the basic solution associated with the tableau, $i = 1, \ldots, n$, and suppose $f \notin \mathbb{Z}^n$. The tableau restricted to these n rows is of the form

$$x = f + \sum_{j=1}^{m} r^j s_j, \quad x \geq 0 \text{ integral}, \; s \geq 0, \; \text{and } s_j \in \mathbb{Z}, j \in I, \tag{1}$$

where $r^j \in \mathbb{R}^n$, $j = 1, \ldots, m$, and I denotes the set of integer nonbasic variables.

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An important question in integer programming is to derive valid inequalities for (1), cutting off the current infeasible solution $x = f$, $s = 0$. We will consider a simplified model where the integrality conditions are relaxed on all nonbasic variables. On the other hand, we can present our results in a more general context, where the constraints $x \geq 0$, $x \in \mathbb{Z}^n$, are replaced by constraints $x \in S$, where S is the set of integral points in some given rational polyhedron such that $\dim(S) = n$, i.e. S contains $n+1$ affinely independent points. Recall that a polyhedron $Ax \leq b$ is *rational* if the matrix A and vector b have rational entries.

So we study the following model, introduced by Johnson [8].

$$x = f + \sum_{j=1}^{m} r^j s_j, \quad x \in S,\, s \geq 0, \tag{2}$$

where $f \in \operatorname{conv}(S) \setminus \mathbb{Z}^n$. Note that every inequality cutting off the point $(f, 0)$ can be expressed in terms of the nonbasic variables s only, and can therefore be written in the form $\sum_{j=1}^{m} \alpha_j s_j \geq 1$.

In this paper we are interested in "formulas" for deriving such inequalities. More formally, we are interested in functions $\psi : \mathbb{R}^n \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{m} \psi(r^j) s_j \geq 1$$

is valid for (2) for every choice of m and vectors $r^1, \ldots, r^m \in \mathbb{R}^n$. We refer to such functions ψ as *valid functions* (with respect to f and S). Note that, if ψ is a valid function and ψ' is a function such that $\psi \leq \psi'$, then ψ' is also valid, and the inequality $\sum_{j=1}^{m} \psi'(r^j) s_j \geq 1$ is implied by $\sum_{j=1}^{m} \psi(r^j) s_j \geq 1$. Therefore we only need to investigate (pointwise) minimal valid functions.

Andersen, Louveaux, Weismantel, Wolsey [1] characterize minimal valid functions for the case $n = 2$, $S = \mathbb{Z}^2$. Borozan and Cornuéjols [6] extend this result to $S = \mathbb{Z}^n$ for any n. These papers and a result of Zambelli [11] show a one-to-one correspondence between minimal valid functions and maximal lattice-free convex sets with f in the interior. These results have been further generalized in [4]. Minimal valid functions for the case $S = \mathbb{Z}^n$ are intersection cuts [2].

Our interest in model (2) arose from a recent paper of Dey and Wolsey [7]. They introduce the notion of *S-free convex set* as a convex set without points of S in its interior, and show the connection between valid functions and S-free convex sets with f in their interior.

A class of valid functions can be defined as follows. A function ψ is *positively homogeneous* if $\psi(\lambda r) = \lambda(\psi r)$ for every $r \in \mathbb{R}^n$ and every $\lambda \geq 0$, and it is *subadditive* if $\psi(r) + \psi(r') \geq \psi(r + r')$ for all $r, r' \in \mathbb{R}^n$. A function ψ is *sublinear* if it is positively homogeneous and subadditive. It is easy to observe that sublinear functions are also convex.

Assume that ψ is a sublinear function such that the set

$$B_\psi = \{x \in \mathbb{R}^n \,|\, \psi(x - f) \leq 1\} \tag{3}$$

is S-free. Note that B_ψ is closed and convex because ψ is convex. Since ψ is positively homogeneous, $\psi(0) = 0$, thus f is in the interior of B_ψ. We claim that ψ is a valid function.

Indeed, given any solution $(\bar{x}, \bar{s})$ to (2), we have

$$\sum_{j=1}^{m} \psi(r^j)\bar{s}_j \geq \psi(\sum_{j=1}^{m} r^j \bar{s}_j) = \psi(\bar{x} - f) \geq 1,$$

where the first inequality follows from sublinearity and the last one follows from the fact that $\bar{x}$ is not in the interior of B_ψ.

Dey and Wolsey [7] show that every minimal valid function ψ is sublinear and B_ψ is an S-free convex set with f in its interior. In this paper, we prove that if ψ is a minimal valid function, then B_ψ is a *maximal S-free convex set.*

In Section 2, we show that maximal S-free convex sets are polyhedra. Therefore a maximal S-free convex set $B \subseteq \mathbb{R}^n$ containing f in its interior can be uniquely written in the form $B = \{x \in \mathbb{R}^n : \ a_i(x - f) \leq 1, \ i = 1, \dots, k\}$. Let $\psi_B : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\psi_B(r) = \max_{i=1,\dots,k} a_i r, \quad \forall r \in \mathbb{R}^n. \tag{4}$$

It is easy to observe that the above function is sublinear and $B = \{x \in \mathbb{R}^n \mid \psi_B(x-f) \leq 1\}$. In Section 3 we will prove that every minimal valid function is of the form ψ_B for some maximal S-free convex set B containing f in its interior. Conversely, if B is a maximal S-free convex set containing f in its interior, then ψ_B is a minimal valid function.

2 Maximal S-free convex sets

Let $S \subseteq \mathbb{Z}^n$ be the set of integral points in some rational polyhedron of $\mathbb{R}^n$. We say that $B \subset \mathbb{R}^n$ is an *S-free convex set* if B is convex and does not contain any point of S in its interior. We say that B is a *maximal S-free convex set* if it is an S-free convex set and it is not properly contained in any S-free convex set. It follows from Zorn's lemma that every S-free convex set is contained in a maximal S-free convex set.

When $S = \mathbb{Z}^n$, an S-free convex set is called a *lattice-free convex set.* The following theorem of Lovász characterizes maximal lattice-free convex sets. A linear subspace or cone in $\mathbb{R}^n$ is *rational* if it can be generated by rational vectors, i.e. vectors with rational coordinates.

Theorem 1. (Lovász [9]) *A set $B \subset \mathbb{R}^n$ is a maximal lattice-free convex set if and only if one of the following holds:*

(i) B is a polyhedron of the form $B = P + L$ where P is a polytope, L is a rational linear space, $\dim(B) = \dim(P) + \dim(L) = n$, B does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of B;

(ii) B is a hyperplane of $\mathbb{R}^n$ that is not rational.

Lovász only gives a sketch of the proof. A complete proof can be found in [4]. The next theorem is an extension of Lovász' theorem to maximal S-free convex sets.

Given a convex set $K \subset \mathbb{R}^n$, we denote by $\text{rec}(K)$ its recession cone and by $\text{lin}(K)$ its lineality space. Given a set $X \subseteq \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by X. Given a k-dimensional linear space V and a subset Λ of V, we say that Λ is a *lattice of* V if there exists a linear bijection $f : \mathbb{R}^k \to V$ such that $\Lambda = f(\mathbb{Z}^k)$.

Theorem 2. *Let S be the set of integral points in some rational polyhedron of $\mathbb{R}^n$ such that $\dim(S) = n$. A set $B \subset \mathbb{R}^n$ is a maximal S-free convex set if and only if one of the following holds:*

- *(i) B is a polyhedron such that $B \cap \mathrm{conv}(S)$ has nonempty interior, B does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets.*
- *(ii) B is a half-space of $\mathbb{R}^n$ such that $B \cap \mathrm{conv}(S)$ has empty interior and the boundary of B is a supporting hyperplane of $\mathrm{conv}(S)$.*
- *(iii) B is a hyperplane of $\mathbb{R}^n$ such that $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ is not rational.*

Furthermore, if (i) holds, the recession cone of $B \cap \mathrm{conv}(S)$ is rational and it is contained in the lineality space of B.

We illustrate case (i) of the theorem in the plane in Figure 2. The question of the polyhedrality of maximal S-free convex sets was raised by Dey and Wolsey [7]. They proved that this is the case for a maximal S-free convex set B, under the assumptions that $B \cap \mathrm{conv}(S)$ has nonempty interior and that the recession cone of $B \cap \mathrm{conv}(S)$ is finitely generated and rational. Theorem 2 settles the question in general.

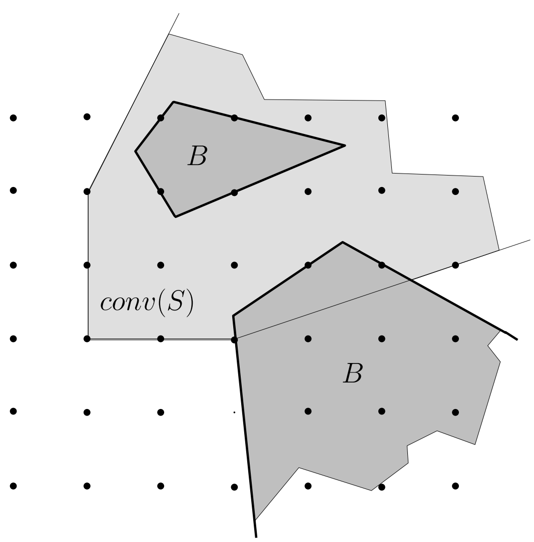

Figure 1: Two examples of S-free sets in the plane (case (i) of Theorem 2). The light gray region indicates $\mathrm{conv}(S)$ and the dark grey regions illustrate the S-free sets. A jagged line indicates that the region extends to infinity.

To prove Theorem 2 we will need the following lemmas. The first one is proved in [4] and is an easy consequence of Dirichlet's theorem.

Lemma 3. *Let $y \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$. For every $\varepsilon > 0$ and $\bar{\lambda} \geq 0$, there exists an integral point at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$.*

Lemma 4. *Let B be an S-free convex set such that $B \cap \operatorname{conv}(S)$ has nonempty interior. For every $r \in \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$, $B + \langle r \rangle$ is S-free.*

Proof. Let $C = \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ and $r \in C \setminus \{0\}$. Suppose by contradiction that there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. We show that $y \in \mathbf{int}(B) + \langle r \rangle$. If not, $(y + \langle r \rangle) \cap \mathbf{int}(B) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $B + \langle r \rangle$, a contradiction. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(B)$, i.e. there exists $\varepsilon > 0$ such that B contains the open ball $B_\varepsilon(\bar{y})$ of radius ε centered at $\bar{y}$. Since $r \in C \subseteq \operatorname{rec}(B)$, it follows that $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset B$. Since $y \in \mathbb{Z}^n$, by Lemma 3 there exists $z \in \mathbb{Z}^n$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \mathbf{int}(B)$. Note that the half-line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ is in $\operatorname{conv}(S)$, since $y \in S$ and $r \in \operatorname{rec}(\operatorname{conv}(S))$. Since $\operatorname{conv}(S)$ is a rational polyhedron, for $\varepsilon > 0$ sufficiently small every integral point at distance at most ε from $\operatorname{conv}(S)$ is in $\operatorname{conv}(S)$. Therefore $z \in S$, a contradiction. □

Proof of Theorem 2. The proof of the "if" part is standard, and it is similar to the proof for the lattice-free case (see [4]). We show the "only if" part. Let B be a maximal S-free convex set. If $\dim(B) < n$, then B is contained in some affine hyperplane K. Since K has empty interior, K is S-free, thus $B = K$ by maximality of B. Next we show that $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ is not rational. Suppose not. Then the linear subspace $L = \langle \operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) \rangle$ is rational. Therefore the projection Λ of $\mathbb{Z}^n$ onto $L^\perp$ is a lattice of $L^\perp$ (see, for example, Barvinok [3] p 284 problem 3). The projection S' of S onto $L^\perp$ is a subset of Λ. Let B' be the projection of B onto $L^\perp$. Then $B' \cap \operatorname{conv}(S')$ is the projection of $B \cap \operatorname{conv}(S)$ onto $L^\perp$. Since B is a hyperplane, $\operatorname{lin}(B) = \operatorname{rec}(B)$. This implies that $B' \cap \operatorname{conv}(S')$ is bounded : otherwise there is an unbounded direction $d \in L^\perp$ in $\operatorname{rec}(B') \cap \operatorname{rec}(\operatorname{conv}(S'))$ and so $d + l \in \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ for some $l \in L$. Since $\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) = \operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Fix $\delta > 0$. Since Λ is a lattice and $S' \subseteq \Lambda$, there is a finite number of points at distance less than δ from the bounded set $B' \cap \operatorname{conv}(S')$ in $L^\perp$. It follows that there exists $\varepsilon > 0$ such that every point of S' has distance at least ε from $B' \cap \operatorname{conv}(S')$. Let $B'' = \{v + w \mid v \in B,\, w \in L^\perp, \|w\| \leq \varepsilon\}$. The set B'' is S-free by the choice of ε, but B'' strictly contains B, contradicting the maximality of B. Therefore (iii) holds when $\dim(B) < n$. Hence we may assume $\dim(B) = n$. If $B \cap \operatorname{conv}(S)$ has empty interior, then there exists a hyperplane separating B and $\operatorname{conv}(S)$ which is supporting for $\operatorname{conv}(S)$. By maximality of B case (ii) follows.

Therefore we may assume that $B \cap \operatorname{conv}(S)$ has nonempty interior. We show that B satisfies (i).

Claim 1. *There exists a rational polyhedron P such that:*
i) $\operatorname{conv}(S) \subset \mathbf{int}(P)$,
ii) The set $K = B \cap P$ is lattice-free,
iii) For every facet F of P, $F \cap K$ is a facet of K,
iv) For every facet F of P, $F \cap K$ contains an integral point in its relative interior.

Since $\operatorname{conv}(S)$ is a rational polyhedron, there exist integral A and b such that $\operatorname{conv}(S) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The set $P' = \{x \in \mathbb{R}^n \mid Ax \leq b + \frac{1}{2}\mathbf{1}\}$ satisfies $i)$. The set $B \cap P'$ is lattice-free since B is S-free and P' does not contain any point in $\mathbb{Z}^n \setminus S$, thus P' also satisfies $ii)$. Let $\bar{A}x \leq \bar{b}$ be the system containing all inequalities of $Ax \leq b + \frac{1}{2}\mathbf{1}$ that define facets of $B \cap P'$. Let $P_0 = \{x \in \mathbb{R}^n \,|\, \bar{A}x \le \bar{b}\}$. Then P_0 satisfies $i), ii), iii)$. See Figure 2 for an illustration.

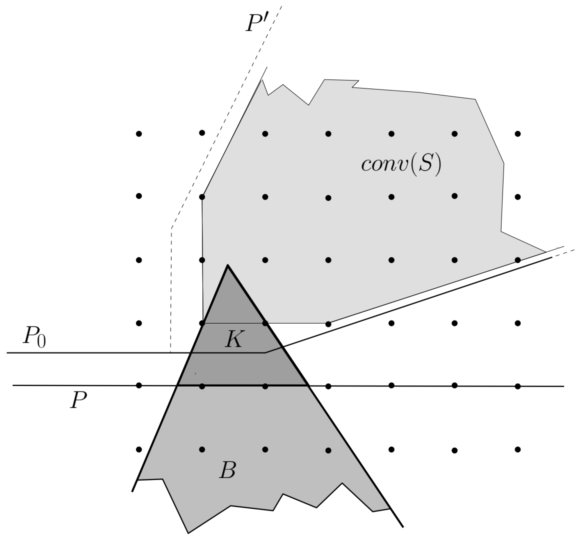

Figure 2: Illustration for Claim 1.

It will be more convenient to write P_0 as intersection of the half-spaces defining the facets of P_0, $P_0 = \cap_{H \in \mathcal{F}_0} H$. We construct a sequence of rational polyhedra $P_0 \subset P_1 \subset \ldots \subset P_t$ such that P_i satisfies $i), ii), iii)$, $i = 1, \ldots, t$, and such that P_t satisfies $iv)$. Given P_i, we construct P_{i+1} as follows. Let $P_i = \cap_{H \in \mathcal{F}_i} H$, where $\mathcal{F}_i$ is the set of half spaces defining facets of P_i. Let $\bar{H}$ be a half-space in $\mathcal{F}_i$ defining a facet of $B \cap P_i$ that does not contain an integral point in its relative interior; if no such $\bar{H}$ exists, then P_i satisfies $iv)$ and we are done. If $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ does not contain any integral point in its interior, let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\}$. Otherwise, since P_i is rational, among all integral points in the interior of $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ there exists one, say $\bar{x}$, at minimum distance from $\bar{H}$. Let H' be the half-space containing $\bar{H}$ with $\bar{x}$ on its boundary. Let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\} \cap \{H'\}$. Observe that H' defines a facet of P_{i+1} since $\bar{x}$ is in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i+1} \setminus \{H'\}} H$ and it is on the boundary of H'. So $i), ii), iii)$ are satisfied and P_{i+1} has fewer facets that violate $iv)$ than P_i. $\diamond$

Let T be a maximal lattice-free convex set containing the set K defined in Claim 1. As remarked earlier, such a set T exists. By Theorem 1, T is a polyhedron with an integral point in the relative interior of each of its facets. Let H be a hyperplane that defines a facet of P. Since $K \cap H$ is a facet of K with an integral point in its relative interior, it follows that H defines a facet of T. This implies that $T \subset P$. Therefore we can write T as

$$T = P \cap \bigcap_{i=1}^{k} H_i, \tag{5}$$

where H_i are halfspaces. Let $\bar{H}_i = \mathbb{R}^n \setminus \mathbf{int}(H_i)$, $i = 1, \ldots, k$.

Claim 2. *B is a polyhedron.*

We first show that, for $i = 1, \ldots, k$, $\mathbf{int}(B) \cap (\bar{H}_i \cap \text{conv}(S)) = \emptyset$. Consider $y \in \mathbf{int}(B) \cap \bar{H}_i$. Since $y \in \bar{H}_i$ and K is contained in T, $y \notin \mathbf{int}(K)$. Since $K = B \cap P$ and $y \in \mathbf{int}(B) \setminus \mathbf{int}(K)$, it follows that $y \notin \mathbf{int}(P)$. Hence $y \notin \text{conv}(S)$ because $\text{conv}(S) \subseteq \mathbf{int}(P)$.

Thus, for $i = 1, \ldots, k$, there exists a hyperplane separating B and $\bar{H}_i \cap \text{conv}(S)$. Hence there exists a halfspace K_i such that $B \subset K_i$ and $\bar{H}_i \cap \text{conv}(S)$ is disjoint from the interior of K_i. We claim that the set $B' = \cap_{i=1}^k K_i$ is S-free. Indeed, let $y \in S$. Then y is not interior of T. Since $y \in \text{conv}(S)$ and $\text{conv}(S) \subseteq \mathbf{int}(P)$, y is in the interior of P. Hence, by (5), there exists $i \in \{1, \ldots, k\}$ such that y is not in the interior of H_i. Thus $y \in \bar{H}_i \cap \text{conv}(S)$. By construction, y is not in the interior of K_i, hence y is not in the interior of B'. Thus B' is an S-free convex set containing B. Since B is maximal, $B' = B$. $\diamond$

Claim 3. $\text{lin}(K) = \text{rec}(K)$.

Let $r \in \text{rec}(K)$. We show $-r \in \text{rec}(K)$. By Lemma 4 applied to $\mathbb{Z}^n$, $K + \langle r \rangle$ is lattice-free. We observe that $B + \langle r \rangle$ is S-free. If not, let $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $S \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P + \langle r \rangle)$, hence $y \in \mathbf{int}(K + \langle r \rangle)$, a contradiction. Hence, by maximality of B, $B = B + \langle r \rangle$. Thus $-r \in \text{rec}(B)$. Suppose that $-r \notin \text{rec}(P)$. Then there exists a facet F of P that is not parallel to r. By construction, $F \cap K$ is a facet of K containing an integral point $\bar{x}$ in its relative interior. The point $\bar{x}$ is then in the interior of $K + \langle r \rangle$, a contradiction. $\diamond$

Claim 4. $\text{lin}(K)$ *is rational.*

Consider the maximal lattice-free convex set T containing K considered earlier. By Theorem 1, $\text{lin}(T) = \text{rec}(T)$, and $\text{lin}(T)$ is rational. Clearly $\text{lin}(T) \supseteq \text{lin}(K)$. Hence, if the claim does not hold, there exists a rational vector $r \in \text{lin}(T) \setminus \text{lin}(K)$. By (5), $r \in \text{lin}(P)$. Since $K = B \cap P$, $r \notin \text{lin}(B)$. Hence $B \subset B + \langle r \rangle$. We will show that $B + \langle r \rangle$ is S-free, contradicting the maximality of B. Suppose there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $\text{conv}(S) \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P) \subseteq \mathbf{int}(P) + \langle r \rangle$. Therefore $y \in \mathbf{int}(B \cap P) + \langle r \rangle$. Since $B \cap P \subseteq T$, then $y \in \mathbf{int}(T) + \langle r \rangle = \mathbf{int}(T)$ where the last equality follows from $r \in \text{lin}(T)$. This contradicts the fact that T is lattice-free. $\diamond$

By Lemma 4 and by the maximality of B, $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{rec}(B) \cap \text{rec}(\text{conv}(S))$.

Claim 5. $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ *is rational.*

Since $\text{lin}(K)$ and $\text{rec}(\text{conv}(S))$ are both rational, we only need to show $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{lin}(K) \cap \text{rec}(\text{conv}(S))$. The "$\supseteq$" direction follows from $B \supseteq K$. For the other direction, note that, since $\text{conv}(S) \subseteq P$, we have $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) \subseteq \text{lin}(B) \cap \text{rec}(P) = \text{lin}(B \cap P) = \text{lin}(K)$, hence $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) \subseteq \text{lin}(K) \cap \text{rec}(\text{conv}(S))$. $\diamond$

Claim 6. *Every facet of B contains a point of S in its relative interior.*

Let L be the linear space generated by $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$. By Claim 5, L is rational. Let B', S', Λ be the projections of B, S, $\mathbb{Z}^n$, respectively, onto $L^\perp$. Since L is rational, Λ is a lattice of $L^\perp$ and $S' = \text{conv}(S') \cap \Lambda$. Also, B' is a maximal S'-free convex set of $L^\perp$, since for any S'-free set D of $L^\perp$, $D + L$ is S-free. Note that $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ implies that $B' \cap \text{conv}(S')$ is bounded. Otherwise there is an unbounded direction $d \in L^\perp$ in $\text{rec}(B') \cap \text{rec}(\text{conv}(S'))$ and so $d + l \in \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ for some

$l \in L$. Since $\text{rec}(B) \cap \text{rec}(\text{conv}(S)) = \text{lin}(B) \cap \text{rec}(\text{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Let $B' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t\}$. Given $\varepsilon > 0$, let $\bar{B} = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \beta_t + \varepsilon\}$. The polyhedron $\text{conv}(S') \cap \bar{B}$ is a polytope since it has the same recession cone as $\text{conv}(S') \cap B'$. The polytope $\text{conv}(S') \cap \bar{B}$ contains points of S' in its interior by the maximality of B'. Since Λ is a lattice of $L^\perp$, $\mathbf{int}(\text{conv}(S') \cap \bar{B})$ has a finite number of points in S', hence there exists one minimizing $\alpha_t x$, say z. By construction, the polyhedron $B'' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \alpha_t z\}$ does not contain any point of S' in its interior and contains B'. By the maximality of B', $B' = B''$ hence B' contains z in its relative interior, and B contains a point of S in its relative interior. □

Corollary 5. *For every maximal S-free convex set B there exists a maximal lattice-free convex set K such that, for every facet F of B, $F \cap K$ is a facet of K.*

Proof. Let K be defined as in Claim 1 in the proof of Theorem 2. It follows from the proof that K is a maximal lattice-free convex set with the desired properties. □

3 Minimal valid functions

In this section we study minimal valid functions. We find it convenient to state our results in terms of an infinite model introduced by Dey and Wolsey [7].

Throughout this section, $S \subseteq \mathbb{Z}^n$ is a set of integral points in some rational polyhedron of $\mathbb{R}^n$ such that $\dim(S) = n$, and f is a point in $\text{conv}(S) \setminus \mathbb{Z}^n$. Let $R_{f,S}$ be the set of all infinite dimensional vectors $s = (s_r)_{r \in \mathbb{R}^n}$ such that

$$\begin{aligned} & f + \sum_{r \in \mathbb{R}^n} r s_r \in S \\ & s_r \geq 0, \quad r \in \mathbb{R}^n \\ & s \text{ has finite support} \end{aligned} \tag{6}$$

where s has *finite support* means that s_r is zero for all but a finite number of $r \in \mathbb{R}^n$.

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *valid* (with respect to f and S) if the linear inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1 \tag{7}$$

is satisfied by every $s \in R_{f,S}$. Note that this definition coincides with the one we gave in the introduction.

Given two functions ψ, ψ' we say that ψ' *dominates* ψ if $\psi'(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$. A valid function ψ is *minimal* if there is no valid function $\psi' \neq \psi$ that dominates ψ.

Theorem 6. *For every valid function ψ, there exists a maximal S-free convex set B with f in its interior such that ψ_B dominates ψ. Furthermore, if B is a maximal S-free convex set containing f in its interior, then ψ_B is a minimal valid function.*

We will need the following lemma.

Lemma 7. *Every valid function is dominated by a sublinear valid function.*

Sketch of proof. Given a valid function ψ, define the following function $\bar{\psi}$. For all $\bar{r} \in \mathbb{R}^n$, let $\bar{\psi}(\bar{r}) = \inf\{\sum_{r\in\mathbb{R}^n} \psi(r)s_r \mid \sum_{r\in\mathbb{R}^n} rs_r = \bar{r},\ s \geq 0$ with finite support$\}$. Following the proof of Lemma 18 in [4] one can show that $\bar{\psi}$ is a valid sublinear function that dominates ψ. □

Given a valid sublinear function ψ, the set $B_\psi = \{x \in \mathbb{R}^n \mid \psi(x-f) \leq 1\}$ is closed, convex, and contains f in its interior. Since ψ is a valid function, B_ψ is S-free. Indeed the interior of B_ψ is $\mathbf{int}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) < 1\}$. Its boundary is $\mathbf{bd}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) = 1\}$, and its recession cone is $\mathrm{rec}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) \leq 0\}$.

Before proving Theorem 6, we need the following general theorem about sublinear functions. Let K be a closed, convex set in $\mathbb{R}^n$ with the origin in its interior. The *polar* of K is the set $K^* = \{y \in \mathbb{R}^n \mid ry \leq 1 \text{ for all } r \in K\}$. Clearly K^* is closed and convex, and since $0 \in \mathbf{int}(K)$, it is well known that K^* is bounded. In particular, K^* is a compact set. Also, since $0 \in K$, $K^{**} = K$. Let

$$\hat{K} = \{y \in K^* \mid \exists x \in K \text{ such that } xy = 1\}. \tag{8}$$

Note that $\hat{K}$ is contained in the relative boundary of K^*. Let $\rho_K : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\rho_K(r) = \sup_{y\in\hat{K}} ry, \quad \text{for all } r \in \mathbb{R}^n. \tag{9}$$

It is easy to show that ρ_K is sublinear.

Theorem 8 (Basu et al. [5])**.** *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. Then $K = \{r \in \mathbb{R}^n \mid \rho_K(r) \leq 1\}$. Furthermore, for every sublinear function σ such that $K = \{r \mid \sigma(r) \leq 1\}$, we have $\rho_K(r) \leq \sigma(r)$ for every $r \in \mathbb{R}^n$.*

Remark 9. *Let $K \subset \mathbb{R}^n$ be a polyhedron containing the origin in its interior. Let $a_1, \ldots, a_t \in \mathbb{R}^n$ such that $K = \{r \in \mathbb{R}^n \mid a_i r \leq 1,\ i = 1, \ldots, t\}$. Then $\rho_K(r) = \max_{i=1,\ldots,t} a_i r$.*

Proof. The polar of K is $K^* = \mathrm{conv}\{0, a_1, \ldots, a_t\}$ (see Theorem 9.1 in Schrijver [10]). Furthermore, $\hat{K}$ is the union of all the facets of K^* that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y\in\hat{K}} yr = \max_{i=1,\ldots,t} a_i r$$

for all $r \in \mathbb{R}^n$. □

Remark 10. *Let B be a closed S-free convex set in $\mathbb{R}^n$ with f in its interior, and let $K = B - f$. Then ρ_K is a valid function.*

Proof: Let $s \in R_{f,S}$. Then $x = f + \sum_{r\in\mathbb{R}^n} rs_r$ is in S, therefore $x \notin \mathbf{int}(B)$ because B is S-free. By Theorem 8, $\rho_K(x-f) \geq 1$. Thus

$$1 \leq \rho_K\Big(\sum_{r\in\mathbb{R}^n} rs_r\Big) \leq \sum_{r\in\mathbb{R}^n} \rho_K(r)s_r,$$

where the second inequality follows from the sublinearity of ρ_K. □

Lemma 11. *Let C be a closed S-free convex set containing f in its interior, and let $K = C - f$. There exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x-f) \leq 1,\ i = 1,\dots,k\}$ such that $a_i \in \mathbf{cl}(\mathrm{conv}(\hat{K}))$ for $i = 1,\dots,k$.*

Proof. Since C is an S-free convex set, it is contained in some maximal S-free convex set T. The set T satisfies one of the statements (i)-(iii) of Theorem 2. By assumption, $f \in \mathrm{conv}(S)$ and f is in the interior of C. Since $\dim(S) = n$, $\mathrm{conv}(S)$ is a full dimensional polyhedron, thus $\mathbf{int}(C \cap \mathrm{conv}(S)) \neq \emptyset$. This implies that $\mathbf{int}(T \cap \mathrm{conv}(S)) \neq \emptyset$, hence case (i) applies.

Thus T is a polyhedron and $\mathrm{rec}(T \cap \mathrm{conv}(S)) = \mathrm{lin}(T) \cap \mathrm{rec}(\mathrm{conv}(S))$ is rational. Let us choose T such that the dimension of $\mathrm{lin}(T)$ is largest possible.

Since T is a polyhedron containing f in its interior, there exists $D \in \mathbb{R}^{t\times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0$, $i = 1,\dots,t$, and $T = \{x \in \mathbb{R}^n \mid D(x-f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x\in C} d_i(x-f) = 1$ where d_i denotes the ith row of D, $i = 1,\dots,t$. By our assumption, $\sup_{r\in K} d_i r = 1$. Therefore $d_i \in K^*$, since $d_i r \leq 1$ for all $r \in K$. Furthermore $d_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r\in K} d_i r = 1$.

Let $P = \{x \in \mathbb{R}^n \mid D(x-f) \leq e\}$. Note that $\mathrm{lin}(P) = \mathrm{lin}(T)$. By our choice of T, $P + \langle r\rangle$ is not S-free for any $r \in \mathrm{rec}(\mathrm{conv}(S)) \setminus \mathrm{lin}(P)$, otherwise P would be contained in a maximal S-free convex set whose lineality space contains $\mathrm{lin}(T) + \langle r\rangle$, a contradiction.

Let $L = \langle \mathrm{rec}(P \cap \mathrm{conv}(S))\rangle$. Since $\mathrm{lin}(P) = \mathrm{lin}(T)$, L is a rational space. Note that $L \subseteq \mathrm{lin}(P)$, implying that $d_i \in L^\perp$ for $i = 1,\dots,t$.

We observe next that we may assume that $P \cap \mathrm{conv}(S)$ is bounded. Indeed, let $\bar{P}$, $\bar{S}$, Λ be the projections onto $L^\perp$ of P, S, and $\mathbb{Z}^n$, respectively. Since L is a rational space, Λ is a lattice of $L^\perp$ and $\bar{S} = \mathrm{conv}(\bar{S}) \cap \Lambda$. Note that $\bar{P} \cap \mathrm{conv}(\bar{S})$ is bounded, since $L \supseteq \mathrm{rec}(P \cap \mathrm{conv}(S))$. If we are given a maximal $\bar{S}$-free convex set $\bar{B}$ in $L^\perp$ such that $\bar{B} = \{x \in L^\perp \mid a_i(x-f) \leq 1,\ i = 1,\dots,h\}$ and $a_i \in \mathrm{conv}\{d_1,\dots,d_t\}$ for $i = 1,\dots,h$, then $B = \bar{B} + L$ is the set $B = \{x \in \mathbb{R}^n \mid a_i(x-f) \leq 1,\ i = 1,\dots,h\}$. Since $\bar{B}$ contains a point of $\bar{S}$ in the relative interior of each of its facets, B contains a point of S in the relative interior of each of its facets, thus B is a maximal S-free convex set.

Thus we assume that $P \cap \mathrm{conv}(S)$ is bounded, so $\dim(L) = 0$. If all facets of P contain a point of S in their relative interior, then P is a maximal S-free convex set, thus the statement of the lemma holds. Otherwise we describe a procedure that replaces one of the inequalities defining a facet of P without any point of S in its relative interior with an inequality which is a convex combination of the inequalities of $D(x-f) \leq e$, such that the new polyhedron thus obtained is S-free and has one fewer facet without points of S in its relative interior. More formally, suppose the facet of P defined by $d_1(x-f) \leq 1$ does not contain any point of S in its relative interior. Given $\lambda \in [0,1]$, let

$$P(\lambda) = \{x \in \mathbb{R}^n \mid [\lambda d_1 + (1-\lambda)d_2](x-f) \leq 1, \quad d_i(x-f) \leq 1 \ \ i = 2,\dots,t\}.$$

Note that $P(1) = P$ and $P(0)$ is obtained from P by removing the inequality $d_1(x-f) \leq 1$. Furthermore, given $0 \leq \lambda' \leq \lambda'' \leq 1$, we have $P(\lambda') \supseteq P(\lambda'')$.

Let $r_1,\dots,r_m$ be generators of $\mathrm{rec}(\mathrm{conv}(S))$. Note that, since $P \cap \mathrm{conv}(S)$ is bounded, for every $j = 1,\dots,m$ there exists $i \in \{1,\dots,t\}$ such that $d_i r_j > 0$. Let $r_1,\dots,r_h$ be the generators of rec(conv(S)) satisfying

$$\begin{aligned} d_1 r_j &> 0 \\ d_i r_j &\le 0 \qquad i = 2, \dots, t. \end{aligned}$$

Note that, if no such generators exist, then $P(0) \cap \text{conv}(S)$ is bounded. Otherwise $P(\lambda) \cap \text{conv}(S)$ is bounded if and only if, for $j = 1, \dots, h$

$$[\lambda d_1 + (1 - \lambda) d_2] r_j > 0.$$

This is the case if and only if $\lambda > \lambda^*$, where

$$\lambda^* = \max_{j=1,\dots,h} \frac{-d_2 r_j}{(d_1 - d_2) r_j}.$$

Let r^* be one of the vectors $r_1, \dots, r_h$ attaining the maximum in the previous equation. Then $r^* \in \text{rec}(P(\lambda^* \cap \text{conv}(S)$.

Note that $P(\lambda^*)$ is not S-free otherwise $P(\lambda^*) + \langle r^* \rangle$ is S-free by Lemma 4, and so is $P + \langle r^* \rangle$, a contradiction.

Thus $P(\lambda^*)$ contains a point of S in its interior. That is, there exists a point $\bar{x} \in S$ such that $[\lambda^* d_1 + (1 - \lambda^*) d_2](\bar{x} - f) < 1$ and $d_i(\bar{x} - f) < 1$ for $i = 2, \dots, t$. Since P is S-free, $d_1(\bar{x} - f) > 1$. Thus there exists $\bar{\lambda} > \lambda^*$ such that $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](\bar{x} - f) = 1$. Note that, since $P(\bar{\lambda}) \cap \text{conv}(S)$ is bounded, there is a finite number of points of S in the interior of $P(\bar{\lambda})$. So we may choose $\bar{x}$ such that $\bar{\lambda}$ is maximum. Thus $P(\bar{\lambda})$ is S-free and $\bar{x}$ is in the relative interior of the facet of $P(\bar{\lambda})$ defined by $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](x - f) \le 1$.

Note that, for $i = 2, \dots, t$, if $d_i(x - f) \le 1$ defines a facet of P with a point of S in its relative interior, then it also defines a facet of $P(\bar{\lambda})$ with a point of S in its relative interior, because $P \subset P(\bar{\lambda})$. Thus repeating the above construction at most t times, we obtain a set B satisfying the lemma. □

Remark 12. *Let C and K be as in Lemma 11. Given any maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \le 1,\ i = 1, \dots, k\}$ containing C, then $a_1, \dots, a_k \in K^*$. If* rec(C) *is not full dimensional, then the origin is not an extreme point of K^*. Since all extreme points of K^* are contained in $\{0\} \cup \hat{K}$, in this case $\mathbf{cl}(\text{conv}(\hat{K})) = K^*$. Therefore, when* rec($C$) *is not full dimensional, every maximal S-free convex set containing C satisfies the statement of Lemma 11.*

Proof of Theorem 6.
We first show that any valid function is dominated by a function of the form ψ_B, for some maximal S-free convex set B containing f in its interior.

Let ψ be a valid function. By Lemma 7, we may assume that ψ is sublinear. Let $K = \{r \in \mathbb{R}^n \mid \psi(r) \le 1\}$, and let $\hat{K}$ be defined as in (8). Note that $K = B_\psi - f$. Thus, by Remark 10, $\sum_{r \in \mathbb{R}^n} \rho_K(r) s_r \ge 1$ is valid for $R_{f,S}$. Since ψ is sublinear, it follows from Theorem 8 that $\rho_K(r) \le \psi(r)$ for every $r \in \mathbb{R}^n$.

By Lemma 11, there exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \le 1,\ i = 1, \dots, k\}$ such that $a_i \in \mathbf{cl}(\text{conv}(\hat{K}))$ for $i = 1, \dots, k$.

Then

$$\psi(r) \geq \rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{y \in \mathbf{cl}(\mathrm{conv}(\hat{K}))} yr \geq \max_{i=1,\dots,k} a_i r = \psi_B(r).$$

This shows that ψ_B dominates ψ for all $r \in \mathbb{R}^n$.

To complete the proof of the theorem, we need to show that, given a maximal S-free convex set B, the function ψ_B is minimal. Consider any valid function ψ dominating ψ_B. Then $B_\psi \supseteq B$ and B_ψ is S-free. By maximality of B, $B = B_\psi$. By Theorem 8 and Remark 9, $\psi_B(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$, proving $\psi = \psi_B$.

□

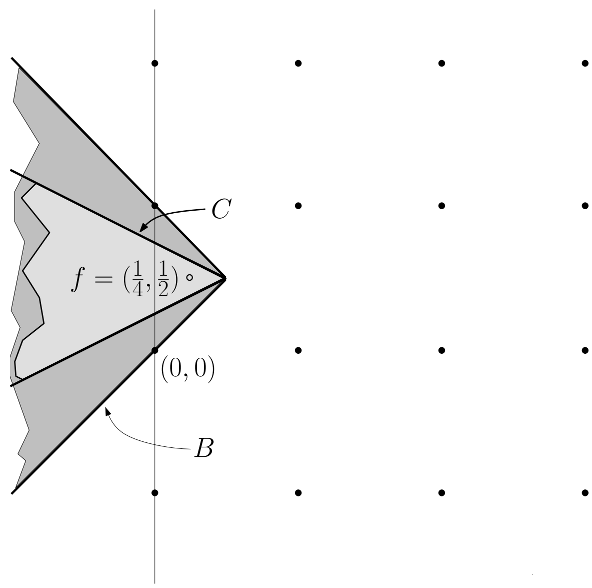

Figure 3: Illustration for Example 13

Example 13. *We illustrate the ideas behind the proof in the following two-dimensional example. Consider* $f = (\frac{1}{4}, \frac{1}{2})$ *and* $S = \{(x_1, x_2) \,|\, x_1 \geq 0)\}$. *See Figure 3. Then the function* $\psi(r) = \max\{4r_1 + 8r_2, 4r_1 - 8r_2\}$ *is a valid linear inequality for* $R_{f,S}$. *The corresponding* B_ψ *is* $\{(x_1, x_2) \,|\, 4(x_1 - \frac{1}{4}) + 8(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 8(x_2 - \frac{1}{2}) \leq 1\}$. *Note that* B_ψ *is not a maximal* S*-free convex set and it corresponds to* C *in Lemma 11. Following the procedure outlined in the proof, we obtain the maximal* S*-free convex set* $B = \{(x_1, x_2) \,|\, 4(x_1 - \frac{1}{4}) + 4(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 4(x_2 - \frac{1}{2}) \leq 1\}$. *Then,* $\psi_B(r) = \max\{4r_1 + 4r_2, 4r_1 - 4r_2\}$ *and* ψ_B *dominates* ψ.

Remark 14. *Note that* ψ *is nonnegative if and only if* $\mathrm{rec}(B_\psi)$ *is not full-dimensional. It follows from Remark 12 that, for every maximal* S*-free convex set* B *containing* B_ψ, *we have* $\psi_B(r) \leq \psi(r)$ *for every* $r \in \mathbb{R}^n$ *when* ψ *is nonnegative.*

A statement similar to the one of Theorem 6 was shown by Borozan-Cornuéjols [6] for a model similar to (6) when $S = \mathbb{Z}^n$ and the vectors s are elements of $\mathbb{R}^{\mathbb{Q}^n}$. In this case, it is easy to show that, for every valid inequality $\sum_{r \in \mathbb{Q}^n} \psi(r) s_r \geq 1$, the function $\psi : \mathbb{Q}^n \to \mathbb{R}$ is nonnegative. Remark 14 explains why in this context it is much easier to prove that minimal inequalities arise from maximal lattice-free convex sets.

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