A Geometric View on Integer Lifting

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April 2012

Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Cutting Plane approach to solving MILP:

- First solve the LP relaxation. Basic optimal solution:

$$x_i \quad = \quad f_i + \sum_{j \in N} r^j x_j \quad \text{for } i \in B.$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add one or more cutting planes (Example: GMI cuts).

Theme of talk: Formulas for cut coefficients of nonbasic variables. Well understood for continuous variables. How does one deal with integer variables?

References on Integer Lifting

This talk

Conforti, Cornuéjols and Zambelli Operations Research 2011
Basu, Conforti, Campelo, Cornuéjols and Zambelli IPCO 2010
Basu, Conforti, Campelo, Cornuéjols, Zambelli MP 2012
Basu, Cornuéjols and Köppe MOR 2012
Cornuéjols, Kis and Molinaro working paper Dec 2011

Related work

Dey and Wolsey IPCO 2008
Dey and Wolsey SIOPT 2010

Formulas for Deriving Cutting Planes

The case of continuous nonbasic variables:

$$\begin{aligned} x &= f + \textstyle\sum_{j=1}^{k} r^j s_j \\ x &\in P \cap \mathbb{Z}^q \\ s &\geq 0 \end{aligned}$$

Every inequality cutting off the point $(\bar{x}, \bar{s}) = (f, 0)$ can be expressed in terms of the nonbasic variables s only, in the form $\sum_{j=1}^{k} \alpha_j s_j \geq 1$.

We are interested in "formulas" for deriving such inequalities.

More formally, we are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$$

is valid for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$.

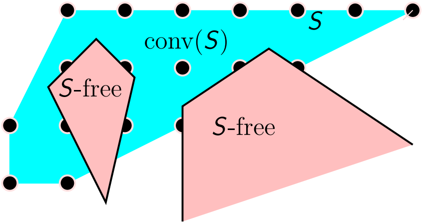
Such functions ψ are called valid functions with respect to f, P.

We are most interested in minimal valid functions.

Well Understood Model: Continuous Nonbasic Variables

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j s_j \\ x &\in S \\ s &\geq 0 \end{aligned}$$

where $S = P \cap \mathbb{Z}^q$ and P is a rational polyhedron.

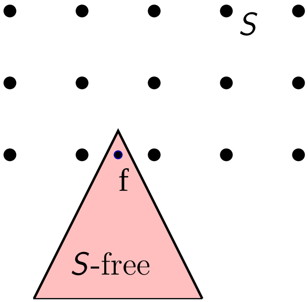


Well Understood Model: Continuous Nonbasic Variables

If $K = \{x \in \mathbb{R}^q : a_i(x - f) \leq 1, i = 1, \ldots, t\}$,
let $\psi_K(r) = max_{i=1,\ldots,t} a_i r$.

THEOREM Basu, Conforti, Cornuéjols, Zambelli SIDMA 2010

For every valid function ψ, there exists a maximal S-free convex set K with f in its interior such that $\psi_K \leq \psi$.
Conversely, if K is a maximal S-free convex set K with f in its interior, then ψ_K is a minimal valid function.



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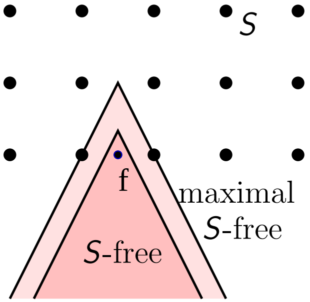

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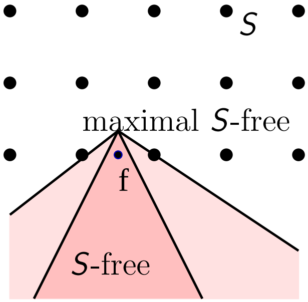


QUESTION: How should we deal with INTEGER nonbasic variables?

Integer Lifting

We now consider a system of the form

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j + \sum_{i=1}^{\ell} \rho^i y_i \\ x & \in & S := P \cap \mathbb{Z}^q \\ s & \geq & 0 \\ y & \geq & 0, \quad y \in \mathbb{Z}^\ell. \end{array}$$

We are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ and $\pi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j + \sum_{i=1}^{\ell} \pi(\rho^i) y_i \geq 1$$

is valid for every choice of integers k, ℓ and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$ and $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^q$.

DEFINITION The function π is called a lifting of ψ.

REMARK If ψ is a valid function and π is a minimal lifting of ψ, then $\pi \leq \psi$.

An Equivalent Formulation

The following formulation is equivalent for all $h : \mathbb{R}^q \longrightarrow \mathbb{Z}$.

$$\begin{aligned}
x &= f + \textstyle\sum_{j=1}^{k} r^j s_j + \sum_{i=1}^{\ell} \rho^i y_i \\
z &= 0 + \textstyle\sum_{j=1}^{k} 0 s_j + \sum_{i=1}^{\ell} h(\rho^i) y_i \\
z &\in \mathbb{Z} \\
x &\in S \\
s &\geq 0 \\
y &\geq 0, \quad y \in \mathbb{Z}^{\ell}.
\end{aligned}$$

Now we relax the integrality of the y variables.

This is a problem of the form that we understand: minimal inequalities correspond to maximal lattice-free convex sets. We have increased the dimension by 1.

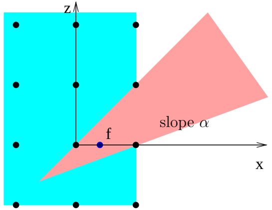
Let $\psi(r^j) := \tilde{\psi}(\begin{pmatrix} r^j \\ 0 \end{pmatrix})$ and $\pi^h(\rho^i) := \tilde{\psi}(\begin{pmatrix} \rho^i \\ h(\rho^i) \end{pmatrix})$

$$\sum_{j=1}^{k} \psi(r^j) s_j + \sum_{i=1}^{\ell} \pi^h(\rho^i) y_i \geq 1$$

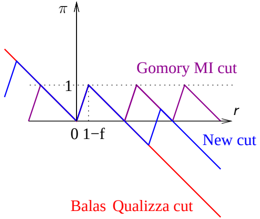
Example   Cornuéjols, Kis and Molinaro 2011

Consider a single basic row, with integer basic variable $x \leq 1$.

Introduce a new basic variable $z \in \mathbb{Z}$.



This yields a new cut that is identical to the Gomory mixed integer cut on the continuous variables but different on the integer variables: $\pi_\alpha(r) = \min\{\frac{-r+\lceil \alpha r \rceil}{f}, \frac{r}{1-f} - \frac{\lfloor \alpha r \rfloor (1-\alpha(1-f))}{\alpha f(1-f)}\}$.



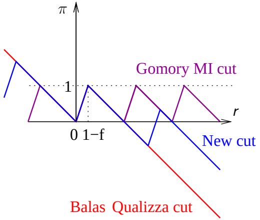
Integer Lifting

Conforti, Cornuéjols, Zambelli OR 2011

QUESTION Starting from a minimal valid function $\psi : \mathbb{R}^q \to \mathbb{R}$, what can we say about a minimal lifting function π?

We already observed that $\pi \leq \psi$. Are there regions R where we can guarantee that $\pi(r) = \psi(r)$ for all $r \in R$?

THEOREM Let ψ be a minimal valid function and π a minimal lifting of ψ. Then there exists $\epsilon > 0$ such that ψ and π coincide on a ball of radius ϵ centered at the origin.

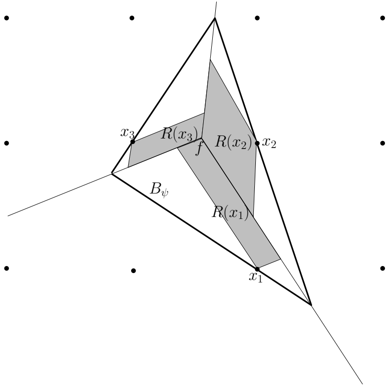


Let R be the region where π and ψ coincide.

What can be said about this region R?

Integer Lifting Basu, Campelo, Conforti, Cornuéjols, Zambelli IPCO 2010

THEOREM Let ψ be minimal and let π be a minimal lifting of ψ. Then $\pi(r) = \psi(r)$ for $r \in R = \bigcup_t R(x_t)$ where the union is taken over all integral points x_t on the boundary of the maximal S-free convex set B_ψ defining ψ.
Conversely, if $r \notin R$, there exists a minimal lifting π where $\pi(r) < \psi(r)$.



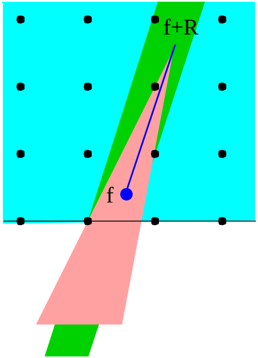
THEOREM Consider a minimal valid function ψ. Assume $S = \mathbb{Z}^n$. Then ψ has a unique minimal lifting π if and only if $R + \mathbb{Z}^q$ covers $\mathbb{R}^q$.

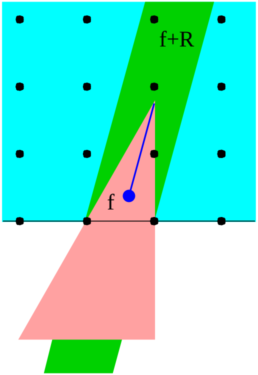
OPEN PROBLEM Does the result hold for general $S := P \cap \mathbb{Z}^n$?

Unique lifting

Conforti, Cornuéjols, Zambelli OR 2011

THEOREM Consider a minimal valid function ψ. Let L be the lineality space of $\text{conv}(S)$. If $R + (\mathbb{Z}^q \cap L)$ covers $\mathbb{R}^q$, then ψ has a unique minimal lifting π.



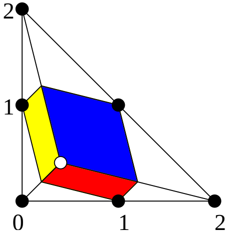


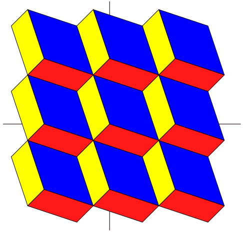
Body with a Unique Lifting

Dey, Wolsey IPCO 2008

THEOREM In the plane, the splits, Type 1 and Type 2 triangles have a unique lifting. The Type 3 triangles and most quadrilaterals do not.

Example: The region R and its integer translates.





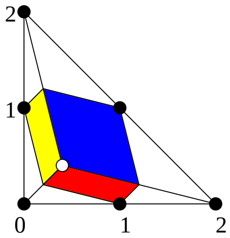
THEOREM Basu, Cornuéjols, Köppe 2012

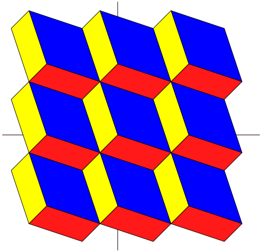
Let K be a maximal $\mathbb{Z}^q$-free simplicial polytope ($q \geq 2$). Then K is either a body with a unique lifting for all $f \in \text{int}(K)$, or a body with multiple liftings for all $f \in \text{int}(K)$.

Bodies with a Unique Lifting

Basu, Cornuéjols, Köppe 2012

THEOREM Let K be a maximal $\mathbb{Z}^q$-free simplex such that each facet of K has exactly one integer point in its relative interior. Then K is a body with a unique lifting if and only if all the vertices of K are integral, i.e., K is a unimodular transformation of $\text{conv}\{0, qe^1, \ldots, qe^q\}$.

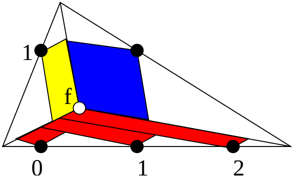


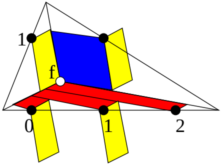


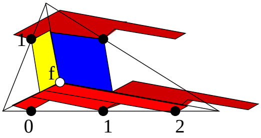
Proof Outline

Basu, Cornuéjols, Köppe 2012

Properties of region R in maximal $\mathbb{Z}^n$-free simplicial polytope:







THEOREM Let K be a maximal $\mathbb{Z}^n$-free simplicial polytope and let f be in its interior. Then $\mathrm{vol}(R/\mathbb{Z}^n)$ is an affine function of the coordinates of f.

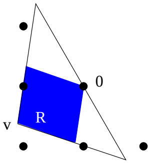
In particular,
the maximum volume occurs when f is at a vertex of K.

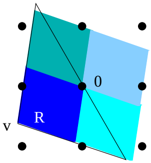
Proof Outline

Basu, Cornuéjols, Köppe 2012

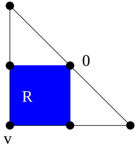
Let K be a maximal $\mathbb{Z}^q$-free simplex with exactly one integral point in the relative interior of each facet.

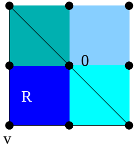
By making an affine unimodular transformation, we can assume that one of these points is 0. Let v be the opposite vertex of K.





LEMMA $\text{vol}(R) \leq 1$, where equality holds if and only if K is a unimodular transformation of the simplex $\text{conv}\{0, qe^1, \ldots, qe^q\}$.





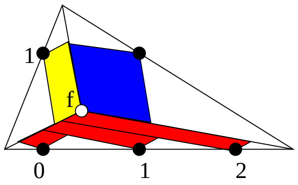
Bodies with a Unique Lifting

Basu, Cornuéjols, Köppe 2012

THEOREM

Let K be a maximal $\mathbb{Z}^q$-free 2-partitionable simplex with hyperplanes H_1, H_2 such that H_1 defines a facet of K and this is the only facet of K with more than one lattice point in its relative interior.

Then K is a body with a unique lifting if and only if $K \cap H_2$ is an affine unimodular transformation of $\text{conv}\{0, (q-1)e^1, \ldots, (q-1)e^{q-1}\}$.



Open Problems

- Generalize the theory of maximal S-free convex sets to

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & S := P \cap (\mathbb{Z}^q \times \mathbb{R}^r) \\ s & \geq & 0 \end{array}$$

- Consider $S := P \cap \mathbb{Z}^q$ and a minimal valid function ψ.
Is it true that
ψ has a unique minimal lifting π if and only if $R + \mathbb{Z}^q$ covers $\mathbb{R}^q$.

- Do the results about bodies with a unique lifting extend to non-simplicial polytopes?

Thank you

Papers available on http://integer.tepper.cmu.edu