LEHMAN MATRICES

Gérard Cornuéjols

Carnegie Mellon University

Pretty Structures 2011, Paris

The problem

Which pairs of square $0, 1$ matrices A, B satisfy

$$AB^T = E + kI$$

where E is the $n \times n$ matrix of all 1s and k is a positive integer.

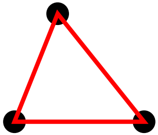
Example : Circulant $n \times n$ matrices C_r^n with r consecutive 1s, for positive integers n and r such that $n = rs + 1$ for some positive integer s.

$$\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & 1 & \\ & 1 & 1 & & 1 \\ 1 & & 1 & 1 & \\ & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Examples

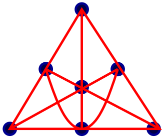
Finite projective planes $A = B$.

$$C_2^3 = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ 1 & & 1 \end{bmatrix}$$



$$AA^T = E + I$$

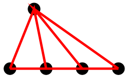
$$F_7 = \begin{bmatrix} 1 & 1 & & 1 & & & \\ & 1 & 1 & & 1 & & \\ & & 1 & 1 & & 1 & \\ & & & 1 & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 1 \end{bmatrix}$$



$$AA^T = E + 2I$$

Finite projective planes

A projective plane is *degenerate* if at least three of any four points belong to the same line.



All the lines of a nondegenerate finite projective plane have the same number of points.

Therefore, point-line incidence matrices A of nondegenerate finite projective planes are exactly the solutions of the equation

$$AA^T = E + kI.$$

We have $n = k^2 + k + 1$.

Number of projective planes for small orders k :

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
#	1	1	1	1	0	1	1	4	0	≥ 1	?	≥ 1	0	?	≥ 22

Bruck and Ryser 1949 Lam 1991

The New Infinite Family of Jonathan Wang JCTA 2011

$$W_2 = C_2^3 = \left[\begin{array}{c|c|c} 1 & 1 & \\ \hline & 1 & 1 \\ \hline 1 & & 1 \end{array}\right] \qquad W_3 = \left[\begin{array}{cc|cc|cc|cc} 1 & & & 1 & 1 & & & \\ 1 & & & 1 & & 1 & & \\ \hline & & 1 & & & 1 & 1 & \\ & & 1 & & & 1 & & 1 \\ \hline 1 & & & & 1 & & & 1 \\ & 1 & & & 1 & & & 1 \\ \hline & 1 & 1 & & & & 1 & \\ & 1 & & 1 & & & 1 & \end{array}\right]$$

$$W_4 = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & & & & 1 & & & & 1 & 1 & & & & & \\ 1 & & & & 1 & & & & 1 & & 1 & & & & \\ 1 & & & & 1 & & & & 1 & & & 1 & & & \\ \hline & & & 1 & & & & 1 & & & & 1 & 1 & & \\ & & & 1 & & & & 1 & & & & 1 & & 1 & \\ & & & 1 & & & & 1 & & & & 1 & & & 1 \\ \hline 1 & & & & & & 1 & & & & 1 & & & & 1 \\ & 1 & & & & & 1 & & & & 1 & & & & 1 \\ & & 1 & & & & 1 & & & & 1 & & & & 1 \\ \hline & & 1 & 1 & & & & & & 1 & & & & 1 & \\ & & 1 & & 1 & & & & & 1 & & & & 1 & \\ & & 1 & & & 1 & & & & 1 & & & & 1 & \\ \hline & 1 & & & & 1 & 1 & & & & & & 1 & & \\ & 1 & & & & 1 & & 1 & & & & & 1 & & \\ & 1 & & & & 1 & & & 1 & & & & 1 & & \end{array}\right]$$

Why are Jonathan Wang's matrices Lehman Matrices ?

$$
\left[\begin{array}{cc|cc|cc|cc}
1 & & & 1 & 1 & & & \\
1 & & & 1 & & 1 & & \\
\hline
 & & 1 & & & 1 & 1 & \\
 & & 1 & & & 1 & & 1 \\
\hline
1 & & & & 1 & & & 1 \\
 & 1 & & & 1 & & & 1 \\
\hline
 & 1 & 1 & & & & 1 & \\
 & 1 & & 1 & & & 1 &
\end{array}\right]
\left[\begin{array}{cc|cc|cc|cc}
 & & 1 & 1 & 1 & & & \\
1 & 1 & & & & 1 & & \\
\hline
 & & & & 1 & 1 & 1 & \\
 & & 1 & 1 & & & & 1 \\
\hline
1 & & & & & & 1 & 1 \\
 & 1 & & & 1 & 1 & & \\
\hline
1 & 1 & 1 & & & & & \\
 & & & 1 & & & 1 & 1
\end{array}\right]^T = E + 2I
$$

In general, $W_k \times Permut(W_k)^T = E + 2I$ where $Permut(W_k)$ is obtained from W_k by permuting the rows and columns in a certain way.

Motivation

Lehman matrices are key to understanding the *set covering problem* $\min\{c^T x : Mx \geq \mathbf{1},\ x \in \{0,1\}^n\}$, where M is a 0,1 matrix.

When can the set covering problem be solved by linear programming?

This can be done for every objective function c exactly when the *set covering polytope* $\{x \in [0,1]^n : Mx \geq \mathbf{1}\}$ is *integral*. When this occurs, the matrix M is said to be *ideal*.

THEOREM Lehman 1991

If M is a minimally nonideal matrix, then
either it is the point-line incidence matrix of a degenerate finite projective plane
or it has a unique core A which is a Lehman matrix :

$$AB^T = E + kI.$$

Motivation

A $0, 1$ matrix M is *Mengerian* if for every nonnegative integral vector c the linear program $\min\{c^T x : Mx \geq \mathbf{1},\ 0 \leq x \leq \mathbf{1}\}$ and its dual both have integral optimal solutions.

Many classical minimax theorems are associated with an underlying Mengerian matrix (e.g. Max Flow Min Cut theorem).

A $0, 1$ matrix is *minimally non-Mengerian* if it is not Mengerian but all its minors are.

Minimally non-Mengerian matrices are either minimally nonideal or ideal.

THEOREM Cornuejols, Guenin, Margot 2000

If a matrix is minimally non-Mengerian and minimally nonideal, then it is a Lehman matrix with $k = 1$.

Motivation

Analogy between the Lehman equation
$AB^T = E + kI$
and the equation
$AB^T = E - I$
that arises in the study of perfect graphs.

Minimally imperfect graphs satisfy $AB^T = E - I$ where A (B respectively) is the maximum clique (maximum stable set respectively) incidence matrix.
Graphs that satisfy this matrix equation are called *partitionable graphs*.

Basic results

THEOREM Bridges and Ryser 1969

Let A be an $n \times n$ Lehman matrix. Then

- A has the same number r of 1s in each row and column,
- B has the same number s of 1s in each row and column and $rs = n + k$,
- A^T is also a Lehman matrix.

REMARK

Let A be an r-regular Lehman matrix.

- If $k = 1$, then $|\det(A)| = r$,
- If A is a finite projective plane, then $|\det(A)| = (r-1)^{\frac{r(r-1)}{2}} r$.

There are Two Lehman matrices with $k = 1$ and $n = 8$

$$C_3^8 = \begin{bmatrix} 1 & 1 & 1 & & & & & \\ & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & & & \\ & & & 1 & 1 & 1 & & \\ & & & & 1 & 1 & 1 & \\ & & & & & 1 & 1 & 1 \\ 1 & & & & & & 1 & 1 \\ 1 & 1 & & & & & & 1 \end{bmatrix} \qquad D_8 = \begin{bmatrix} 1 & & 1 & & 1 & & & \\ & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & & & \\ & 1 & & 1 & & 1 & & \\ & & & & 1 & 1 & 1 & \\ & & & & & 1 & 1 & 1 \\ 1 & & & & & & 1 & 1 \\ 1 & 1 & & & & & & 1 \end{bmatrix}$$

D_8 was first discovered by Ding and is obtained from C_3^8 by adding a $0, \pm 1$ matrix of rank 1.

REMARK D_8 is Wang's matrix W_3 after permutation of rows and columns.

Lehman Matrices Related to Circulants C_r^n

Define the *level* of a r-regular $n \times n$ Lehman matrix A to be the minimum rank of $A' - C_r^n$ over all matrices A' isomorphic to A.

For example,
the circulant matrices C_r^n have level 0
and the matrix D_8 above has level 1.

To demonstrate that the notion of level is natural, we appeal to information complexity
(also known as Kolmogorov complexity).

A *parameter* is any $\alpha \in \{1, \ldots, n\}$.
We say that an $n \times n$ matrix A can be *described with k parameters* $\mathcal{P} = \{p_1, \ldots, p_k\}$ if there exists an algorithm that, given $\mathcal{P}$, constructs a matrix isomorphic to A.

THEOREM

If A is an $n \times n$ Lehman matrix of level t with $k = 1$, then A can be described with $O(t^4)$ parameters.

THEOREM Cornuéjols, Guenin, Tuncel 2009

A 0,1 matrix A is a Lehman matrix of level one if and only if A is isomorphic to $C_r^n + \Sigma$ where Σ is a $0, \pm 1$ matrix with four blocks.

$$
C_r^n + \Sigma = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & & 1 & 1 & 1 & & & & & \\
 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & 1 & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & & & & \\
 & & 1 & 1 & 1 & & 0 & 0 & 0 & 1 & 1 & & & \\
 & & & & & & & 1 & 1 & 1 & 1 & 1 & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & 1 & \\
 & & & & & & & & & 1 & 1 & 1 & 1 & 1 \\
1 & & & & & & & & & & 1 & 1 & 1 & 1 \\
1 & 1 & & & & & & & & & & 1 & 1 & 1 \\
1 & 1 & 1 & & & & & & & & & & 1 & 1 \\
1 & 1 & 1 & 1 & & & & & & & & & & 1
\end{bmatrix}
$$

Two parameters : Number of rows in a block $n_R \in \{1, \ldots, r-1\}$ and vertical shift tr with $t \in \{1, \ldots, s-1\}$. In the example, $n_R = 2$ and $t = 1$.

Top left point $(1, 1 + n_R)$; Columns $n_C = r - n_R$; Horizontal shift $tr - 1$.

Nearly self-dual Lehman matrices

Examples : C_2^5 and

$$P_{10} = \begin{bmatrix} & & & 1 & & & & & 1 & 1 \\ & & & 1 & & & 1 & 1 & & \\ & & & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & & & & \\ & & 1 & & & & & 1 & 1 & \\ & & 1 & & & & 1 & & & 1 \\ & 1 & & & & 1 & & & 1 & \\ & 1 & & & 1 & & & & & 1 \\ 1 & & & & 1 & & 1 & & & \\ 1 & & & & & 1 & & 1 & & \end{bmatrix}$$

A Lehman matrix A is *nearly self-dual* if :

- $A = A^T$ and
- its dual is $B = A + I$.

THEOREM

Let A be a nearly self-dual Lehman matrix which is r-regular. Then $r = 2, 3, 7$ or 57.

Hoffman and Singleton 1960 gave a construction for $r = 7$.
It is not known whether there is an example with $r = 57$.

Minimally nonideal matrices and Seymour's conjecture

The point-line matrices of degenerate finite projective planes are minimally nonideal.

The cores of most other known minimally nonideal matrices are Lehman matrices with $k = 1$.

We know only three exceptions : F_7, P_{10} and its dual. These three matrices play a central role in Seymour's conjecture about ideal binary matrices.

A 0,1 matrix is *binary* if the sum modulo 2 of any three rows is greater than or equal to at least one row of the matrix.

Seymour's conjecture 1977 states that
there are only three minimally nonideal binary matrices :
Their cores are F_7, P_{10} and its dual.

Open questions

Question 1 : Are there other infinite families of Lehman matrices with $k \geq 2$ beside nondegenerate finite projective planes ?

Question 2 : Is a Lehman matrix with $k = 1$ always the core of some minimally nonideal matrix ?

Question 3 : Is F_7 the only nondegenerate finite projective plane whose point-line matrix is the core of a minimally nonideal matrix ?

Beth Novick 1990 answered this question positively when "the core of" is removed from the statement.

Paper available on http ://integer.tepper.cmu.edu/