Deciding Emptiness of the Gomory-Chvátal Closure is NP-Complete, Even for a Rational Polyhedron Containing No Integer Point

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Abstract. Gomory-Chvátal cuts are prominent in integer programming. The Gomory-Chvátal closure of a polyhedron is the intersection of all half spaces defined by its Gomory-Chvátal cuts. In this paper, we show that it is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a rational polyhedron is empty, even when this polyhedron contains no integer point. This implies that the problem of deciding whether the Gomory-Chvátal closure of a rational polyhedron P is identical to the integer hull of P is $\mathcal{NP}$-hard. Similar results are also proved for the $\{-1, 0, 1\}$-cuts and $\{0, 1\}$-cuts, two special types of Gomory-Chvátal cuts with coefficients restricted in $\{-1, 0, 1\}$ and $\{0, 1\}$, respectively.

Keywords: Integer programming, Gomory-Chvátal cuts, Gomory-Chvátal closure, integer hull, computational complexity

1 Introduction

Throughout this paper we will assume a knowledge of elementary integer programming definitions and results. One may use the book by Conforti et al. [4] as a reference.

We consider the integer program: min wx s.t. $Ax \le b$, $x \in \mathbb{Z}^n$, where $b \in \mathbb{Z}^m, w \in \mathbb{Z}^n$ and $A \in \mathbb{Z}^{m\times n}$. The polyhedron associated with the linear programming relaxation of the integer program is denoted by $P \equiv \{x \in \mathbb{R}^n : Ax \le b\}$. Polyhedra of this form, where $b \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$, are called *rational polyhedra*. The convex hull of all feasible solutions of the integer program is a polyhedron called the *integer hull* and denoted by P_I. An inequality of the form $cx \le \lfloor d \rfloor$ is called a *Gomory-Chvátal cut* of P if $cx \le d$ is valid for every $x \in P$, where $c \in \mathbb{Z}^n$. Gomory-Chvátal cuts were originally proposed by Gomory [9] as a method for solving integer programming and combinatorial optimization problems. Chvátal [3] introduced a notion of closure associated with these cuts. The *Gomory-Chvátal closure* of P is $P' \equiv \{x \in P : \; cx \le \lfloor d \rfloor \; \forall c \in \mathbb{Z}^n$ and $d \in$

$\mathbb{R}$ such that $cx \leq d$ is valid for $P\}$. Clearly, $P_I \subseteq P' \subseteq P$, and, from the theory of Gomory-Chvátal cuts, the second inclusion is strict when $P \neq P_I$.

The *separation problem* for the Gomory-Chvátal closure of a rational polyhedron (GC-Sep) is the following: Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$, either give a Gomory-Chvátal cut of P such that x^* violates the cut, or conclude that $x^* \in P'$. The *optimization problem* over the Gomory-Chvátal closure of a rational polyhedron (GC-Opt) is: Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{Z}^n$, either find a point $x^* \in P'$ that optimizes the function cx, or conclude that the optimal value of cx over P' is unbounded, or conclude that $P' = \emptyset$. It follows from a general result of Grötschel, Lovász and Schrijver [10] that solving GC-Sep in polynomial time is equivalent to solving GC-Opt in polynomial time. Eisenbrand [6] proved that GC-Sep is $\mathcal{NP}$-hard, which implies the $\mathcal{NP}$-hardness of GC-Opt.

In this paper, we show a stronger result: Given a rational polyhedron P such that $P_I = \emptyset$, it is $\mathcal{NP}$-complete to decide whether $P' = \emptyset$.

A Gomory-Chvátal cut is called a $\{-1,0,1\}$-*cut* (or $\{0,1\}$-*cut*) if the vector of variable coefficients $c \in \{-1,0,1\}^n$ (or $\{0,1\}^n$). The $\{-1,0,1\}$-*closure* (or $\{0,1\}$-*closure*) of P is $P'_{\{-1,0,1\}} \equiv \{x \in P : \ cx \leq \lfloor d \rfloor \ \forall c \in \{-1,0,1\}^n$ and $d \in \mathbb{R}$ such that $cx \leq d$ is valid for $P\}$ (or $P'_{\{0,1\}} \equiv \{x \in P : \ cx \leq \lfloor d \rfloor \ \forall c \in \{0,1\}^n$ and $d \in \mathbb{R}$ such that $cx \leq d$ is valid for $P\}$). We show that, given a polyhedron P such that $P_I = \emptyset$, it is $\mathcal{NP}$-complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ (or $P'_{\{0,1\}} = \emptyset$).

We borrowed some ideas from Mahajan and Ralphs [13] to construct polyhedra P in the proof of our $\mathcal{NP}$-completeness results. In their paper the following disjunctive infeasibility problem is proved to be $\mathcal{NP}$-complete: Given a polyhedron $P \subset \mathbb{R}^n$, does there exist $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$ such that $\{x \in P : \ \pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1\} = \emptyset$? The polyhedra P used in [13] are simplices, whereas our constructed polyhedra are convex hulls of $n+2$ or $n+3$ vectors in $\mathbb{R}^n$. The well-known partition problem is reduced to the disjunctive infeasibility problem in Mahajan and Ralphs' proof, whereas, in our proofs, the single constraint integer programming feasibility problem is reduced to the emptiness problem of Gomory-Chvátal closure, and the partition problem is reduced to the emptiness problems of $\{-1,0,1\}$-closure and $\{0,1\}$-closure.

The rest of the paper is organized as follows. In section 2, we prove the $\mathcal{NP}$-completeness of deciding whether the Gomory-Chvátal closure is empty. In section 3, we prove the $\mathcal{NP}$-completeness of deciding whether the $\{-1,0,1\}$-closure (or $\{0,1\}$-closure) is empty. Lastly, in section 4, we present conclusions and open questions.

2 Deciding Emptiness of the Gomory-Chvátal Closure

In this section, we prove two results. First we show that it is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a rational polyhedron P is empty.

We then show that this problem is $\mathcal{NP}$-complete, even when the polyhedron P is known to contain no integer point. We first observe that these problems are in the complexity class $\mathcal{NP}$.

Lemma 1. *Deciding whether the Gomory-Chvátal closure of a rational polyhedron is empty belongs to the complexity class $\mathcal{NP}$.*

Proof. Let $P \equiv \{x \in \mathbb{R}^n : \ Ax \le b\}$ be a rational polyhedron, where $b \in \mathbb{Z}^m$, and $A \in \mathbb{Z}^{m\times n}$. Chvátal [3] showed that there is only a finite number of inequalities needed to describe the Gomory-Chvátal closure, namely inequalities $uAx \le \lfloor ub \rfloor$ where $u \in \mathbb{R}^m$ is a vector satisfying $uA \in \mathbb{Z}^n$ and $0 \le u < 1$. Note that the integer vectors $d \equiv uA$ in these inequalities have components satisfying $-\sum_{i=1}^m |a_{ij}| \le d_j \le \sum_{i=1}^m |a_{ij}|$. Similarly, $d_0 \equiv \lfloor ub \rfloor$ satisfies $-\sum_{i=1}^m |b_i| \le d_0 \le \sum_{i=1}^m |b_i|$. Therefore the above inequalities are described by coefficients whose encoding size is polynomial in the size of the input. To certify that the Gomory-Chvátal closure is empty, we appeal to Helly's theorem: If the Gomory-Chvátal closure is empty, there exist n+1 of these inequalities whose intersection is empty. A list of $n+1$ such inequalities is a polynomial certificate that the Gomory-Chvátal closure is empty. □

Theorem 1. *It is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a rational polyhedron is empty.*

Proof. The theorem will be proved by polynomially reducing the following single constraint integer programming feasibility problem, which is known to be $\mathcal{NP}$-complete ([12]), to the problem of deciding whether $P' = \emptyset$ for a rational polyhedron P.

Single Constraint Integer Programming Feasibility Problem: Given a finite set of non-negative integers $\{a_i\}_{i=1}^s$ and a non-negative integer b, is there a set of non-negative integers $\{x_i\}_{i=1}^s$ satisfying $\sum_{i=1}^s a_i x_i = b$?

We consider the Single Constraint Integer Programming Feasibility Problem with $s = n-1$ and $n \ge 3$. We assume without loss of generality that the greatest common divisor of $a_1, a_2, \cdots, a_{n-1}$ is 1, and $2 < a_1 < a_2 < \cdots < a_{n-1} < b$. So $b \ge n+2$. Let $r = n + 1 + \frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbb{Z}_+$. We will show:

Reduction:

The Single Constraint Integer Programming Feasibility Problem can be polynomially reduced to the problem of deciding whether $P' = \emptyset$ for the polyhedron $P \subseteq \mathbb{R}^n$ that is the convex hull of the following $n+3$ vectors: $v_1 = (\frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b})$, $v_2 = (0, \frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b}), \cdots, v_{n-1} = (0, \cdots, 0, \frac{1}{2b}, \frac{1}{2b})$, $v_n = (0, \cdots, 0, \frac{1}{2} + \frac{3}{2r})$, $v_{n+1} = (a_1, a_2, \cdots, a_{n-1}, -b + \frac{1}{2})$, $v_{n+2} = ((1-r)a_1, (1-r)a_2, \cdots, (1-r)a_{n-1}, (r-1)b + 1)$, and $v_{n+3} = (0, \cdots, 0, \frac{1}{2r})$.

To show that this reduction is correct, we prove the following two claims; we then observe that converting the vectors $v_1, v_2, \cdots, v_{n+3}$ into an inequality description of P can be done in polynomial time.

Claim 1. There is a set of non-negative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ only if $P' = \emptyset$.

Proof. Consider an inequality $cx \leq q$, where $c = (c_1, c_2, \cdots, c_n)$, $c_i = -w_i$ for $1 \leq i \leq n-1$, $c_n = -1$, and $q = \max\{-\frac{1}{2b}, -\frac{1}{2r}\} = -\frac{1}{2b}$. Because $1 < a_i < b$ for $1 \leq i \leq n-1$ and $\sum_{i=1}^{n-1} a_i w_i = b$, it is easy to verify that $-1 < cv_i \leq -\frac{1}{2b}$ for $1 \leq i \leq n-1$. In addition, $-1 < cv_n < -\frac{1}{2}$, $cv_{n+1} = -\frac{1}{2}$, $cv_{n+2} = -1$, and $cv_{n+3} = -\frac{1}{2r}$. So $cx \leq q$ is valid for P, and the associated Gomory-Chvátal inequality is $cx \leq \lfloor q \rfloor$ $(= -1)$. We can see that v_{n+2} is the only vector in P that satisfies the inequality $cx \leq \lfloor q \rfloor$. Consider the inequality $fx \leq g$, where $fx = x_n$ and $g = (r-1)b + 1$. It can be easily checked that every v_i satisfies $fx \leq g$, so $fx \leq g$ is valid for P, and $fx \leq \lfloor g \rfloor$ $(= \lfloor rb \rfloor - b + 1)$ is a Gomory-Chvátal inequality of P. Since $rb \notin \mathbb{Z}_+$, $fx \leq \lfloor g \rfloor$ is violated by v_{n+2}. Now we can conclude that $P' = \emptyset$ and Claim 1 is proved.

Claim 2. There is a set of non-negative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ if $P' = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \cdots, 0, \frac{1}{2r} + \frac{1}{2})$. So $v_0 \in P$ because $v_0 = \alpha v_n + (1-\alpha) v_{n+3}$ for some $0 < \alpha < 1$. Let $cx \leq \lfloor q \rfloor$ be a Gomory-Chvátal inequality of P that is violated by v_0, where $c = (c_1, c_2, \cdots, c_n) \in \mathbb{Z}^n$ and $cx \leq q$ is valid for P. Then $c_n \neq 0$, otherwise we would have $0 = cv_0 \leq q$, contradicting that $cv_0 > \lfloor q \rfloor$.

Let $\Delta \equiv \sum_{i=1}^{n-1} c_i a_i - c_n b$. First, we show that $c_n \leq -1$ by deriving contradiction in the following two cases:

Case 1. $c_n \geq 1$ and $\Delta \geq 1$. If $\frac{c_n}{2r} \leq \Delta - 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} \leq \Delta - 1 + \frac{c_n}{2} < \lfloor \Delta + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, which contradicts that $cv_0 > \lfloor q \rfloor$. If $\frac{c_n}{2r} > \Delta - 1$, then $cv_n - cv_0 = \frac{c_n}{r} > 2(\Delta - 1)$. If $\Delta \geq 2$, then $cv_n - cv_0 > 2$, so $cv_0 < cv_n - 2 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction to that $cv_0 > \lfloor q \rfloor$. If $\Delta = 1$, then $\frac{c_n}{r} \geq 1$. Otherwise, if $\frac{c_n}{r} < 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < \frac{1}{2} + \frac{c_n}{2} \leq \lfloor 1 + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. Since $\frac{c_n}{r} \geq 1$, $cv_n - cv_0 \geq 1$, therefore $cv_0 \leq cv_n - 1 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction again.

Case 2. $c_n \geq 1$ and $\Delta \leq 0$. Because $r > n+1 > 3$, $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < c_n \leq (1-r)\Delta + c_n = cv_{n+2}$. Hence, $cv_0 < \lfloor cv_{n+2} \rfloor \leq \lfloor q \rfloor$, a contradiction.

It is easy to see that $c_n = -1$. Otherwise, if $c_n \leq -2$, then $cv_{n+3} - cv_0 = -\frac{c_n}{2} \geq 1$, which implies $cv_0 \leq cv_{n+3} - 1 < \lfloor cv_{n+3} \rfloor \leq \lfloor q \rfloor$, a contradiction.

Now we show that $\Delta = 0$. If $\Delta \geq 1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2} < 0 < \Delta - \frac{1}{2} = cv_{n+1}$, implying $cv_0 < \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. If $\Delta \leq -1$, then, because $r > 3$, $cv_0 < 0 < (1-r)\Delta - 1 = cv_{n+2}$, a contradiction again.

We claim that $c_i \leq 0$ for $i = 1, 2, \cdots, n-1$. Otherwise, if $c_i \geq 1$ for some $1 \leq i \leq n-1$, then $cv_0 < 0 \leq \frac{c_i}{2b} - \frac{1}{2b} = cv_i$, a contradiction.

Now let $w_i = -c_i$ for $1 \leq i \leq n-1$. Then Claim 2 is proved.

To complete the proof, it suffices to show that a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbb{Z}^{m \times n}$ and $\tilde{b} \in \mathbb{Z}^m$, can be obtained in polynomial time from the vectors $v_1, v_2, \cdots, v_{n+3}$. We can see from the coordinates of $v_1, v_2, \cdots, v_n$ and v_{n+3} that P is a n-dimensional polyhedron. Let i be a counter looping through $n, n+1$ and $n+2$. For every i vectors of $v_1, v_2, \cdots, v_{n+3}$, check if they are on a unique hyperplane by solving linear equations. If yes, then further check if the $n+3-i$ other vectors are all on one side of the hyperplane. If yes again, then the equation $\tilde{c}x = \tilde{d}$ of the hyperplane with integral $\tilde{c}$ and $\tilde{d}$ whose greatest common divisor is 1 yields a linear inequality of $\tilde{A}x \leq \tilde{b}$. One can easily see that this process takes polynomial time and the size of $\tilde{A}$ and $\tilde{b}$ is polynomial in the size of $v_1, v_2, \cdots, v_{n+3}$. □

Theorem 2. *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its Gomory-Chvátal closure is empty.*

Proof. We build on Theorem 1 and show that the polytope P that was used in the reduction contains no integer point.

This is equivalent to showing that $P^d \equiv P \cap \{x \in \mathbb{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [-b+1, nb+1]$. Let P_1 be the convex hull of $v_1, v_2, \cdots, v_n, v_{n+1}$ and v_{n+3}, and let P_2 be the convex hull of $v_1, v_2, \cdots, v_n, v_{n+2}$ and v_{n+3}. Since $v_0 = \frac{r-1}{r} v_{n+1} + \frac{1}{r} v_{n+2}$ and $v_0 = \alpha v_n + (1-\alpha) v_{n+3}$ for some $0 < \alpha < 1$, it is sufficient to show: (a) $P_1^d \equiv P_1 \cap \{x \in \mathbb{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [-b+1, 0]$; (b) $P_2^d \equiv P_2 \cap \{x \in \mathbb{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [1, nb+1]$.

We first prove (a). Because $v_1, v_2, \cdots, v_n, v_{n+3} \in \{x \in \mathbb{R}^n : x_n > 0\}$ and $v_{n+1} \in \{x \in \mathbb{R}^n : x_n < 0\}$, it is easy to verify by calculation that P_1^d, where d is an integer in $[-b+1, 0]$, is the convex hull of the $n+1$ vectors:

$$
\begin{array}{c}
(\frac{1-2bd}{2b^2-b+1}a_1 + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, \frac{1-2bd}{2b^2-b+1}a_2, \frac{1-2bd}{2b^2-b+1}a_3, \cdots, \frac{1-2bd}{2b^2-b+1}a_{n-1}, d),\\
(\frac{1-2bd}{2b^2-b+1}a_1, \frac{1-2bd}{2b^2-b+1}a_2 + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, \frac{1-2bd}{2b^2-b+1}a_3, \cdots, \frac{1-2bd}{2b^2-b+1}a_{n-1}, d),\\
\cdots\cdots\cdots\cdots\cdots\\
(\frac{1-2bd}{2b^2-b+1}a_1, \frac{1-2bd}{2b^2-b+1}a_2, \cdots, \frac{1-2bd}{2b^2-b+1}a_{n-2}, \frac{1-2bd}{2b^2-b+1}a_{n-1} + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, d),\\
(\frac{r+3-2rd}{2rb+3}a_1, \frac{r+3-2rd}{2rb+3}a_2, \frac{r+3-2rd}{2rb+3}a_3, \cdots, \frac{r+3-2rd}{2rb+3}a_{n-1}, d),\\
(\frac{1-2rd}{2rb+1-r}a_1, \frac{1-2rd}{2rb+1-r}a_2, \frac{1-2rd}{2rb+1-r}a_3, \cdots, \frac{1-2rd}{2rb+1-r}a_{n-1}, d).
\end{array}
$$

Indeed, the first n vectors above are obtained by intersecting the hyperplane $x_n = d$ with the line segment $v_i v_{n+1}$, for $i = 1, 2, \cdots, n$, and the last vector is obtained by intersecting the hyperplane $x_n = d$ with the line segment $v_{n+3} v_{n+1}$

Since $P_1^d \subset \{x \in \mathbb{R}^n : x_n = d\}$, we only need to consider the convex hull of the following $n+1$ vectors in $\mathbb{R}^{n-1}$: $\frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_1$, $\frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_2, \cdots, \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_{n-1}$, $\frac{r+3-2rd}{2rb+3}a$ and $\frac{1-2rd}{2rb+1-r}a$, where $a \equiv (a_1, a_2, \cdots, a_{n-1})$ and e_i is the i-th unit vector. Let $\tilde{P}_1^d$ denote the convex hull. Since $r < b$ and $d \geq -b+1$, it is easy to verify that $0 < \frac{1-2bd}{2b^2-b+1} < \frac{1-2rd}{2rb+1-r} < \frac{r+3-2rd}{2rb+3} < 1$. So $\tilde{P}_1^d \subseteq Q_1^d$, where Q_1^d is the convex hull of the $n+1$ vectors:

$$
\begin{aligned}
z_0 &\equiv \frac{1-2bd}{2b^2-b+1}a,\\
z_1 &\equiv \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_1,\\
z_2 &\equiv \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_2,\\
&\cdots\cdots\cdots
\end{aligned}
$$

$$z_{n-1} \equiv \tfrac{1-2bd}{2b^2-b+1}a + \tfrac{b+d-\frac{1}{2}}{2b^2-b+1}e_{n-1},$$
$$z_n \equiv \tfrac{r+3-2rd}{2rb+3}a.$$

To prove (a), it suffices to show the following claim.

Claim 3. There is no integer point in Q_1^d.

Proof. By contradiction, suppose $\tilde{v} \in Q_1^d \cap \mathbb{Z}^{n-1}$. Then there must exist a vector $\tilde{v}' \equiv \beta_0 a$, where $0 < \beta_0 < 1$, such that $\|\tilde{v} - \tilde{v}'\|_\infty \equiv \max_{1 \le i \le n-1} |\tilde{v}_i - \tilde{v}'_i| \le \frac{b+d-\frac{1}{2}}{2b^2-b+1} \le \frac{b-\frac{1}{2}}{2b^2-b+1} < \frac{1}{2b}$. From the construction of P, it is easy to see that $0 < \tilde{v}_i < a_i$ for $i = 1, 2, \cdots, n-1$. Because the greatest common divisor of $a_1, a_2, \cdots, a_{n-1}$ is 1, there exists no integer point on the line segment connecting 0 and a except for the two end points. Therefore, there exists $1 \le i_0 \le n-2$ such that $(\tilde{v}_{i_0}, \tilde{v}_{n-1})$ is not on the line segment connecting $(0,0)$ and (a_{i_0}, a_{n-1}) in $\mathbb{R}^2$. To derive contradiction, we show below that $\|(\tilde{v}_{i_0}, \tilde{v}_{n-1}) - (\tilde{v}'_{i_0}, \tilde{v}'_{n-1})\|_\infty = \max\left(|\tilde{v}_{i_0} - \tilde{v}'_{i_0}|,\ |\tilde{v}_{n-1} - \tilde{v}'_{n-1}|\right) \ge \frac{1}{2(b-1)}$.

Let L denote the line segment connecting $(0,0)$ and (a_{i_0}, a_{n-1}) in $\mathbb{R}^2$. We know that $(\tilde{v}'_{i_0}, \tilde{v}'_{n-1})$ is on L. Because the integer points between 0 and (a_{i_0}, a_{n-1}) that are not on L are symmetric across $(\frac{a_{i_0}}{2}, \frac{a_{n-1}}{2})$, we may assume without loss of generality that $\frac{\tilde{v}_{n-1}}{\tilde{v}_{i_0}} < \frac{a_{n-1}}{a_{i_0}}$. It is not hard to see that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\tilde{v}_{i_0}, \tilde{v}_{n-1})$ is attained at a point on the segment L connecting $(\frac{\tilde{v}_{n-1} a_{i_0}}{a_{n-1}}, \tilde{v}_{n-1})$ and $(\tilde{v}_{i_0}, \frac{\tilde{v}_{i_0} a_{n-1}}{a_{i_0}})$. Since $\tilde{v}_{i_0} > \frac{\tilde{v}_{n-1} a_{i_0}}{a_{n-1}}$, $\|(\tilde{v}_{i_0}, \tilde{v}_{n-1}) - (\frac{\tilde{v}_{n-1} a_{i_0}}{a_{n-1}}, \tilde{v}_{n-1})\|_\infty \ge \frac{1}{a_{n-1}} \ge \frac{1}{b-1}$. Because $a_{n-1} \ge a_{i_0}$, $\|(\tilde{v}_{i_0}, \tilde{v}_{n-1}) - (\tilde{v}_{i_0}, \frac{\tilde{v}_{i_0} a_{n-1}}{a_{i_0}})\|_\infty \ge \frac{1}{b-1}$. So it follows that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\tilde{v}_{i_0}, \tilde{v}_{n-1})$ is no less than $\frac{1}{2(b-1)}$. Therefore, $\|(\tilde{v}_{i_0}, \tilde{v}_{n-1}) - (\tilde{v}'_{i_0}, \tilde{v}'_{n-1})\|_\infty \ge \frac{1}{2(b-1)}$. Claim 3 is proved.

Next we prove (b). Because $v_1, v_2, \cdots, v_n, v_{n+3} \in \{x \in \mathbb{R}^n :\ x_n < 1\}$ and $v_{n+2} \in \{x \in \mathbb{R}^n :\ x_n > 1\}$, we know by calculation that P_2^d, where d is an integer in $[1, nb+1]$, is the convex hull of the $n+1$ vectors:

$$\left(\tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_1 + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_2, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_3, \cdots, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_{n-1}, d\right),$$
$$\left(\tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_1, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_2 + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_3, \cdots, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_{n-1}, d\right),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\left(\tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_1, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_2, \cdots, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_{n-2}, \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_{n-1} + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, d\right),$$
$$\left(\tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_1, \tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_2, \tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_3, \cdots, \tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_{n-1}, d\right),$$
$$\left(\tfrac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}a_1, \tfrac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}a_2, \tfrac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}a_3, \cdots, \tfrac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}a_{n-1}, d\right).$$

So we just need to prove that the convex hull of the following $n+1$ vectors in $\mathbb{R}^{n-1}$, denoted by $\tilde{P}_2^d$, contains no integer points:

$$\tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a+\tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_1,\ \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a+\tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_2,\cdots,$$
$$\tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a+\tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_{n-1},\ \tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a,\ \tfrac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}a.$$

The following properties can be verified by calculation, using $d \leq nb+1$ and $b > r$:

1. $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} < \frac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}} < \frac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}} < 0$, and each of the three terms strictly decreases as d increases.
2. For $d = kb+h$, where integers k and h satisfy $0 \leq k \leq \lfloor r-1 \rfloor = n$ and $0 < h < b$, $\frac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_i < -ka_i$ and $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_i + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < -ka_i$ for $i = 1, 2, \cdots, n-1$.
3. For $d = kb$, where integer k satisfies $1 \leq k \leq \lfloor r-1 \rfloor = n$, $\frac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a_i < -(k-1)a_i$, $-ka_i < \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_i$, and $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a_i + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < -(k-1)a_i$ for $i = 1, 2, \cdots, n-1$.

By the above property 1, $\tilde{P}_2^d \subseteq Q_2^d$, where Q_2^d is the convex hull of the $n+1$ vectors:

$$\begin{aligned}
y_0 &\equiv \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a,\\
y_1 &\equiv \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_1,\\
y_2 &\equiv \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_2,\\
&\cdots\cdots\cdots\\
y_{n-1} &\equiv \tfrac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \tfrac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_{n-1},\\
y_n &\equiv \tfrac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}a.
\end{aligned}$$

To prove (b), it suffices to show that Q_2^d contains no integer points. Given the properties 2 and 3 and the fact that $\frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < \frac{1}{2b}$, the proof is very similar to that of Claim 3. The theorem is proved. □

3 Deciding Emptiness of the $\{-1,0,1\}$-Closure (or $\{0,1\}$-Closure) of a Rational Polyhedron with No Integer Point

In this section, we first prove $\mathcal{NP}$-completeness of deciding emptiness of the $\{-1,0,1\}$-closure of a rational polyhedron containing no integer point, and then prove the same for the $\{0,1\}$-closure as a corollary.

Theorem 3. *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its $\{-1,0,1\}$-closure is empty.*

Proof. We will prove the theorem by polynomially reducing the following partition problem, which is known to be $\mathcal{NP}$-complete [7], to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for a polyhedron P with no integer points.

Partition Problem: Given a finite set of positive integers $S = \{a_i\}_{i=1}^s$, is there a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$?

Let $b \equiv \frac{1}{2}\sum_{1\le i\le s} a_i$. We assume without loss of generality that $a_i < b$ for $i = 1, 2, \cdots, s$ and that the greatest common divisor of $a_1, a_2, \cdots, a_s$ is 1. We also assume without loss of generality that $s \ge 8$ (because the Partition Problem with fixed s can be formulated as an integer program with only s binary variables, which can be solved in polynomial time [11]).

First, we prove that b can be assumed to be greater than or equal to $s+3$. Note that a Partition Problem with $b < s+3$ can be polynomially converted to another Partition Problem with $S' = S \cup \{\sum_{1\le i\le s} a_i + 1, \sum_{1\le i\le s} a_i + 1\}$. Let $s' = |S'|$. So $s' = s+2$. It is easy to see that the Partition Problem with S has a feasible partition if and only if the Partition Problem with S' has a feasible partition. Let $b' \equiv b + (\sum_{1\le i\le s} a_i + 1)$. Then $b' = \frac{3}{2}\sum_{1\le i\le s} a_i + 1 \ge \frac{3s}{2} + 1 = s + 5 + (\frac{s}{2} - 4) \ge (s+2) + 3 = s' + 3$.

Now we consider the Partition Problem with $s = n-1$, $n \ge 9$ and $b \ge n+2$. Let $r = n + 1 + \frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbb{Z}_+$. We only need to show that the Partition Problem can be polynomially reduced to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for the same polyhedron P as constructed in the proof of Theorem 2, *i.e.*, the convex hull of the $n+3$ vectors: $v_1 = (\frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b})$, $v_2 = (0, \frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b})$, $\cdots$, $v_{n-1} = (0, \cdots, 0, \frac{1}{2b}, \frac{1}{2b})$, $v_n = (0, \cdots, 0, \frac{1}{2} + \frac{3}{2r})$, $v_{n+1} = (a_1, a_2, \cdots, a_{n-1}, -b + \frac{1}{2})$, $v_{n+2} = ((1-r)a_1, (1-r)a_2, \cdots, (1-r)a_{n-1}, (r-1)b+1)$, and $v_{n+3} = (0, \cdots, 0, \frac{1}{2r})$. It suffices to prove the following two claims.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ only if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Consider an inequality $cx \le q$, where $c = (c_1, c_2, \cdots, c_n)$, $c_i = -1$ for $i \in K$, $c_i = 0$ for $i \in S\setminus K$, $c_n = -1$, and $q = -\frac{1}{2b}$. One can easily verify: $-1 < cv_i < -\frac{1}{2b}$ for $1 \le i \le n-1$, $cv_n = -\frac{1}{2} - \frac{3}{2r}$, and because $\sum_{i\in K} a_i = b$, $cv_{n+1} = -\frac{1}{2}$ and $cv_{n+2} = -1$. Hence, $cx \le q$ is valid for P. Since $c_i = -1$ or 0, the inequality $cx \le \lfloor q \rfloor$ $(= -1)$ is a $\{-1,0,1\}$-cut of P. From the value of cv_i for $1 \le i \le n+2$, we see that v_{n+2} is the only vector in P that satisfies the inequality $cx \le \lfloor q \rfloor$. Since $rb \notin \mathbb{Z}_+$ and $b = \sum_{i\in K} a_i$, there exists some $j \in K$ satisfying $ra_j \notin \mathbb{Z}_+$. Consider the inequality $fx \le g$, where $fx = -x_j$ and $g = (r-1)a_j$. Apparently, $g > 0$ and $g \notin \mathbb{Z}_+$. In addition, $fv_i < 0$ for $1 \le i \le n+1$ and $fv_{n+2} = g > 0$. It is obvious that the inequality $fx \le \lfloor g \rfloor$ $(= \lfloor (r-1)a_j \rfloor)$ is a $\{-1,0,1\}$-cut of P and that v_{n+2} violates this inequality. Therefore, $P'_{\{-1,0,1\}} = \emptyset$ and Claim 1 is proved.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \cdots, 0, \frac{1}{2r} + \frac{1}{2}) \in P$. Since $P'_{\{-1,0,1\}} = \emptyset$, v_0 violates an inequality $cx \le \lfloor q \rfloor$, where $c = (c_1, c_2, \cdots, c_n) \in \{-1,0,1\}^n$ and $cx \le q$ is valid for P. It is easy to see that $c_n \ne 0$. Otherwise, $0 = cv_0 \le q$, which contradicts that v_0 violates $cx \le \lfloor q \rfloor$.

Now we show that $\sum_{1\leq i\leq n-1} c_i a_i = c_n b$. If $c_n = 1$, then $cv_0 = \frac{1}{2r}+\frac{1}{2}$. In this case, if $\sum_{1\leq i\leq n-1} c_i a_i \geq c_n b+1$, then $\frac{3}{2} \leq cv_{n+1} \leq q$; if $\sum_{1\leq i\leq n-1} c_i a_i \leq c_n b-1$, then $n+1 < r \leq cv_{n+2} \leq q$. Hence, $cv_0 < 1 < q$, contradicting $cv_0 > \lfloor q \rfloor$. If $c_n = -1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2}$. In this case, if $\sum_{1\leq i\leq n-1} c_i a_i \geq c_n b + 1$, then $\frac{1}{2} \leq cv_{n+1} \leq q$; if $\sum_{1\leq i\leq n-1} c_i a_i \leq c_n b - 1$, then $n-1 < r-2 \leq cv_{n+2} \leq q$. So, $cv_0 < 0 < q$, a contradiction to $cv_0 > \lfloor q \rfloor$.

It is true that $c_n = -1$. Otherwise, $cv_0 = \frac{1}{2r} + \frac{1}{2}$ and $cv_{n+2} = 1 \leq q$, contradicting that v_0 violates $cx \leq \lfloor q \rfloor$.

We now claim that $c_i \leq 0$ for $1 \leq i \leq n-1$. Otherwise, suppose $c_j = 1$ for some $1 \leq j \leq n-1$. Then $cv_j = 0$. Since $cv_0 = -\frac{1}{2r} - \frac{1}{2}$ and $0 = cv_j \leq q$, a contradiction similar to the early ones can be derived. Therefore, Claim 2 is proved.

Using the same approach as shown in the end of the proof of Theorem 2, it is straight to polynomially obtain a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbb{Z}^{m\times n}$ and $\tilde{b} \in \mathbb{Z}^m$, from the vectors $v_1, v_2, \cdots, v_{n+3}$. The theorem is proved. □

Corollary 1. *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its $\{0,1\}$-closure is empty.*

Proof. The proof is similar to that of Theorem 3, hence we omit the details and only point out the differences.

The Partition Problem is polynomially reduced to the problem of deciding whether $P'_{\{0,1\}} = \emptyset$ for the polyhedron $P \subseteq \mathbb{R}^n$ that is the convex hull of the $n+3$ vectors: $v_1 = (-\frac{1}{2b}, 0, \cdots, 0, -\frac{1}{2b})$, $v_2 = (0, -\frac{1}{2b}, 0, \cdots, 0, -\frac{1}{2b}), \cdots, v_{n-1} = (0, \cdots, 0, -\frac{1}{2b}, -\frac{1}{2b})$, $v_n = (0, \cdots, 0, -\frac{1}{2} - \frac{3}{2r})$, $v_{n+1} = (-a_1, -a_2, \cdots, -a_{n-1}, b - \frac{1}{2})$, $v_{n+2} = ((r-1)a_1, (r-1)a_2, \cdots, (r-1)a_{n-1}, (1-r)b - 1)$, and $v_{n+3} = (0, \cdots, 0, -\frac{1}{2r})$.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\backslash K} a_i$ only if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 1 is similar to that of Claim 1 in the proof of Theorem 3 except that $c_i = 1$ for $i \in K$, $c_i = 0$ for $i \in S\backslash K$, $c_n = 1$, and $fx = x_j$.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\backslash K} a_i$ if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 2 is similar to but simpler than that of Claim 2 in the proof of Theorem 3. Here are two differences: First, we let $v_0 \equiv (0, 0, \cdots, 0, -\frac{1}{2r} - \frac{1}{2})$. Second, to show by contradiction that $\sum_{1\leq i\leq n-1} c_i a_i = c_n b$, we only consider the case that $c_n = 1$, and a contradiction can be derived due to $cv_0 < 0 < q$. □

4 Conclusions

In this paper, we proved that the problem of deciding whether the Gomory-Chvátal closure of a rational polyhedron P is empty is $\mathcal{NP}$-complete, even when

P is known to contain no integer point. Similar results are also proved for the $\{-1,0,1\}$-closure and $\{0,1\}$-closure of polyhedron.

There are several questions to which we have not found an answer yet. First, what if our attention is restricted to the polyhedra in the unit cube (denoted by $[0,1]^n$)? Namely,

(i) Is it $\mathcal{NP}$-complete to decide whether $P' = \emptyset$ for $P \subseteq [0,1]^n$?
(ii) Is it $\mathcal{NP}$-complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ for $P \subseteq [0,1]^n$?
(iii) Is it $\mathcal{NP}$-complete to decide whether $P'_{\{0,1\}} = \emptyset$ for $P \subseteq [0,1]^n$?

An interesting class of rational polyhedra is those for which $P' = P_I$. A well-known example in this family is due to Edmonds [5] for 1-matchings of undirected graphs $G = (V,E)$: $P = \{x \in \mathbb{R}^{|E|} : \ x(\delta(v)) \leq 1 \ \forall v \in V, \ x_e \geq 0 \ \forall e \in E\}$, where $\delta(v)$ is the set of edges incident on node v. Edmonds proposed a polynomial-time algorithm for solving GC-Opt for 1-matchings, and Padberg and Rao [14] devised a polynomial-time separation algorithm for b-matching polytopes, implying polynomial-time solvability of GC-Sep for 1-matchings. The question of deciding whether $P' = P_I$ for a rational polytope P is not known to be in $\mathcal{NP}$ and Theorem 2 implies that it is $\mathcal{NP}$-hard. But it is an open question whether the separation problem for the Gomory-Chvátal closure of polyhedra P that satisfy $P' = P_I$ is polynomially solvable, and similarly for the associated optimization problem.

(iv) Is there a polynomial algorithm to find a point in P_I or show that $P_I = \emptyset$ when we know that $P' = P_I$?
(v) Is there a polynomial algorithm to optimize over P_I when we know that $P' = P_I$?

We believe that the answers to the last two questions are positive. As evidence, we observe that the problem of deciding whether $P_I = \emptyset$ when we know that $P' = P_I$ is in the complexity class $\mathcal{NP} \cap$ co-$\mathcal{NP}$. We already observed (Lemma 1) that the problem is in $\mathcal{NP}$. To prove that it is in co-$\mathcal{NP}$, it suffices to exhibit a point $x \in \mathbb{Z}^n$ that satisfies $Ax \leq b$. It is well known that, if such a point exists, there is one whose encoding is polynomial in the size of the input [1]. Therefore a polynomial co-$\mathcal{NP}$ certificate exists for $P' = \emptyset$ when $P' = P_I$. On the other hand no obvious co-$\mathcal{NP}$ certificate is known for $P' = \emptyset$ in general.

As an example, consider the maximum weight stable set problem in a graph $G = (V,E)$, $\max\{wx : x \in P_I\}$ where $P = \{x \in \mathbb{R}^V_+ : x_i + x_j \leq 1 \text{ for } ij \in E\}$. We note that this problem is $\mathcal{NP}$-hard in general, but that it can be solved in polynomial time when $P' = P_I$. Indeed, Campelo and Cornuéjols [2] showed that P' is entirely described by the inequalities defining P together with the odd circuit inequalities $\sum_{i \in C} x_i \leq \frac{|C|-1}{2}$ for vertex sets C of odd cardinality that induce a circuit of G. The graphs for which these inequalities completely describe the stable set polytope P_I are called *t-perfect graphs.* These graphs are discussed in Chapter 68 of Schrijver's book [15]. Theorem 68.1 states that a maximum-weight stable set in a t-perfect graph can be found in polynomial time. This follows from the equivalence of optimization and separation [10] and the fact that the separation of odd circuit inequalities can be done in polynomial time by reduction to shortest path problems [8].

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