Date for Revised Manuscript: November 10, 2017

When the Gomory-Chvátal Closure Coincides with the Integer Hull

Gérard Cornuéjols · Yanjun Li

Abstract Gomory-Chvátal cuts are prominent in integer programming. The Gomory-Chvátal closure of a polyhedron P is the intersection of the half spaces defined by all its Gomory-Chvátal cuts. This set contains the integer hull of P, *i.e.*, the convex hull of the integer points in P. In this paper, we focus on polyhedra P with the property that their Gomory-Chvátal closure is identical to the integer hull of P. Boyd and Pulleyblank showed that deciding whether there exists an integer point in such a polyhedron is in the complexity class $\mathcal{NP} \cap$ co-$\mathcal{NP}$. This raises the possibility that integer linear programs with such constraint sets P might be polynomially solvable. From the recognition perspective, we give a negative result by proving that it is $\mathcal{NP}$-hard to decide whether a rational polyhedron belongs to this class. Specifically, we show that it is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a given rational polyhedron P is empty, even when it is known that the integer hull of P is empty. We prove similar results for the $\{-1, 0, 1\}$-cuts and $\{0, 1\}$-cuts, two special classes of Gomory-Chvátal cuts with variable coefficients restricted in $\{-1, 0, 1\}$ and $\{0, 1\}$, respectively.

Keywords Integer programming · Gomory-Chvátal cuts · Gomory-Chvátal closure · Integer hull · Computational complexity

Mathematics Subject Classification (2000) 90C10 · 90C57 · 90C60

1 Introduction

Throughout this paper we will assume a knowledge of elementary integer programming definitions and results. One may use the book by Conforti et al. [8] as a reference.

This work was supported in part by NSF grant CMMI1560828 and ONR grant N00014-15-12082.

Gérard Cornuéjols
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA
E-mail: gc0v@andrew.cmu.edu

Yanjun Li (Corresponding Author)
Krannert School of Management, Purdue University, West Lafayette, IN 47906, USA
E-mail: li14@purdue.edu
Phone: +001-765-494-4525
Fax: +001-765-494-9658

We consider the integer program: min wx s.t. $Ax \le b$, $x \in \mathbf{Z}^n$, where $b \in \mathbf{Z}^m$, $w \in \mathbf{Z}^n$ and $A \in \mathbf{Z}^{m\times n}$. Let $P \equiv \{x \in \mathbf{R}^n : \; Ax \le b\}$ denote the feasible region of the linear programming relaxation of this integer program. Polyhedra of this form, where $b \in \mathbf{Z}^m$ and $A \in \mathbf{Z}^{m\times n}$, are called *rational polyhedra*. The convex hull of $P \cap \mathbf{Z}^n$ is called the *integer hull* of P and is denoted by P_I. Meyer [29] showed that the integer hull P_I of a rational polyhedron P is itself a rational polyhedron.

An inequality of the form $cx \le \lfloor d \rfloor$ is called a *Gomory-Chvátal cut* of P if $c \in \mathbf{Z}^n$ and $cx \le d$ is valid for every $x \in P$. It follows from this definition that Gomory-Chvátal cuts are valid for $P \cap \mathbf{Z}^n$. Gomory-Chvátal cuts were originally proposed by Gomory [20] as a method for solving integer programs. Chvátal [7] introduced a notion of closure associated with these cuts. The *Gomory-Chvátal closure* of P is $P' \equiv \{x \in P : \; cx \le \lfloor d \rfloor \; \forall c \in \mathbf{Z}^n$ and $d \in \mathbf{R}$ such that $cx \le d$ is valid for $P\}$. Clearly, $P_I \subseteq P' \subseteq P$, and, from the theory of Gomory-Chvátal cuts, the second inclusion is strict when $P \neq P_I$, implying that P' is typically a tighter relaxation of P_I than P. Chvátal [7] showed that the Gomory-Chvátal closure of a rational polytope is again a rational polytope, and Schrijver [32] showed, more generally, that the Gomory-Chvátal closure of a rational polyhedron is again a rational polyhedron. Hartmann, Queyranne and Wang [23] gave a necessary and sufficient condition for a facet-defining inequality of P_I to be a Gomory-Chvátal cut. The number of Gomory-Chvátal cuts needed to describe the Gomory-Chvátal closure of a rational polyhedron can be very large. For example, for the matching problem, P is defined by the degree and nonnegativity constraints; its Gomory-Chvátal closure P' is the matching polytope, which comprises the blossom constraints in addition to the constraints defining P. For the matching problem on a complete graph, the blossom constraints define facets of P' [31] and their number is exponential in the size of P.

Optimizing a linear function over the Gomory-Chvátal closure P' of a rational polyhedron P often gives a better bound on the optimal objective value of the integer program than optimizing over P. The *optimization problem* over the Gomory-Chvátal closure of P (GC-Opt) is: Given $P \subseteq \mathbf{R}^n$ and a vector $c \in \mathbf{Z}^n$, either find a point $x^* \in P'$ that optimizes the function cx, or conclude that the optimal value of cx over P' is unbounded, or conclude that $P' = \emptyset$. The *separation problem* for the Gomory-Chvátal closure of P (GC-Sep) is: Given $P \subseteq \mathbf{R}^n$ and a point $x^* \in \mathbf{R}^n$, either find a Gomory-Chvátal cut of P such that x^* violates the cut, or conclude that $x^* \in P'$. It follows from a general result of Grötschel, Lovász and Schrijver [22] that GC-Sep can be solved in polynomial time if and only if GC-Opt can be solved in polynomial time.

A negative result proved by Eisenbrand [15] is that GC-Sep is $\mathcal{NP}$-hard, which implies the $\mathcal{NP}$-hardness of GC-Opt as well. Eisenbrand's proof is through a polynomial reduction to GC-Sep of the weighted binary clutter problem, known to be $\mathcal{NP}$-complete. Despite this complexity result, the quality of the bound provided by solving GC-Opt has been tested empirically. Fischetti and Lodi [16] were able to estimate the fraction of the integrality gap closed by the Gomory-Chvátal closure on some hard instances from MIPLIB 3.0. They formulated GC-Sep through a mixed integer programming model and solved it using a general-purpose mixed integer programming solver. On these MIPLIB instances, they reported that by optimizing over the Gomory-Chvátal closure instead of the standard linear programming relaxation, the fraction of the gap closed is often quite significant, in excess of 80% on about half the instances. In fact, the gap was closed completely on over a quarter of these MIPLIB instances.

In this paper, we study the rational polyhedra P whose Gomory-Chvátal closure P' is identical to the integer hull P_I. There are two questions about this type of polyhedra that we think are fundamental, one on optimization and the other on recognition:

(Q1) Given a rational polyhedron $P = \{x \in \mathbf{R}^n : \ Ax \leq b\}$ for which we know that $P' = P_I$, are GC-Opt and GC-Sep polynomially solvable?

(Q2) Given a rational polyhedron P, can one decide in polynomial time whether $P' = P_I$?

Question (Q1) is a "promise problem", *i.e.*, the algorithms only have to output a correct answer in polynomial time if the input satisfies some specified property, here $P' = P_I$. We do not have a positive or negative answer for (Q1). However, Boyd and Pulleyblank [2] observed that, given a rational polyhedron P for which we know that $P' = P_I$, the feasibility problem (that is, whether or not $P' = P_I = \emptyset$) belongs to the complexity class $\mathcal{NP} \cap$ co-$\mathcal{NP}$. This can be regarded as strong evidence that the feasibility problem is probably not $\mathcal{NP}$-complete, implying that the two problems in (Q1) are probably not $\mathcal{NP}$-hard [19]. We present three known integer linear programs, whose linear programming relaxation polyhedron P satisfies $P' = P_I$, that can be solved in polynomial time.

For (Q2), we do not know whether the decision problem belongs to the complexity class $\mathcal{NP}$ but we will prove that it is $\mathcal{NP}$-hard. Specifically, we provide the following negative answer to (Q2): Given a rational polyhedron P such that $P_I = \emptyset$, it is $\mathcal{NP}$-complete to decide whether $P' = \emptyset$. This result clearly implies Eisenbrand's [15] result that GC-Opt is $\mathcal{NP}$-hard. We also prove similar results for the $\{-1, 0, 1\}$-cuts and $\{0, 1\}$-cuts. A Gomory-Chvátal cut is called a $\{-1, 0, 1\}$-*cut* (or $\{0, 1\}$-*cut*) if the vector of variable coefficients $c \in \{-1, 0, 1\}^n$ (or $\{0, 1\}^n$). The $\{-1, 0, 1\}$-*closure* (or $\{0, 1\}$-*closure*) of P is $P'_{\{-1,0,1\}} \equiv \{x \in P : \ cx \leq \lfloor d \rfloor \ \forall c \in \{-1, 0, 1\}^n$ and $d \in \mathbf{R}$ such that $cx \leq d$ is valid for $P\}$ (or $P'_{\{0,1\}} \equiv \{x \in P : \ cx \leq \lfloor d \rfloor \ \forall c \in \{0, 1\}^n$ and $d \in \mathbf{R}$ such that $cx \leq d$ is valid for $P\}$). We show that, given a polyhedron P such that $P_I = \emptyset$, it is $\mathcal{NP}$-complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ (or $P'_{\{0,1\}} = \emptyset$).

We borrowed some ideas from Mahajan and Ralphs [28] to construct the polyhedra P in the proof of our $\mathcal{NP}$-completeness results. In their paper the following disjunctive infeasibility problem is proved to be $\mathcal{NP}$-complete: Given a polyhedron $P \subset \mathbf{R}^n$, does there exist $\pi \in \mathbf{Z}^n$ and $\pi_0 \in \mathbf{Z}$ such that $\{x \in P : \pi x \leq \pi_0 \text{ or } \pi x \geq \pi_0 + 1\} = \emptyset$? The polyhedra P used in [28] are simplices, whereas the polyhedra in our construction are convex hulls of $n+3$ vectors in $\mathbf{R}^n$. Mahajan and Ralphs' proof reduces the well-known partition problem to the disjunctive infeasibility problem, whereas, in our proofs, we reduce the single constraint integer programming feasibility problem to the emptiness problem of the Gomory-Chvátal closure, and the partition problem to the emptiness problems of the $\{-1, 0, 1\}$-closure and $\{0, 1\}$-closure.

The remaining sections of this paper are organized as follows. In Section 2, we will present a short proof of the result of Boyd and Pulleyblank that the feasibility problem for integer linear programs whose associated polyhedron P has the property that $P' = P_I$, is in $\mathcal{NP} \cap$ co-$\mathcal{NP}$, and we will give examples of the integer programs with this property that can be solved with polynomial-time algorithms. In Section 3, we will prove the $\mathcal{NP}$-completeness of deciding whether the Gomory-Chvátal closure of a rational polyhedron with no integer point is empty. In Section 4, we will extend the result of Section 3 to the $\{-1, 0, 1\}$-closure and $\{0, 1\}$-closure. Lastly, we conclude with open questions in Section 5.

2 Optimizing over the Gomory-Chvátal closure of a rational polyhedron P for which $P' = P_I$

The polynomial solvability of GC-Opt for polyhedra P that satisfy $P' = P_I$ is an open question, and similarly for the associated GC-Sep problem. The optimization problem is related to the following feasibility problem: Given a rational polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for which we know that $P' = P_I$, decide whether $P_I = \emptyset$. The optimization problem is polynomially solvable only if the feasibility problem is polynomially solvable, but it is unclear whether the corresponding "if" statement is true.

Boyd and Pulleyblank [2] showed that the feasibility problem is in the complexity class $\mathcal{NP} \cap$ co-$\mathcal{NP}$. See also Schrijver [33] Corollary 23.5a. To our knowledge, Boyd and Pulleyblank had not published the result in a refereed journal until 2009, but they had proved it in the early 1980s and it was known since then. In fact, one can prove with standard arguments that deciding whether the Gomory-Chvátal closure of a rational polyhedron is empty belongs to the complexity class $\mathcal{NP}$. To prove that, given a rational polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for which we know that $P' = P_I$, the problem of deciding whether $P_I = \emptyset$ is in co-$\mathcal{NP}$, it is sufficient to exhibit a point $x \in \mathbf{Z}^n$ that satisfies $Ax \leq b$ when such a point exists. It is well known that, if such a point exists, there is one whose encoding is polynomial in the size of the input [1]. Therefore, a polynomial co-$\mathcal{NP}$ certificate exists for $P_I = \emptyset$ when $P' = P_I$. Note that, although a polynomial co-$\mathcal{NP}$ certificate exists for $P' = \emptyset$ when $P' = P_I$, no polynomial co-$\mathcal{NP}$ certificate is known for $P' = \emptyset$ in general.

Next, we present some known integer linear programs whose linear programming relaxation polyhedron P has the property that $P' = P_I$.

A well-known example in this class is due to Edmonds [13]. A matching in an undirected graph $G = (V, E)$ is a set of edges no two of which are incident to the same node. Let $P \equiv \{x \in \mathbf{R}^{|E|} : \ x(\delta(v)) \leq 1 \ \forall v \in V, \ x_e \geq 0 \ \forall e \in E\}$, where $\delta(v)$ denotes the set of edges incident to node v and $x(S) \equiv \sum_{e \in S} x_e$. Edmonds showed via an algorithmic method that $P' = P_I$, and that this polytope is defined by the blossom inequalities, in addition to the inequalities in the description of P. In particular, Edmonds gave a polynomial-time primal-dual algorithm for solving GC-Opt for the matching problem. Edmonds' result on the matching polytope holds in more generality. For example, Edmonds and Johnson [14] showed that the node-edge incidence matrices of bi-directed graphs have *strong* Chvátal rank 1 (*i.e.*, $b \leq Ax \leq d, u \leq x \leq l$ has Chvátal rank 1 for every choice of integer vectors b, d, l, u). Padberg and Rao [30] devised a polynomial-time separation algorithm for the b-matching polytope, implying polynomial-time solvability of GC-Sep for the matching problem. The separation oracle given by Padberg and Rao is an efficient algorithm for the minimum odd-cut problem, which is the following: Given an edge-capacitated undirected graph $G = (V, E)$, find a cut $\delta(U)$ of minimum capacity among all node sets U such that $|U|$ is odd (here $\delta(U)$ denotes the set of edges with exactly one endnode in U). Recently, Chandrasekaran, Végh and Vempala [6] developed a linear-programming-based cutting plane algorithm for solving GC-Opt for the matching problem. The algorithm sequentially adds violated blossom inequalities. By maintaining half-integral intermediate linear programming solutions, they are able to give a polynomial bound on the number of iterations needed for the convergence of their cutting plane algorithm. A similar algorithm to that of Chandrasekaran *et al.* was previously given in Bunch's Ph.D. thesis [3], for the more general b-matching problem. This was acknowledged by Chandrasekaran *et al.* in their paper.

As a second example, consider the maximum weight stable set problem $\max\{wx : x \in P_I\}$ in a graph $G = (V, E)$, where $P \equiv \{x \in \mathbf{R}^V_+ : x_i + x_j \leq 1 \text{ for } ij \in E\}$ is the so-called edge formulation. We note that this problem is $\mathcal{NP}$-hard in general, but that it can be solved in polynomial time when $P' = P_I$. In fact, Gerards and Schrijver [18] proved that P' is entirely described by the inequalities defining P together with the odd circuit inequalities $\sum_{i \in C} x_i \leq \frac{|C|-1}{2}$ for node sets C of odd cardinality that induce a circuit of G. Campelo and Cornuéjols [4] showed that the $\{0, 1/2\}$-Chvátal closure [5], the split closure [10], and the intersection of all possible corner polyhedra [21] of P are all identical to $\{x \in \mathbf{R}^V_+ : \ x_i + x_j \leq 1 \text{ for every } ij \in E, \ \sum_{i \in C} x_i \leq \frac{|C|-1}{2} \text{ for every induced odd cycle } C \text{ of } G\}$. The graphs for which these inequalities completely describe the stable set polytope P_I are called *t-perfect graphs*. These graphs are discussed in Chapter 68 of Schrijver's book [34]. Theorem 68.1 states that a maximum-weight stable set in a t-perfect graph can be found in polynomial time. This follows from the equivalence of optimization and separation [22] and the fact that the separation of odd circuit inequalities can be done in polynomial time by reduction to a shortest path problem [18].

The third example is along the line of characterizing the edge-node incidence matrices of bi-directed graphs that have strong Chvátal rank 1. A necessary and sufficient condition proved by Gerards and Schrijver [18] is that the bi-directed graphs contain no odd-K_4 minor. Conforti, Gerards and Zambelli [9] gave another class of matrices with strong Chvátal rank 1, which is the family of all matrices obtained by multiplying by 2 any subset of the columns of edge-node incidence matrices of bipartite bi-directed graphs. Del Pia and Zambelli [11] showed that the corresponding systems are totally dual integral. Del Pia, Musitelli and Zambelli [12] gave an excluded-minor characterization of the matrices with strong Chvátal rank 1 among those in some class obtained by multiplying by 2 some columns of node-edge incidence matrices of bi-directed graphs.

The fourth example is with regard to the problem of finding a maximum weight simple 2-matching that contains no triangles in an edge-weighted subcubic graph, where a simple 2-matching in a graph is a subgraph all of whose nodes have degree 0, 1 or 2, and a subcubic graph is a graph in which every node has degree at most 3. From a natural integer programming formulation, $P = \{x \in \mathbf{R}_+^E : \ x(\delta(v)) \leq 2$ for every $v \in V, \ x_e \leq 1$ for every $e \in E, \ x(T) \leq 2$ for every triangle T of $G\}$. Hartvigsen and Li [24] showed that $P' = P_I$, which is determined by the tri-comb inequalities and the inequalities in the description of P, where tri-combs are more general than blossoms and a special case of combs. To prove this result, they developed an Edmonds-type primal-dual algorithm for solving the GC-Opt in polynomial time. Kobayashi [25] gave a polynomial-time algorithm for solving the same problem that comprises two basic algorithms: a steepest ascent algorithm and a classical maximum weight 2-matching algorithm. The correctness of his method is justified using some fundamental results from the theory of discrete convex functions on jump systems.

3 Deciding Emptiness of the Gomory-Chvátal Closure of a Rational Polyhedron with No Integer Point

In this section, we prove our main result: It is $\mathcal{NP}$-complete to decide emptiness of the Gomory-Chvátal closure of a rational polyhedron containing no integer point. Our proof proceeds in three steps. First, we note the remark given in the previous section that deciding whether the Gomory-Chvátal closure of a rational polyhedron is empty is in $\mathcal{NP}$. Second, we prove in Lemma 1 that it is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a rational polyhedron is empty. Third, we prove in Lemma 2 that the polyhedron used in the proof of Lemma 1 contains no integer point.

Lemma 1 *It is $\mathcal{NP}$-complete to decide whether the Gomory-Chvátal closure of a rational polyhedron is empty.*

Proof The lemma will be proved by polynomially reducing the following single constraint integer programming feasibility problem, which is known to be

$\mathcal{NP}$-complete [27], to the problem of deciding whether $P' = \emptyset$ for a rational polyhedron P.

Single Constraint Integer Programming Feasibility Problem: Given a finite set of nonnegative integers $\{a_i\}_{i=1}^s$ and a nonnegative integer b, is there a set of nonnegative integers $\{x_i\}_{i=1}^s$ satisfying $\sum_{i=1}^s a_i x_i = b$?

We consider the Single Constraint Integer Programming Feasibility Problem with $s = n-1$ and $n \geq 3$. We assume without loss of generality that the greatest common divisor of $a_1, a_2, \cdots, a_{n-1}$ is 1, and $2 < a_1 < a_2 < \cdots < a_{n-1} < b$. So $b \geq n+2$. Let $r = n+1+\frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbf{Z}$. We will show:

Reduction:

The Single Constraint Integer Programming Feasibility Problem can be polynomially reduced to the problem of deciding whether $P' = \emptyset$ for the polyhedron $P \subseteq \mathbf{R}^n$ that is the convex hull of the following $n+3$ vectors:

$$
\begin{aligned}
&v_1 \equiv (\tfrac{1}{2b}, 0, \cdots, 0, \tfrac{1}{2b}),\\
&v_2 \equiv (0, \tfrac{1}{2b}, 0, \cdots, 0, \tfrac{1}{2b}),\\
&\qquad \cdots\cdots\cdots\\
&v_{n-1} \equiv (0, \cdots, 0, \tfrac{1}{2b}, \tfrac{1}{2b}),\\
&v_n \equiv (0, \cdots, 0, \tfrac{1}{2} + \tfrac{3}{2r}),\\
&v_{n+1} \equiv (a_1, a_2, \cdots, a_{n-1}, -b + \tfrac{1}{2}),\\
&v_{n+2} \equiv ((1-r)a_1, (1-r)a_2, \cdots, (1-r)a_{n-1}, (r-1)b + 1),\\
&v_{n+3} \equiv (0, \cdots, 0, \tfrac{1}{2r}).
\end{aligned}
$$

An informal explanation of P is given at the end of Section 3.

To show that this reduction is correct, we prove the following two claims; we then observe that converting the vectors $v_1, v_2, \cdots, v_{n+3}$ into an inequality description of P can be done in polynomial time.

Claim 1. There is a set of nonnegative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ only if $P' = \emptyset$.

Proof. Consider an inequality $cx \leq q$, where $c = (c_1, c_2, \cdots, c_n)$, $c_i = -w_i$ for $1 \leq i \leq n-1$, $c_n = -1$, and $q = \max\,\{-\frac{1}{2b}, -\frac{1}{2r}\} = -\frac{1}{2b}$. Because $1 < a_i < b$ for $1 \leq i \leq n-1$ and $\sum_{i=1}^{n-1} a_i w_i = b$, it is easy to verify that $-1 < cv_i \leq -\frac{1}{2b}$ for $1 \leq i \leq n-1$. In addition, $-1 < cv_n < -\frac{1}{2}$, $cv_{n+1} = -\frac{1}{2}$, $cv_{n+2} = -1$, and $cv_{n+3} = -\frac{1}{2r}$. So $cx \leq q$ is valid for P, and the associated Gomory-Chvátal inequality is $cx \leq \lfloor q \rfloor$ $(= -1)$. We can see that v_{n+2} is the only vector in P that satisfies the inequality $cx \leq \lfloor q \rfloor$. Since $cv_{n+2} = \lfloor q \rfloor$ and $cv_i > \lfloor q \rfloor$ for $i = 1, 2, \cdots, n+1, n+3$, it follows that no other point in P satisfies $cx \leq \lfloor q \rfloor$. Consider the inequality $fx \leq g$, where $fx = x_n$ and $g = (r-1)b+1$. It can be easily checked that every v_i satisfies $fx \leq g$, so $fx \leq g$ is valid for P, and $fx \leq \lfloor g \rfloor$ $(= \lfloor rb \rfloor - b + 1)$ is a Gomory-Chvátal inequality of P. Since $rb \notin \mathbf{Z}$, $fx \leq \lfloor g \rfloor$ is violated by v_{n+2}. Now we can conclude that $P' = \emptyset$ and Claim 1 is proved.

Claim 2. There is a set of nonnegative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ if $P' = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \cdots, 0, \frac{1}{2r} + \frac{1}{2})$. We know $v_0 \in P$ because $v_0 = \alpha v_n + (1-\alpha)v_{n+3}$ for some $0 < \alpha < 1$. Let $cx \leq \lfloor q \rfloor$ be a Gomory-Chvátal inequality of P that is violated by v_0, where $c = (c_1, c_2, \cdots, c_n) \in \mathbf{Z}^n$ and $cx \leq q$ is valid for P. Then $c_n \neq 0$, otherwise we would have $0 = cv_0 \leq q$, contradicting that $cv_0 > \lfloor q \rfloor$.

Let $\Delta \equiv \sum_{i=1}^{n-1} c_i a_i - c_n b$. First, we show that $c_n \leq -1$ by deriving a contradiction in the following two cases:

Case 1. $c_n \geq 1$ and $\Delta \geq 1$. If $\frac{c_n}{2r} \leq \Delta - 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} \leq \Delta - 1 + \frac{c_n}{2} < \lfloor \Delta + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, which contradicts that $cv_0 > \lfloor q \rfloor$. If $\frac{c_n}{2r} > \Delta - 1$, then $cv_n - cv_0 = \frac{c_n}{r} > 2(\Delta - 1)$. If $\Delta \geq 2$, then $cv_n - cv_0 > 2$, so $cv_0 < cv_n - 2 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction to that $cv_0 > \lfloor q \rfloor$. If $\Delta = 1$, then $\frac{c_n}{r} \geq 1$. Otherwise, if $\frac{c_n}{r} < 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < \frac{1}{2} + \frac{c_n}{2} \leq \lfloor 1 + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. Since $\frac{c_n}{r} \geq 1$, $cv_n - cv_0 \geq 1$, therefore $cv_0 \leq cv_n - 1 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction again.

Case 2. $c_n \geq 1$ and $\Delta \leq 0$. Because $r > n + 1 > 3$, $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < c_n \leq (1 - r)\Delta + c_n = cv_{n+2}$. Hence, $cv_0 < \lfloor cv_{n+2} \rfloor \leq \lfloor q \rfloor$, a contradiction.

It is easy to see that $c_n = -1$. Otherwise, if $c_n \leq -2$, then $cv_{n+3} - cv_0 = -\frac{c_n}{2} \geq 1$, which implies $cv_0 \leq cv_{n+3} - 1 < \lfloor cv_{n+3} \rfloor \leq \lfloor q \rfloor$, a contradiction.

Now we show that $\Delta = 0$. If $\Delta \geq 1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2} < 0 < \Delta - \frac{1}{2} = cv_{n+1}$, implying $cv_0 < \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. If $\Delta \leq -1$, then, because $r > 3$, $cv_0 < 0 < (1 - r)\Delta - 1 = cv_{n+2}$, a contradiction again.

We claim that $c_i \leq 0$ for $i = 1, 2, \cdots, n - 1$. Otherwise, if $c_i \geq 1$ for some $1 \leq i \leq n - 1$, then $cv_0 < 0 \leq \frac{c_i}{2b} - \frac{1}{2b} = cv_i$, a contradiction.

Now let $w_i = -c_i$ for $1 \leq i \leq n - 1$. Then Claim 2 is proved.

To complete the proof, it suffices to show that a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbf{Z}^{m \times n}$ and $\tilde{b} \in \mathbf{Z}^m$, can be obtained in polynomial time from the vectors $v_1, v_2, \cdots, v_{n+3}$. We can see from the coordinates of $v_1, v_2, \cdots, v_n$ and v_{n+3} that P is a n-dimensional polyhedron. Let i be a counter looping through $n, n + 1$ and $n + 2$. For every i vectors of $v_1, v_2, \cdots, v_{n+3}$, check if they are on a unique hyperplane by solving linear equations. If yes, then further check if the $n + 3 - i$ other vectors are all on one side of the hyperplane. If yes again, then the equation $\tilde{c}x = \tilde{d}$ of the hyperplane with integral $\tilde{c}$ and $\tilde{d}$ whose greatest common divisor is 1 yields a linear inequality of $\tilde{A}x \leq \tilde{b}$. One can easily see that this process takes polynomial time and the size of $\tilde{A}$ and $\tilde{b}$ is polynomial in the size of $v_1, v_2, \cdots, v_{n+3}$. $\square$

Lemma 2 *The polyhedron P used in the proof of Lemma 1 contains no integer point.*

Proof Proving that P contains no integer point is equivalent to proving that $P^d \equiv P \cap \{x \in \mathbf{R}^n : \ x_n = d\}$ contains no integer points for every integer $d \in [-b + 1, nb + 1]$. Let P_1 be the convex hull of $v_1, v_2, \cdots, v_n, v_{n+1}$ and v_{n+3}, and let P_2 be the convex hull of $v_1, v_2, \cdots, v_n, v_{n+2}$ and v_{n+3}. We claim that $P = P_1 \cup P_2$. First, $P_1 \cup P_2 \subseteq P$ follows easily from $P_1 \subset P$ and $P_2 \subset P$. To show $P \subset P_1 \cup P_2$, consider a vector $v \in P$. So $v = \sum_{i=1}^{n+3} \alpha_i v_i$, where $\sum_{i=1}^{n+3} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, 2, \cdots, n+3$, *i.e.*, v is a convex combination of $v_1, v_2, \cdots, v_{n+3}$. We assume $\alpha_{n+1} > 0$ and $\alpha_{n+2} > 0$, otherwise, $v \in P_1$ or P_2, and the proof is done. Let $\alpha_{n+1}^{n+2} \equiv \alpha_{n+1} + \alpha_{n+2}$, and let $\hat{v} \equiv \frac{\alpha_{n+1}}{\alpha_{n+1}^{n+2}} v_{n+1} + \frac{\alpha_{n+2}}{\alpha_{n+1}^{n+2}} v_{n+2}$. Then $\hat{v}$ is a convex combination of v_{n+1} and v_{n+2}, $\alpha_{n+1} v_{n+1} + \alpha_{n+2} v_{n+2} = \alpha_{n+1}^{n+2} \hat{v}$, and v is a convex combination of $v_1, v_2, \cdots, v_n, \hat{v}, v_{n+3}$. Since $v_0 = \frac{r-1}{r} v_{n+1} + \frac{1}{r} v_{n+2}$, v_0 is also a convex combination of v_{n+1} and v_{n+2}. Therefore, if $\frac{\alpha_{n+1}}{\alpha_{n+1}^{n+2}} \geq \frac{r-1}{r}$, then $\hat{v}$ is a convex combination of v_{n+1} and v_0, otherwise, $\hat{v}$ is a convex combination of v_0 and v_{n+2}. One can see that $v_0 = \alpha v_n + (1-\alpha) v_{n+3}$ for some $0 < \alpha < 1$, *i.e.*, v_0 is a convex combination of v_n and v_{n+3}. So, if $\hat{v}$ is a convex combination of v_{n+1} and v_0, then $\hat{v}$ is a convex combination of v_n, v_{n+1} and v_{n+3}, hence, we see that $v \in P_1$. Similarly, if $\hat{v}$ is a convex combination of v_0 and v_{n+2}, then $v \in P_2$. The proof of the claim is completed.

Now it is sufficient to show: (a) $P_1^d \equiv P_1 \cap \{x \in \mathbf{R}^n : \ x_n = d\}$ contains no integer points for every integer $d \in [-b+1, 0]$; (b) $P_2^d \equiv P_2 \cap \{x \in \mathbf{R}^n : \ x_n = d\}$ contains no integer points for every integer $d \in [1, nb+1]$.

We first prove (a). Because $v_1, v_2, \cdots, v_n, v_{n+3} \in \{x \in \mathbf{R}^n : \ x_n > 0\}$ and $v_{n+1} \in \{x \in \mathbf{R}^n : \ x_n < 0\}$, it follows that P_1^d, where d is an integer in $[-b+1, 0]$, is the convex hull of the $n+1$ vectors obtained by intersecting the hyperplane $x_n = d$ with the line segments joining each of $v_1, v_2, \cdots, v_n, v_{n+3}$ with the point v_{n+1}. It is easy to verify by calculation that these vectors are:

$$(\alpha_0 a, d) + \beta e_i, \quad i = 1, 2, \cdots, n-1,$$
$$(\alpha_1 a, d),$$
$$(\alpha_2 a, d),$$

where $a \equiv (a_1, a_2, \cdots, a_{n-1})$, e_i is the i-th unit vector in $\mathbf{R}^n$, $\alpha_0 \equiv \frac{1-2bd}{2b^2-b+1}$, $\alpha_1 \equiv \frac{r+3-2rd}{2rb+3}$, $\alpha_2 \equiv \frac{1-2rd}{2rb+1-r}$, and $\beta \equiv \frac{b+d-\frac{1}{2}}{2b^2-b+1}$.

Since $P_1^d \subset \{x \in \mathbf{R}^n : \ x_n = d\}$, we only need to consider the convex hull of the following $n+1$ vectors in $\mathbf{R}^{n-1}$: $\alpha_0 a + \beta e_i$ (where e_i is the i-th unit vector in $\mathbf{R}^{n-1}$, and $i = 1, 2, \cdots, n-1$), $\alpha_1 a$ and $\alpha_2 a$. Let $\tilde{P}_1^d$ denote the convex hull. Since $r < b$ and $-b+1 \leq d \leq 0$, it is easy to verify that $0 < \alpha_0 < \alpha_2 < \alpha_1 < 1$. So $\tilde{P}_1^d \subseteq Q_1^d$, where Q_1^d is the convex hull of the $n+1$ vectors:

$$z_0 \equiv \alpha_0 a,$$
$$z_i \equiv \alpha_0 a + \beta e_i, \quad i = 1, 2, \cdots, n-1,$$
$$z_n \equiv \alpha_1 a.$$

To prove (a), it suffices to show the following claim:

Claim 1. There is no integer point in Q_1^d.

Proof. By contradiction, suppose $\bar{v} \in Q_1^d \cap \mathbf{Z}^{n-1}$. Then there must exist a vector $v' \equiv \beta_0 a$, where $0 < \beta_0 < 1$, such that $\|\bar{v} - v'\|_\infty \equiv \max_{1 \leq i \leq n-1} |\bar{v}_i -$

$v_i'| \leq \beta \leq \frac{b-\frac{1}{2}}{2b^2-b+1} < \frac{1}{2b}$. From the construction of P, it is easy to see that $0 < \bar{v}_i < a_i$ for $i = 1, 2, \cdots, n-1$. Because the greatest common divisor of $a_1, a_2, \cdots, a_{n-1}$ is 1, there exists no integer point on the line segment connecting 0 and a except for the two end points. Therefore, there exists $1 \leq k \leq n-2$ such that $(\bar{v}_k, \bar{v}_{n-1})$ is not on the line segment connecting $(0,0)$ and (a_k, a_{n-1}) in $\mathbf{R}^2$. This is because otherwise if $(\bar{v}_i, \bar{v}_{n-1})$ is on the line segment connecting $(0,0)$ and (a_i, a_{n-1}) for every $1 \leq i \leq n-2$, then $\frac{\bar{v}_i}{a_i} = \frac{\bar{v}_{n-1}}{a_{n-1}}$ for every i, implying that $\bar{v}$ is on the line segment connecting 0 and a, which contradicts the previous statement. Now, to derive contradiction to the inequality $\|\bar{v} - v'\|_\infty < \frac{1}{2b}$, we show below that $\|(\bar{v}_k, \bar{v}_{n-1}) - (v_k', v_{n-1}')\|_\infty = \max(|\bar{v}_k - v_k'|,\ |\bar{v}_{n-1} - v_{n-1}'|) \geq \frac{1}{2(b-1)}$.

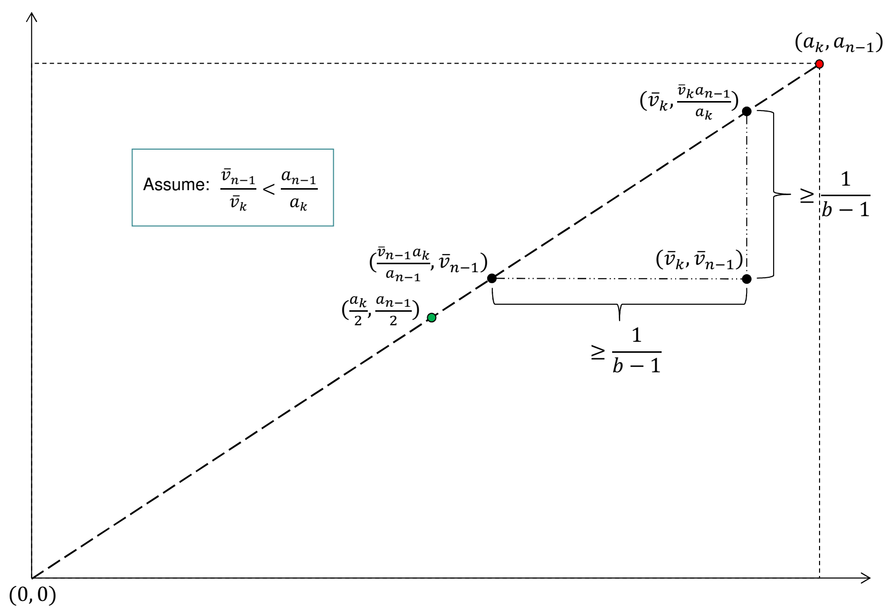

Fig. 1 Proving Claim 1 in the proof of Lemma 2

Let L denote the line segment connecting $(0,0)$ and (a_k, a_{n-1}) in $\mathbf{R}^2$; see Fig.1 for illustration. We know that (v_k', v_{n-1}') is on L. Because the integer points between 0 and (a_k, a_{n-1}) that are not on L are symmetric across $(\frac{a_k}{2}, \frac{a_{n-1}}{2})$, we may assume without loss of generality that $\frac{\bar{v}_{n-1}}{\bar{v}_k} < \frac{a_{n-1}}{a_k}$. It is not hard to see that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\bar{v}_k, \bar{v}_{n-1})$ is attained at a point on the line segment connecting $(\frac{\bar{v}_{n-1}a_k}{a_{n-1}}, \bar{v}_{n-1})$ and $(\bar{v}_k, \frac{\bar{v}_k a_{n-1}}{a_k})$. Since $\bar{v}_k > \frac{\bar{v}_{n-1}a_k}{a_{n-1}}$, $\|(\bar{v}_k, \bar{v}_{n-1}) - (\frac{\bar{v}_{n-1}a_k}{a_{n-1}}, \bar{v}_{n-1})\|_\infty \geq \frac{1}{a_{n-1}} \geq \frac{1}{b-1}$. Because $a_{n-1} \geq a_k$, $\|(\bar{v}_k, \bar{v}_{n-1}) - (\bar{v}_k, \frac{\bar{v}_k a_{n-1}}{a_k})\|_\infty \geq \frac{1}{b-1}$. So it follows that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\bar{v}_k, \bar{v}_{n-1})$ is no less than $\frac{1}{2(b-1)}$. Therefore, $\|(\bar{v}_k, \bar{v}_{n-1}) - (v_k', v_{n-1}')\|_\infty \geq \frac{1}{2(b-1)}$. Claim 1 is proved.

Next we prove (b). Because $v_1, v_2, \cdots, v_n, v_{n+3} \in \{x \in \mathbf{R}^n : \ x_n < 1\}$ and $v_{n+2} \in \{x \in \mathbf{R}^n : \ x_n > 1\}$, we know by calculation that P_2^d, where d is an integer in $[1, nb+1]$, is the convex hull of the $n+1$ vectors:

$$(\alpha_0 a, d) + \beta e_i, \quad i = 1, 2, \cdots, n-1,$$
$$(\alpha_1 a, d),$$
$$(\alpha_2 a, d),$$

where $a \equiv (a_1, a_2, \cdots, a_{n-1})$, e_i is the i-th unit vector in $\mathbf{R}^n$, $\alpha_0 \equiv \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}$, $\alpha_1 \equiv \frac{(r-1)(\frac{1}{2}+\frac{3}{2r}-d)}{(r-1)b+\frac{1}{2}-\frac{3}{2r}}$, $\alpha_2 \equiv \frac{(r-1)(\frac{1}{2r}-d)}{(r-1)b+1-\frac{1}{2r}}$, and $\beta \equiv \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}$. So we just need to prove that the convex hull of the following $n+1$ vectors in $\mathbf{R}^{n-1}$, denoted by $\tilde{P}_2^d$, contains no integer points: $\alpha_0 a + \beta e_i$ (where e_i is the i-th unit vector in $\mathbf{R}^{n-1}$, and $i = 1, 2, \cdots, n-1$), $\alpha_1 a$ and $\alpha_2 a$.

The following properties can be verified by calculation, using $d \leq nb+1$, $b > r$, and $r = n + 1 + \frac{1}{2b}$. A few brief explanations are put in brackets to justify some of the inequalities.

1. $\alpha_0 < \alpha_2 < \alpha_1 < 0$, and each of the three terms strictly decreases as d increases.
   [The second inequality in the property 1 holds because of the equivalent inequality: $0 < 2(r^2+2r)((r-1)b-d) + 2r^2 + 4r$.]

2. For $d = kb + h$, where integers k and h satisfy $0 \leq k \leq \lfloor r-1 \rfloor = n$ and $1 \leq h \leq b-1$, $\alpha_1 a_i < -k a_i$ and $\alpha_0 a_i + \beta < -k a_i$ for $i = 1, 2, \cdots, n-1$.
   [The last inequality in the property 2 can be derived through the following sequence of inequalities, each derivable from the previous one: $0 < (5b-3)(r-k-1)$, $(r-1)b < a_i(k-r+1)(1-2b) + kb$ (because $a_i \geq 3$), $(r-1)b - kb - h + 1 < a_i(k(1-2b) + (2hb-1)(r-1))$.]

3. For $d = kb$, where integer k satisfies $1 \leq k \leq \lfloor r-1 \rfloor = n$, $\alpha_1 a_i < -(k-1)a_i$, $-k a_i < \alpha_0 a_i$, and $\alpha_0 a_i + \beta < -(k-1)a_i$ for $i = 1, 2, \cdots, n-1$.
   [The last inequality in the property 3 can be derived through the following sequence of inequalities, each derivable from the previous one: $(b+3)(r-1)+1 < 24b(r-1)+kb$, $(r-1)b+1 < 6(r-1)b^2 - 3(k-1)(2b-1) - 3(r-1) + kb$ (because $b \geq 5$), $(r-1)b+1 < a_i(2(r-1)b^2 - (k-1)(2b-1) - (r-1)) + kb$ (because $a_i \geq 3$).]

By the above property 1, $\tilde{P}_2^d \subseteq Q_2^d$, where Q_2^d is the convex hull of the $n+1$ vectors:

$$y_0 \equiv \alpha_0 a,$$
$$y_i \equiv \alpha_0 a + \beta e_i, \quad i = 1, 2, \cdots, n-1,$$
$$y_n \equiv \alpha_1 a.$$

To prove (b), it suffices to show that Q_2^d contains no integer points. Given the properties 2 and 3 and the fact that $\beta < \frac{1}{2b}$, the proof is similar to that of

Claim 1. In fact, in the proof of Claim 1, $Q_1^d \subset \{x \in \mathbf{R}^{n-1} : 0 \le x \le a\}$, where $-b+1 \le d \le 0$. In the proof of the same for Q_2^d, the fact that each of α_0, α_1 and α_2 strictly decreases as d increases, together with the properties 2 and 3, imply that $Q_2^d \subset \{x \in \mathbf{R}^{n-1} : -(k+1)a \le x \le -ka\}$ for every $d = kb+h$ and $1 \le h \le b$, where $0 \le k \le n$. Note that $\{x \in \mathbf{R}^{n-1} : -(k+1)a \le x \le -ka\}$ varies as k goes from 0 to n. Despite this complicacy, the fact that $\beta < \frac{1}{2b}$ enables us to prove that Q_2^d for each integer k in $[0, n]$ contains no integer points very analogously to how Claim 1 is proved. □

In conclusion, we obtain the following result from Lemmas 1 and 2.

Theorem 1 *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its Gomory-Chvátal closure is empty.*

Now we give an informal description of the polytope P used in the proof of Lemma 1 for reduction, which may shed some light on the rationale behind the choice of the polytope. As noted in the proof of Lemma 2, $P = P_1 \cup P_2$, where $P_1 = \text{Conv}(v_1, v_2, \cdots, v_n, v_{n+1}, v_{n+3})$, $P_2 = \text{Conv}(v_1, v_2, \cdots, v_n, v_{n+2}, v_{n+3})$, and $\text{Conv}(\cdot)$ denotes the convex hull operator. By the definitions of $v_1, v_2, \cdots$, v_n and v_{n+3}, $\text{Conv}(v_1, v_2, \cdots, v_n, v_{n+3})$ is a small polytope in the unit cube (denoted by $[0,1]^n$) that contains no integer points. With the choice of $a_1, a_2, \cdots$, a_{n-1}, b and n, v_{n+1} can be far below the plane $x_n = 0$ and distant from $[0,1]^n$, v_{n+2} can be far above the plane $x_n = 0$ and distant from $[0,1]^n$, and the line between v_{n+1} and v_{n+2} is through $\text{Conv}(v_1, v_2, \cdots, v_n, v_{n+3})$. So the polytope P can stretch very thin and slim with two very sharp extreme points v_{n+1} and v_{n+2}. This allows for P to contain no integer points. In addition, this special shape of P makes it hard to decide if there is a Gomory-Chvátal cut of P that can remove v_{n+1}, $\text{Conv}(v_1, v_2, \cdots, v_n, v_{n+3})$, and many other points from P except for the part of P in the proximity of v_{n+2}. This is important because such a Gomory-Chvátal cut exists if and only if $P' = \emptyset$.

4 Deciding Emptiness of the $\{-1,0,1\}$-Closure (or $\{0,1\}$-Closure) of a Rational Polyhedron with No Integer Point

In this section, we first prove $\mathcal{NP}$-completeness of deciding emptiness of the $\{-1,0,1\}$-closure of a rational polyhedron containing no integer point, and then prove the same for the $\{0,1\}$-closure as a corollary.

Theorem 2 *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its $\{-1,0,1\}$-closure is empty.*

Proof We will prove the theorem by polynomially reducing the following partition problem, which is known to be $\mathcal{NP}$-complete [17], to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for a polyhedron P with no integer points.

Partition Problem: Given a finite set of positive integers $S = \{a_i\}_{i=1}^s$, is there a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$?

Let $b \equiv \frac{1}{2}\sum_{1\le i\le s} a_i$. We assume without loss of generality that $a_i < b$ for $i = 1, 2, \cdots, s$ and that the greatest common divisor of $a_1, a_2, \cdots, a_s$ is 1. We also assume without loss of generality that $s \ge 8$ (because the Partition Problem with fixed s can be formulated as an integer program with only s binary variables, which can be solved in polynomial time [26]).

First, we prove that b can be assumed to be greater than or equal to $s+3$. Note that a Partition Problem with $b < s+3$ can be polynomially converted to another Partition Problem with $S' = S \cup \{\sum_{1\le i\le s} a_i + 1, \sum_{1\le i\le s} a_i + 1\}$. Let $s' = |S'|$. So $s' = s+2$. It is easy to see that the Partition Problem with S has a feasible partition if and only if the Partition Problem with S' has a feasible partition. Let $b' \equiv b + (\sum_{1\le i\le s} a_i + 1)$. Then $b' = \frac{3}{2}\sum_{1\le i\le s} a_i + 1 \ge \frac{3s}{2} + 1 = s + 5 + (\frac{s}{2} - 4) \ge (s+2) + 3 = s' + 3$.

Now we consider the Partition Problem with $s = n-1, n \ge 9$ and $b \ge n+2$. Let $r = n + 1 + \frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbf{Z}$. We only need to show that the Partition Problem can be polynomially reduced to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for the same polyhedron P as constructed in the proof of Lemma 1, *i.e.*, the convex hull of the $n+3$ vectors: $v_1 = (\frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b})$, $v_2 = (0, \frac{1}{2b}, 0, \cdots, 0, \frac{1}{2b})$, $\cdots$, $v_{n-1} = (0, \cdots, 0, \frac{1}{2b}, \frac{1}{2b})$, $v_n = (0, \cdots, 0, \frac{1}{2} + \frac{3}{2r})$, $v_{n+1} = (a_1, a_2, \cdots, a_{n-1}, -b + \frac{1}{2})$, $v_{n+2} = ((1-r)a_1, (1-r)a_2, \cdots, (1-r)a_{n-1}, (r-1)b+1)$, and $v_{n+3} = (0, \cdots, 0, \frac{1}{2r})$. Recall from Lemma 2 that the polyhedron P contains no integer point. It suffices to prove the following two claims.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ only if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Consider an inequality $cx \le q$, where $c = (c_1, c_2, \cdots, c_n)$, $c_i = -1$ for $i \in K$, $c_i = 0$ for $i \in S\setminus K$, $c_n = -1$, and $q = -\frac{1}{2b}$. One can easily verify: $-1 < cv_i \le -\frac{1}{2b}$ for $1 \le i \le n-1$, $cv_n = -\frac{1}{2} - \frac{3}{2r}$, $cv_{n+3} = -\frac{1}{2r}$, and because $\sum_{i\in K} a_i = b$, $cv_{n+1} = -\frac{1}{2}$ and $cv_{n+2} = -1$. Hence, $cx \le q$ is valid for P. Since $c_i = -1$ or 0, the inequality $cx \le \lfloor q \rfloor$ $(= -1)$ is a $\{-1,0,1\}$-cut of P. From the value of cv_i for $1 \le i \le n+3$, we see that v_{n+2} is the only vector in P that satisfies the inequality $cx \le \lfloor q \rfloor$. Since $rb \notin \mathbf{Z}$ and $b = \sum_{i\in K} a_i$, there exists some $j \in K$ satisfying $ra_j \notin \mathbf{Z}$. Consider the inequality $fx \le g$, where $fx = -x_j$ and $g = (r-1)a_j$. Apparently, $g > 0$ and $g \notin \mathbf{Z}$. In addition, $fv_i \le 0$ for $i = 1, 2, \cdots, n+1, n+3$ and $fv_{n+2} = g > 0$. Then it is obvious that the inequality $fx \le \lfloor g \rfloor$ $(= \lfloor (r-1)a_j \rfloor)$ is a $\{-1,0,1\}$-cut of P and that v_{n+2} violates this inequality. Therefore, $P'_{\{-1,0,1\}} = \emptyset$ and Claim 1 is proved.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \cdots, 0, \frac{1}{2r} + \frac{1}{2}) \in P$. Since $P'_{\{-1,0,1\}} = \emptyset$, v_0 violates an inequality $cx \le \lfloor q \rfloor$, where $c = (c_1, c_2, \cdots, c_n) \in \{-1,0,1\}^n$ and $cx \le q$ is valid for P. It is easy to see that $c_n \ne 0$. Otherwise, $0 = cv_0 \le q$, which contradicts that v_0 violates $cx \le \lfloor q \rfloor$.

Now we show that $\sum_{1\leq i\leq n-1} c_i a_i = c_n b$. If $c_n = 1$, then $cv_0 = \frac{1}{2r} + \frac{1}{2}$. In this case, if $\sum_{1\leq i\leq n-1} c_i a_i \geq c_n b + 1$, then $\frac{3}{2} \leq cv_{n+1} \leq q$; if $\sum_{1\leq i\leq n-1} c_i a_i \leq c_n b - 1$, then $n+1 < r \leq cv_{n+2} \leq q$. Hence, $cv_0 < 1 < q$, contradicting $cv_0 > \lfloor q \rfloor$. If $c_n = -1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2}$. In this case, if $\sum_{1\leq i\leq n-1} c_i a_i \geq c_n b + 1$, then $\frac{1}{2} \leq cv_{n+1} \leq q$; if $\sum_{1\leq i\leq n-1} c_i a_i \leq c_n b - 1$, then $n-1 < r-2 \leq cv_{n+2} \leq q$. So, $cv_0 < 0 < q$, a contradiction to $cv_0 > \lfloor q \rfloor$.

It is true that $c_n = -1$. Otherwise, $cv_0 = \frac{1}{2r} + \frac{1}{2}$ and $cv_{n+2} = 1 \leq q$, contradicting that v_0 violates $cx \leq \lfloor q \rfloor$.

We now claim that $c_i \leq 0$ for $1 \leq i \leq n-1$. Otherwise, suppose $c_j = 1$ for some $1 \leq j \leq n-1$. Then $cv_j = 0$. Since $cv_0 = -\frac{1}{2r} - \frac{1}{2}$ and $0 = cv_j \leq q$, a contradiction similar to the early ones can be derived. Therefore, Claim 2 is proved.

As shown in the end of the proof of Lemma 1, it is straight to polynomially obtain a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbf{Z}^{m\times n}$ and $\tilde{b} \in \mathbf{Z}^m$, from the vectors $v_1, v_2, \cdots, v_{n+3}$. The theorem is proved. □

Corollary 1 *Given a rational polyhedron containing no integer point, it is $\mathcal{NP}$-complete to decide whether its $\{0,1\}$-closure is empty.*

Proof The proof is similar to that of Theorem 2, hence we omit the details and only point out the differences.

The Partition Problem is polynomially reduced to the problem of deciding whether $P'_{\{0,1\}} = \emptyset$ for the polyhedron $P \subseteq \mathbf{R}^n$ that is the convex hull of the $n+3$ vectors: $v_1 = (-\frac{1}{2b}, 0, \cdots, 0, -\frac{1}{2b})$, $v_2 = (0, -\frac{1}{2b}, 0, \cdots, 0, -\frac{1}{2b})$, $\cdots$, $v_{n-1} = (0, \cdots, 0, -\frac{1}{2b}, -\frac{1}{2b})$, $v_n = (0, \cdots, 0, -\frac{1}{2} - \frac{3}{2r})$, $v_{n+1} = (-a_1, -a_2, \cdots, -a_{n-1}, b - \frac{1}{2})$, $v_{n+2} = ((r-1)a_1, (r-1)a_2, \cdots, (r-1)a_{n-1}, (1-r)b - 1)$, and $v_{n+3} = (0, \cdots, 0, -\frac{1}{2r})$.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ only if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 1 is similar to that of Claim 1 in the proof of Theorem 2 except that $c_i = 1$ for $i \in K$, $c_i = 0$ for $i \in S\setminus K$, $c_n = 1$, and $fx = x_j$.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i\in K} a_i = \sum_{i\in S\setminus K} a_i$ if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 2 is similar to but simpler than that of Claim 2 in the proof of Theorem 2. Here are two differences: First, we let $v_0 \equiv (0, 0, \cdots, 0, -\frac{1}{2r} - \frac{1}{2})$. Second, to show by contradiction that $\sum_{1\leq i\leq n-1} c_i a_i = c_n b$, we only consider the case that $c_n = 1$, and a contradiction can be derived due to $cv_0 < 0 < q$. □

5 Conclusion

In this paper, we addressed two questions, (Q1) and (Q2), about integer linear programs with the property that the Gomory-Chvátal closure of the linear constraints is identical to their integer hull. To answer (Q2), we proved that the problem of deciding whether the Gomory-Chvátal closure of a rational polyhedron P is empty is $\mathcal{NP}$-complete, even when P is known to contain no integer point. Similar results are also proved for the $\{-1,0,1\}$-closure and $\{0,1\}$-closure of polyhedron. There are several questions related to (Q2) to which we have not found an answer yet. In particular, (Q2) is open when we restrict our attention to polyhedra in the unit cube. We state three natural questions of this type:

(i) Is it $\mathcal{NP}$-complete to decide whether $P' = \emptyset$ for a polytope $P \subseteq [0,1]^n$ that contains no integer point?
(ii) Is it $\mathcal{NP}$-complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ for a polytope $P \subseteq [0,1]^n$ that contains no integer point?
(iii) Is it $\mathcal{NP}$-complete to decide whether $P'_{\{0,1\}} = \emptyset$ for a polytope $P \subseteq [0,1]^n$ that contains no integer point?

Given a rational polyhedron P with the promise that $P_I = \emptyset$, Theorem 1 shows that it is $\mathcal{NP}$-complete to decide whether $P' = \emptyset$. However the following question, which is related to (Q1), is still open: Given a rational polyhedron P with the promise that $P' = P_I$, can one decide in polynomial time whether $P' = P_I = \emptyset$, or is this an "intermediate problem" in $\mathcal{NP} \cap$ co-$\mathcal{NP}$ that is not polynomially solvable?

Acknowledgements We thank Michele Conforti and Dabeen Lee for pointing out the papers [1] and [2]. We also thank one referee who pointed out a few important papers to our attention and helped us improve the presentation of this paper.

References

1. Borosh, I., Treybig, L.B.: Bounds on positive integral solutions to linear Diophantine equations. Proceedings of the American Mathematical Society **55**, 299–304 (1976)
2. Boyd, S., Pulleyblank, W.R.: Facet generating techniques. In: Cook, W., Lovász, L., Vygen, J., editors, Research Trends in Combinatorial Optimization, Springer, Berlin, 33–55 (2009)
3. Bunch, P.R.: A simplex based primal-dual algorithm for the perfect b-matching problem - a study in combinatorial optimization algorithm in engineering. Ph.D. thesis, Purdue University, 1997.
4. Campelo, M., Cornuéjols, G.: Stable sets, corner polyhedra and the Chvátal closure. Operations Research Letters **37**, 375–378 (2009)
5. Caprara, A., Fischetti, M.: $\{0, 1/2\}$-Chvátal-Gomory cuts. Mathematical Programming **74**, 221–235 (1996)
6. Chandrasekaran, K., Végh, L., Vempala, S.: The cutting plane method is polynomial for perfect matchings. Mathematics of Operations Research **41**, 23–48 (2016)
7. Chvátal, V.: Edmonds polytope and a hierarchy of combinatorial problems. Discrete Mathematics **4**, 305–337 (1973)
8. Conforti, M., Cornuéjols, G., Zambelli, G.: Integer Programming. Springer, Switzerland (2014)
9. Conforti, M., Gerards, A.M.H., Zambelli, G.: Mixed-integer vertex covers on bipartite graphs. In the Proceedings of IPCO 2007, Lecture Notes in Computer Science **4513**, 324–336 (2007)
10. Cook, W., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programs. Mathematical Programming **47**, 155–174 (1990)

11. Del Pia, A., Zambelli, G.: Half-integral vertex covers on bipartite bidirected graphs: total dual integrality and cut-rank. SIAM Journal on Discrete Mathematics **23**, 1281–1296 (2009)
12. Del Pia, A., Musitelli, A., Zambelli, G.: A class of matrices with the Edmonds-Johnson property arising from bidirected graphs. Conditionally accepted by Journal of Combinatorial Theory, Series B (2017).
13. Edmonds, J.: Maximum matching and a polyhedron with 0,1-vertices. Journal of Research of the National Bureau of Standards B **69**, 125–130 (1965)
14. Edmonds, J., Johnson, E.L.: Matching, Euler tours and the Chinese postman. Mathematical Programming **5**, 88-124 (1973)
15. Eisenbrand, F.: On the membership problem for the elementary closure of a polyhedron. Combinatorica **19**, 297–300 (1999)
16. Fischetti, M., Lodi, A.: Optimizing over the first Chvátal closure. Mathematical Programming **110**, 3–20 (2007)
17. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, San Francisco (1979)
18. Gerards, A.M.H., Schrijver, A.: Matrices with the Edmonds-Johnson property. Combinatorica **6**, 365–379 (1986)
19. Goldreich, O.: P, NP, and NP-Completeness: The Basics of Computational Complexity. Cambridge University Press (2010)
20. Gomory, R.E.: Outline of an algorithm for integer solutions to linear programs. Bulletin of the American Mathematical Society **64**, 275–278 (1958)
21. Gomory, R.E.: Some polyhedra related to combinatorial problems. Linear Algebra and Its Applications **2**, 451–558 (1969)
22. Grötschel, M., Lovász, L., Schrijver, A.: The ellipsoid method and its consequences in combinatorial optimization. Combinatorica **1**, 169–197 (1981)
23. Hartmann, M. E., Queyranne, M., Wang, Y.: On the Chvátal rank of certain inequalities. IPCO, Cornuéjols, G., Burkard, R.E., Woeginger, G.J. (Eds.), Lecture Notes in Computer Science **1610**, 218–233 (1999)
24. Hartvigsen, D., Li, Y.: Polyhedron of triangle-free simple 2-matchings in subcubic graphs. Mathematical Programming **138**, 43–82 (2013)
25. Kobayashi, Y.: A simple algorithm for finding a maximum triangle-free 2-matching in subcubic graphs. Discrete Optimization **7**, 197–202 (2010)
26. Lenstra, Jr., H.W.: Integer programming with a fixed number of variables. Mathematics of Operations Research **8**, 538–548 (1983)
27. Lueker, G.S.: Two NP-complete problems in nonnegative integer programming. Report No. 178, Department of Computer Science, Princeton University, Princeton, N.J., (1975)
28. Mahajan, A., Ralphs, T.: On the complexity of selecting disjunctions in integer programming. SIAM Journal on Optimization **20**, 2181–2198 (2010)
29. Meyer, R.R.: On the existence of optimal solutions to integer and mixed integer programming problems. Mathematical Programming **7**, 223–235 (1974)
30. Padberg, M.W., Rao, M.R.: Odd minimum cut-sets and b-matchings. Mathematics of Operations Research **7**, 67–80 (1982)
31. Pulleyblank, W.R., Edmonds, J.: Facets of 1-matching polyhedra, in Hypergraph Seminar (C. Berge and D. Ray-Chaudhuri, eds.), Springer, Berlin, 214–242 (1974)
32. Schrijver, A.: On cutting planes. Annals of Discrete Mathematics **9**, 291–296 (1980)
33. Schrijver, A.: Theory of Linear and Integer Programming, Wiley, Chichester (1986)
34. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency. Springer, Berlin (2003)