

When the Gomory-Chvátal Closure Coincides with the Integer Hull

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Abstract Gomory-Chv atal cuts are prominent in integer programming. The Gomory-Chv atal closure of a polyhedron P is the intersection of the half spaces defined by all its Gomory-Chv atal cuts. This set contains the integer hull of P , *i.e.*, the convex hull of the integer points in P . In this paper, we focus on polyhedra P with the property that their Gomory-Chv atal closure is identical to the integer hull of P . Boyd and Pulleyblank showed that deciding whether there exists an integer point in such a polyhedron is in the complexity class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$. This raises the possibility that integer linear programs with such constraints sets P might be polynomially solvable. We exhibit a few known integer linear programs with such constraints that can be solved in polynomial time. From the recognition perspective, we give a negative result by proving that it is \mathcal{NP} -hard to decide whether a rational polyhedron belongs to this class. Specifically, we show that it is \mathcal{NP} -complete to decide whether the Gomory-Chv atal closure of a given rational polyhedron P is empty, even when it is known that the integer hull of P is empty. We prove similar results for the $\{-1, 0, 1\}$ -cuts and $\{0, 1\}$ -cuts, two special classes of Gomory-Chv atal cuts with variable coefficients restricted in $\{-1, 0, 1\}$ and $\{0, 1\}$, respectively.

Keywords Integer programming · Gomory-Chv atal cuts · Gomory-Chv atal closure · Integer hull · Computational complexity

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1 Introduction

Throughout this paper we will assume a knowledge of elementary integer programming definitions and results. One may use the book by Conforti et al. [7] as a reference.

We consider the integer program: $\min wx$ s.t. $Ax \leq b$, $x \in \mathbf{Z}^n$, where $b \in \mathbf{Z}^m$, $w \in \mathbf{Z}^n$ and $A \in \mathbf{Z}^{m \times n}$. Let $P \equiv \{x \in \mathbf{R}^n : Ax \leq b\}$ denote the feasible region of the linear programming relaxation of this integer program. Polyhedra of this form, where $b \in \mathbf{Z}^m$ and $A \in \mathbf{Z}^{m \times n}$, are called *rational polyhedra*. The convex hull of $P \cap \mathbf{Z}^n$ is called the *integer hull* of P and is denoted by P_I . Meyer [24] showed that the integer hull P_I of a rational polyhedron P is itself a rational polyhedron.

An inequality of the form $cx \leq \lfloor d \rfloor$ is called a *Gomory-Chv atal cut* of P if $c \in \mathbf{Z}^n$ and $cx \leq d$ is valid for every $x \in P$. It follows from this definition that Gomory-Chv atal cuts are valid for $P \cap \mathbf{Z}^n$. Gomory-Chv atal cuts were originally proposed by Gomory [15] as a method for solving integer programs. Chv atal [6] introduced a notion of closure associated with these cuts. The *Gomory-Chv atal closure* of P is $P' \equiv \{x \in P : cx \leq \lfloor d \rfloor \forall c \in \mathbf{Z}^n \text{ and } d \in \mathbf{R} \text{ such that } cx \leq d \text{ is valid for } P\}$. Clearly, $P_I \subseteq P' \subseteq P$, and, from the theory of Gomory-Chv atal cuts, the second inclusion is strict when $P \neq P_I$, implying that P' is typically a tighter relaxation of P_I than P . Chv atal [6] showed that the Gomory-Chv atal closure of a rational polytope is again a rational polytope, and Schrijver [27] showed, more generally, that the Gomory-Chv atal closure of a rational polyhedron is again a rational polyhedron. Hartmann, Queyranne and Wang [18] gave a necessary and sufficient condition for a facet-defining inequality of P_I to be a Gomory-Chv atal cut. The number of Gomory-Chv atal cuts needed to describe the Gomory-Chv atal closure of a rational polyhedron can be very large. For example, for the matching problem, P is defined by the degree and nonnegativity constraints; its Gomory-Chv atal closure P' is the matching polytope, which comprises of the blossom constraints in addition to the constraints defining P . For the matching problem on a complete graph, the blossom constraints define facets of P' [26] and their number is exponential in the size of P .

Optimizing a linear function over the Gomory-Chv atal closure P' of a rational polyhedron P often gives a better bound on the optimal objective value of the integer program than optimizing over P . The *optimization problem* over the Gomory-Chv atal closure of P (GC-Opt) is: Given $P \subseteq \mathbf{R}^n$ and a vector $c \in \mathbf{Z}^n$, either find a point $x^* \in P'$ that optimizes the function cx , or conclude that the optimal value of cx over P' is unbounded, or conclude that $P' = \emptyset$. The *separation problem* for the Gomory-Chv atal closure of P (GC-Sep) is: Given $P \subseteq \mathbf{R}^n$ and a point $x^* \in \mathbf{R}^n$, either find a Gomory-Chv atal cut of P such that x^* violates the cut, or conclude that $x^* \in P'$. It follows from a general result of Gr otscchel, Lov asz and Schrijver [17] that GC-Sep can be solved in polynomial time if and only if GC-Opt can be solved in polynomial time.

A negative result proved by Eisenbrand [10] is that GC-Sep is \mathcal{NP} -hard, which implies the \mathcal{NP} -hardness of GC-Opt as well. Eisenbrand's proof is through a polynomial reduction to GC-Sep of the weighted binary clutter problem, known to be \mathcal{NP} -complete. Despite this complexity result, the quality of the bound provided by solving GC-Opt has been tested empirically. Fischetti and Lodi [11] were able to estimate the fraction of the integrality gap closed by the Gomory-Chvátal closure on some hard instances from MIPLIB 3.0. They formulated GC-Sep through a mixed integer programming model and solved it using a general-purpose mixed integer programming solver. On these MIPLIB instances, they reported that by optimizing over the Gomory-Chvátal closure instead of the standard linear programming relaxation, the fraction of the gap closed is often quite significant, in excess of 80% on about half the instances. In fact, the gap was closed completely on over a quarter of these MIPLIB instances.

In this paper, we study the rational polyhedra P whose Gomory-Chvátal closure P' is identical to the integer hull P_I . There are two questions about this type of polyhedra that we think are fundamental, one on optimization and the other on recognition:

- (Q1) Given a rational polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for which we know that $P' = P_I$, are GC-Opt and GC-Sep polynomially solvable?
- (Q2) Given a rational polyhedron P , can one decide in polynomial time whether $P' = P_I$?

Question (Q1) is a “promise problem”, *i.e.*, the algorithms only have to output a correct answer in polynomial time if the input satisfies some specified property, here $P' = P_I$. We do not have a positive or negative answer for (Q1). However, Boyd and Pulleyblank [2] observed that, given a rational polyhedron P for which we know that $P' = P_I$, the feasibility problem (that is, whether or not $P' = P_I = \emptyset$) belongs to the complexity class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$. This can be regarded as strong evidence that the feasibility problem is probably not \mathcal{NP} -complete, implying that the two problems in (Q1) are probably not \mathcal{NP} -hard [14]. We present three known integer linear programs, whose linear programming relaxation polyhedron P satisfies $P' = P_I$, that can be solved in polynomial time.

For (Q2), we do not know whether the decision problem belongs to the complexity class \mathcal{NP} but we will prove that it is \mathcal{NP} -hard. Specifically, we provide the following negative answer to (Q2): Given a rational polyhedron P such that $P_I = \emptyset$, it is \mathcal{NP} -complete to decide whether $P' = \emptyset$. This result clearly implies Eisenbrand's [10] result that GC-Opt is \mathcal{NP} -hard. We also prove similar results for the $\{-1, 0, 1\}$ -cuts and $\{0, 1\}$ -cuts. A Gomory-Chvátal cut is called a $\{-1, 0, 1\}$ -cut (or $\{0, 1\}$ -cut) if the vector of variable coefficients $c \in \{-1, 0, 1\}^n$ (or $\{0, 1\}^n$). The $\{-1, 0, 1\}$ -closure (or $\{0, 1\}$ -closure) of P is $P'_{\{-1, 0, 1\}} \equiv \{x \in P : cx \leq \lfloor d \rfloor \forall c \in \{-1, 0, 1\}^n \text{ and } d \in \mathbf{R} \text{ such that } cx \leq d \text{ is valid for } P\}$ (or $P'_{\{0, 1\}} \equiv \{x \in P : cx \leq \lfloor d \rfloor \forall c \in \{0, 1\}^n \text{ and } d \in \mathbf{R} \text{ such that } cx \leq d \text{ is valid for } P\}$). We show that, given

a polyhedron P such that $P_I = \emptyset$, it is \mathcal{NP} -complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ (or $P'_{\{0,1\}} = \emptyset$).

We borrowed some ideas from Mahajan and Ralphs [23] to construct the polyhedra P in the proof of our \mathcal{NP} -completeness results. In their paper the following disjunctive infeasibility problem is proved to be \mathcal{NP} -complete: Given a polyhedron $P \subset \mathbf{R}^n$, does there exist $\pi \in \mathbf{Z}^n$ and $\pi_0 \in \mathbf{Z}$ such that $\{x \in P : \pi x \leq \pi_0 \text{ or } \pi x \geq \pi_0 + 1\} = \emptyset$? The polyhedra P used in [23] are simplices, whereas the polyhedra in our construction are convex hulls of $n + 3$ vectors in \mathbf{R}^n . Mahajan and Ralphs' proof reduces the well-known partition problem to the disjunctive infeasibility problem, whereas, in our proofs, we reduce the single constraint integer programming feasibility problem to the emptiness problem of the Gomory-Chv atal closure, and the partition problem to the emptiness problems of the $\{-1, 0, 1\}$ -closure and $\{0, 1\}$ -closure.

The remaining sections of this paper are organized as follows. In Section 2, we will show that the feasibility problem for integer linear programs whose associated polyhedron P has the property that $P' = P_I$, is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$, and we will give examples of integer programs with this property that can be solved with polynomial-time algorithms. In Section 3, we will prove the \mathcal{NP} -completeness of deciding whether the Gomory-Chv atal closure of a rational polyhedron with no integer point is empty. In Section 4, we will extend the result of Section 3 to the $\{-1, 0, 1\}$ -closure and $\{0, 1\}$ -closure. Lastly, we conclude with open questions in Section 5.

2 Optimizing over the Gomory-Chv atal closure of a rational polyhedron P for which $P' = P_I$

The polynomial solvability of GC-Opt for polyhedra P that satisfy $P' = P_I$ is an open question, and similarly for the associated GC-Sep problem. The optimization problem is closely related to the following feasibility problem: Given a rational polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for which we know that $P' = P_I$, decide whether $P_I = \emptyset$. Through a standard binary search argument on the objective value, the optimization problem is polynomially solvable if and only if the feasibility problem is polynomially solvable. Boyd and Pulleyblank [2] showed that the feasibility problem is in the complexity class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$. See also Schrijver [28] Corollary 23.5a. We give a short proof of this result (Theorem 1) for completeness. The main step the proof is the following lemma.

Lemma 1 *Deciding whether the Gomory-Chv atal closure of a rational polyhedron is empty belongs to the complexity class \mathcal{NP} .*

Proof Let $P \equiv \{x \in \mathbf{R}^n : Ax \leq b\}$ be a rational polyhedron, where $b \in \mathbf{Z}^m$, and $A \in \mathbf{Z}^{m \times n}$. Chv atal [6] showed that there is only a finite number of inequalities needed to describe the Gomory-Chv atal closure of P , namely inequalities $uAx \leq \lfloor ub \rfloor$ where $u \in \mathbf{R}^m$ is a vector satisfying $uA \in \mathbf{Z}^n$ and

$0 \leq u < 1$. Note that the integer vectors $d \equiv uA$ in these inequalities have components satisfying $-\sum_{i=1}^m |a_{ij}| \leq d_j \leq \sum_{i=1}^m |a_{ij}|$. Similarly, $d_0 \equiv \lfloor ub \rfloor$ satisfies $-\sum_{i=1}^m |b_i| \leq d_0 \leq \sum_{i=1}^m |b_i|$. Therefore the above inequalities are described by coefficients whose encoding size is polynomial in the size of the input. To certify that the Gomory-Chvátal closure of P is empty, we appeal to Helly's theorem: If the Gomory-Chvátal closure of P is empty, there exist $n+1$ of these inequalities whose intersection is empty. A list of $n+1$ such inequalities is a polynomial certificate that the Gomory-Chvátal closure of P is empty. \square

Theorem 1 *Given a rational polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ for which we know that $P' = P_I$, the problem of deciding whether $P_I = \emptyset$ is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.*

Proof We already know from Lemma 1 that the problem is in \mathcal{NP} . To prove that it is in $\text{co-}\mathcal{NP}$, it is sufficient to exhibit a point $x \in \mathbf{Z}^n$ that satisfies $Ax \leq b$ when such a point exists. It is well known that, if such a point exists, there is one whose encoding is polynomial in the size of the input [1]. Therefore, a polynomial $\text{co-}\mathcal{NP}$ certificate exists for $P_I = \emptyset$ when $P' = P_I$. \square

Note that, although a polynomial $\text{co-}\mathcal{NP}$ certificate exists for $P' = \emptyset$ when $P' = P_I$, no polynomial $\text{co-}\mathcal{NP}$ certificate is known for $P' = \emptyset$ in general.

Next, we present three known integer linear programs whose linear programming relaxation polyhedron P has the property that $P' = P_I$. The respective polyhedral description of P' as well as some polynomial-time algorithms for solving the integer linear program are reviewed.

A well-known example in this class is due to Edmonds [9]. A matching in an undirected graph $G = (V, E)$ is a set of edges no two of which are incident to the same node. Let $P \equiv \{x \in \mathbf{R}^{|E|} : x(\delta(v)) \leq 1 \forall v \in V, x_e \geq 0 \forall e \in E\}$, where $\delta(v)$ denotes the set of edges incident to node v and $x(S) \equiv \sum_{e \in S} x_e$. Edmonds showed via an algorithmic method that $P' = P_I$, and that this polytope is defined by the blossom inequalities, in addition to the inequalities in the description of P . In particular, Edmonds gave a polynomial-time primal-dual algorithm for solving GC-Opt for the matching problem. Padberg and Rao [25] devised a polynomial-time separation algorithm for the b -matching polytope, implying polynomial-time solvability of GC-Sep for the matching problem. The separation oracle given by Padberg and Rao is an efficient algorithm for the minimum odd-cut problem, which is the following: Given an edge-capacitated undirected graph $G = (V, E)$, find a cut $\delta(U)$ of minimum capacity among all node sets U such that $|U|$ is odd (here $\delta(U)$ denotes the set of edges with exactly one endnode in U). Recently, Chandrasekaran, Végé and Vempala [5] developed a linear-programming-based cutting plane algorithm for solving GC-Opt for the matching problem. The algorithm sequentially adds violated blossom inequalities. By maintaining half-integral intermediate linear programming solutions, they are able to give a polynomial bound on the number of iterations needed for the convergence of their cutting plane algorithm.

As a second example, consider the maximum weight stable set problem $\max\{wx : x \in P_I\}$ in a graph $G = (V, E)$, where $P \equiv \{x \in \mathbf{R}_+^V : x_i + x_j \leq 1 \text{ for } ij \in E\}$ is the so-called edge formulation. We note that this problem is \mathcal{NP} -hard in general, but that it can be solved in polynomial time when $P' = P_I$. Indeed, Campelo and Cornu ejols [3] showed that P' is entirely described by the inequalities defining P together with the odd circuit inequalities $\sum_{i \in C} x_i \leq \frac{|C|-1}{2}$ for node sets C of odd cardinality that induce a circuit of G . The main result in [3] is that the $\{0, 1/2\}$ -Chv atal closure [4], the Gomory-Chv atal closure, the split closure [8], and the intersection of all possible corner polyhedra [16] of P are all identical to $\{x \in \mathbf{R}_+^V : x_i + x_j \leq 1 \text{ for every } ij \in E, \sum_{i \in C} x_i \leq \frac{|C|-1}{2} \text{ for every induced odd cycle } C \text{ of } G\}$. The graphs for which these inequalities completely describe the stable set polytope P_I are called *t-perfect graphs*. These graphs are discussed in Chapter 68 of Schrijver's book [29]. Theorem 68.1 states that a maximum-weight stable set in a *t*-perfect graph can be found in polynomial time. This follows from the equivalence of optimization and separation [17] and the fact that the separation of odd circuit inequalities can be done in polynomial time by reduction to a shortest path problem [13].

The third example is with regard to the problem of finding a maximum weight simple 2-matching that contains no triangles in an edge-weighted subcubic graph, where a simple 2-matching in a graph is a subgraph all of whose nodes have degree 0, 1 or 2, and a subcubic graph is a graph in which every node has degree at most 3. From a natural integer programming formulation, $P = \{x \in \mathbf{R}_+^E : x(\delta(v)) \leq 2 \text{ for every } v \in V, x_e \leq 1 \text{ for every } e \in E, x(T) \leq 2 \text{ for every triangle } T \text{ of } G\}$. Hartvigsen and Li [19] showed that $P' = P_I$, which is determined by the tri-comb inequalities and the inequalities in the description of P , where tri-combs are more general than blossoms and a special case of combs. To prove this result, they developed an Edmonds-type primal-dual algorithm for solving the GC-Opt in polynomial time. Kobayashi [20] gave a polynomial-time algorithm for solving the same problem that comprises of two basic algorithms: a steepest ascent algorithm and a classical maximum weight 2-matching algorithm. The correctness of his method is justified using some fundamental results from the theory of discrete convex functions on jump systems.

3 Deciding Emptiness of the Gomory-Chv atal Closure of a Rational Polyhedron with No Integer Point

In this section, we prove our main result: It is \mathcal{NP} -complete to decide emptiness of the Gomory-Chv atal closure of a rational polyhedron containing no integer point. Our proof proceeds in three steps. First, we note that Lemma 1 proved in the previous section shows that deciding whether the Gomory-Chv atal closure of a rational polyhedron is empty is in \mathcal{NP} . Second, we prove in Lemma 2 that it is \mathcal{NP} -complete to decide whether the Gomory-Chv atal

closure of a rational polyhedron is empty. Third, we prove in Lemma 3 that the polyhedron used in the proof of Lemma 2 contains no integer point.

Lemma 2 *It is \mathcal{NP} -complete to decide whether the Gomory-Chvátal closure of a rational polyhedron is empty.*

Proof The lemma will be proved by polynomially reducing the following single constraint integer programming feasibility problem, which is known to be \mathcal{NP} -complete [22], to the problem of deciding whether $P' = \emptyset$ for a rational polyhedron P .

Single Constraint Integer Programming Feasibility Problem: Given a finite set of nonnegative integers $\{a_i\}_{i=1}^s$ and a nonnegative integer b , is there a set of nonnegative integers $\{x_i\}_{i=1}^s$ satisfying $\sum_{i=1}^s a_i x_i = b$?

We consider the Single Constraint Integer Programming Feasibility Problem with $s = n - 1$ and $n \geq 3$. We assume without loss of generality that the greatest common divisor of a_1, a_2, \dots, a_{n-1} is 1, and $2 < a_1 < a_2 < \dots < a_{n-1} < b$. So $b \geq n + 2$. Let $r = n + 1 + \frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbf{Z}_+$. We will show:

Reduction:

The Single Constraint Integer Programming Feasibility Problem can be polynomially reduced to the problem of deciding whether $P' = \emptyset$ for the polyhedron $P \subseteq \mathbf{R}^n$ that is the convex hull of the following $n + 3$ vectors:

$$\begin{aligned} v_1 &\equiv \left(\frac{1}{2b}, 0, \dots, 0, \frac{1}{2b}\right), \\ v_2 &\equiv \left(0, \frac{1}{2b}, 0, \dots, 0, \frac{1}{2b}\right), \\ &\dots\dots\dots \\ v_{n-1} &\equiv \left(0, \dots, 0, \frac{1}{2b}, \frac{1}{2b}\right), \\ v_n &\equiv \left(0, \dots, 0, \frac{1}{2} + \frac{3}{2r}\right), \\ v_{n+1} &\equiv \left(a_1, a_2, \dots, a_{n-1}, -b + \frac{1}{2}\right), \\ v_{n+2} &\equiv \left((1-r)a_1, (1-r)a_2, \dots, (1-r)a_{n-1}, (r-1)b + 1\right), \\ v_{n+3} &\equiv \left(0, \dots, 0, \frac{1}{2r}\right). \end{aligned}$$

To show that this reduction is correct, we prove the following two claims; we then observe that converting the vectors v_1, v_2, \dots, v_{n+3} into an inequality description of P can be done in polynomial time.

Claim 1. There is a set of nonnegative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ only if $P' = \emptyset$.

Proof. Consider an inequality $cx \leq q$, where $c = (c_1, c_2, \dots, c_n)$, $c_i = -w_i$ for $1 \leq i \leq n - 1$, $c_n = -1$, and $q = \max\{-\frac{1}{2b}, -\frac{1}{2r}\} = -\frac{1}{2b}$. Because $1 < a_i < b$ for $1 \leq i \leq n - 1$ and $\sum_{i=1}^{n-1} a_i w_i = b$, it is easy to verify that $-1 < cv_i \leq -\frac{1}{2b}$ for $1 \leq i \leq n - 1$. In addition, $-1 < cv_n < -\frac{1}{2}$, $cv_{n+1} = -\frac{1}{2}$, $cv_{n+2} = -1$, and $cv_{n+3} = -\frac{1}{2r}$. So $cx \leq q$ is valid for P , and the associated Gomory-Chvátal inequality is $cx \leq \lfloor q \rfloor (= -1)$. We can see that v_{n+2} is the

only vector in P that satisfies the inequality $cx \leq \lfloor q \rfloor$. Consider the inequality $fx \leq g$, where $fx = x_n$ and $g = (r-1)b + 1$. It can be easily checked that every v_i satisfies $fx \leq g$, so $fx \leq g$ is valid for P , and $fx \leq \lfloor g \rfloor = \lfloor rb \rfloor - b + 1$ is a Gomory-Chv atal inequality of P . Since $rb \notin \mathbf{Z}_+$, $fx \leq \lfloor g \rfloor$ is violated by v_{n+2} . Now we can conclude that $P' = \emptyset$ and Claim 1 is proved.

Claim 2. There is a set of nonnegative integers $\{w_i\}_{i=1}^{n-1}$ satisfying $\sum_{i=1}^{n-1} a_i w_i = b$ if $P' = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \dots, 0, \frac{1}{2r} + \frac{1}{2})$. We know $v_0 \in P$ because $v_0 = \alpha v_n + (1-\alpha)v_{n+3}$ for some $0 < \alpha < 1$. Let $cx \leq \lfloor q \rfloor$ be a Gomory-Chv atal inequality of P that is violated by v_0 , where $c = (c_1, c_2, \dots, c_n) \in \mathbf{Z}^n$ and $cx \leq q$ is valid for P . Then $c_n \neq 0$, otherwise we would have $0 = cv_0 \leq q$, contradicting that $cv_0 > \lfloor q \rfloor$.

Let $\Delta \equiv \sum_{i=1}^{n-1} c_i a_i - c_n b$. First, we show that $c_n \leq -1$ by deriving contradiction in the following two cases:

Case 1. $c_n \geq 1$ and $\Delta \geq 1$. If $\frac{c_n}{2r} \leq \Delta - 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} \leq \Delta - 1 + \frac{c_n}{2} < \lfloor \Delta + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, which contradicts that $cv_0 > \lfloor q \rfloor$. If $\frac{c_n}{2r} > \Delta - 1$, then $cv_n - cv_0 = \frac{c_n}{r} > 2(\Delta - 1)$. If $\Delta \geq 2$, then $cv_n - cv_0 > 2$, so $cv_0 < cv_n - 2 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction to that $cv_0 > \lfloor q \rfloor$. If $\Delta = 1$, then $\frac{c_n}{r} \geq 1$. Otherwise, if $\frac{c_n}{r} < 1$, then $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < \frac{1}{2} + \frac{c_n}{2} \leq \lfloor 1 + \frac{c_n}{2} \rfloor = \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. Since $\frac{c_n}{r} \geq 1$, $cv_n - cv_0 \geq 1$, therefore $cv_0 \leq cv_n - 1 < \lfloor cv_n \rfloor \leq \lfloor q \rfloor$, a contradiction again.

Case 2. $c_n \geq 1$ and $\Delta \leq 0$. Because $r > n + 1 > 3$, $cv_0 = \frac{c_n}{2r} + \frac{c_n}{2} < c_n \leq (1-r)\Delta + c_n = cv_{n+2}$. Hence, $cv_0 < \lfloor cv_{n+2} \rfloor \leq \lfloor q \rfloor$, a contradiction.

It is easy to see that $c_n = -1$. Otherwise, if $c_n \leq -2$, then $cv_{n+3} - cv_0 = -\frac{c_n}{2} \geq 1$, which implies $cv_0 \leq cv_{n+3} - 1 < \lfloor cv_{n+3} \rfloor \leq \lfloor q \rfloor$, a contradiction.

Now we show that $\Delta = 0$. If $\Delta \geq 1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2} < 0 < \Delta - \frac{1}{2} = cv_{n+1}$, implying $cv_0 < \lfloor cv_{n+1} \rfloor \leq \lfloor q \rfloor$, a contradiction. If $\Delta \leq -1$, then, because $r > 3$, $cv_0 < 0 < (1-r)\Delta - 1 = cv_{n+2}$, a contradiction again.

We claim that $c_i \leq 0$ for $i = 1, 2, \dots, n-1$. Otherwise, if $c_i \geq 1$ for some $1 \leq i \leq n-1$, then $cv_0 < 0 \leq \frac{c_i}{2b} - \frac{1}{2b} = cv_i$, a contradiction.

Now let $w_i = -c_i$ for $1 \leq i \leq n-1$. Then Claim 2 is proved.

To complete the proof, it suffices to show that a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbf{Z}^{m \times n}$ and $\tilde{b} \in \mathbf{Z}^m$, can be obtained in polynomial time from the vectors v_1, v_2, \dots, v_{n+3} . We can see from the coordinates of v_1, v_2, \dots, v_n and v_{n+3} that P is a n -dimensional polyhedron. Let i be a counter looping through $n, n+1$ and $n+2$. For every i vectors of v_1, v_2, \dots, v_{n+3} , check if they are on a unique hyperplane by solving linear equations. If yes, then further check if the $n+3-i$ other vectors are all on one side of the hyperplane. If yes again, then the equation $\tilde{c}x = \tilde{d}$ of the hyperplane with integral \tilde{c} and \tilde{d} whose greatest common divisor is 1 yields a linear inequality of $\tilde{A}x \leq \tilde{b}$. One can easily see that this process takes polynomial time and the size of \tilde{A} and \tilde{b} is polynomial in the size of v_1, v_2, \dots, v_{n+3} . \square

Lemma 3 *The polyhedron P used in the proof of Lemma 2 contains no integer point.*

Proof Proving that P contains no integer point is equivalent to proving that $P^d \equiv P \cap \{x \in \mathbf{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [-b + 1, nb + 1]$. Let P_1 be the convex hull of $v_1, v_2, \dots, v_n, v_{n+1}$ and v_{n+3} , and let P_2 be the convex hull of $v_1, v_2, \dots, v_n, v_{n+2}$ and v_{n+3} . Since $v_0 = \frac{r-1}{r}v_{n+1} + \frac{1}{r}v_{n+2}$ and $v_0 = \alpha v_n + (1 - \alpha)v_{n+3}$ for some $0 < \alpha < 1$, it is sufficient to show: (a) $P_1^d \equiv P_1 \cap \{x \in \mathbf{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [-b + 1, 0]$; (b) $P_2^d \equiv P_2 \cap \{x \in \mathbf{R}^n : x_n = d\}$ contains no integer points for every integer $d \in [1, nb + 1]$.

We first prove (a). Because $v_1, v_2, \dots, v_n, v_{n+3} \in \{x \in \mathbf{R}^n : x_n > 0\}$ and $v_{n+1} \in \{x \in \mathbf{R}^n : x_n < 0\}$, it is easy to verify by calculation that P_1^d , where d is an integer in $[-b + 1, 0]$, is the convex hull of the $n + 1$ vectors:

$$\begin{aligned} & \left(\frac{1-2bd}{2b^2-b+1}a_1 + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, \frac{1-2bd}{2b^2-b+1}a_2, \frac{1-2bd}{2b^2-b+1}a_3, \dots, \frac{1-2bd}{2b^2-b+1}a_{n-1}, d \right), \\ & \left(\frac{1-2bd}{2b^2-b+1}a_1, \frac{1-2bd}{2b^2-b+1}a_2 + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, \frac{1-2bd}{2b^2-b+1}a_3, \dots, \frac{1-2bd}{2b^2-b+1}a_{n-1}, d \right), \\ & \dots \dots \dots \\ & \left(\frac{1-2bd}{2b^2-b+1}a_1, \frac{1-2bd}{2b^2-b+1}a_2, \dots, \frac{1-2bd}{2b^2-b+1}a_{n-2}, \frac{1-2bd}{2b^2-b+1}a_{n-1} + \frac{b+d-\frac{1}{2}}{2b^2-b+1}, d \right), \\ & \left(\frac{r+3-2rd}{2rb+3}a_1, \frac{r+3-2rd}{2rb+3}a_2, \frac{r+3-2rd}{2rb+3}a_3, \dots, \frac{r+3-2rd}{2rb+3}a_{n-1}, d \right), \\ & \left(\frac{1-2rd}{2rb+1-r}a_1, \frac{1-2rd}{2rb+1-r}a_2, \frac{1-2rd}{2rb+1-r}a_3, \dots, \frac{1-2rd}{2rb+1-r}a_{n-1}, d \right). \end{aligned}$$

Indeed, the first n vectors above are obtained by intersecting the hyperplane $x_n = d$ with the line segment $v_i v_{n+1}$, for $i = 1, 2, \dots, n$, and the last vector is obtained by intersecting the hyperplane $x_n = d$ with the line segment $v_{n+3} v_{n+1}$.

Since $P_1^d \subset \{x \in \mathbf{R}^n : x_n = d\}$, we only need to consider the convex hull of the following $n + 1$ vectors in \mathbf{R}^{n-1} : $\frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_1$, $\frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_2$, \dots , $\frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_{n-1}$, $\frac{r+3-2rd}{2rb+3}a$ and $\frac{1-2rd}{2rb+1-r}a$, where $a \equiv (a_1, a_2, \dots, a_{n-1})$ and e_i is the i -th unit vector. Let \tilde{P}_1^d denote the convex hull. Since $r < b$ and $d \geq -b + 1$, it is easy to verify that $0 < \frac{1-2bd}{2b^2-b+1} < \frac{1-2rd}{2rb+1-r} < \frac{r+3-2rd}{2rb+3} < 1$. So $\tilde{P}_1^d \subseteq Q_1^d$, where Q_1^d is the convex hull of the $n + 1$ vectors:

$$\begin{aligned} z_0 & \equiv \frac{1-2bd}{2b^2-b+1}a, \\ z_1 & \equiv \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_1, \\ z_2 & \equiv \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_2, \\ & \dots \dots \dots \\ z_{n-1} & \equiv \frac{1-2bd}{2b^2-b+1}a + \frac{b+d-\frac{1}{2}}{2b^2-b+1}e_{n-1}, \\ z_n & \equiv \frac{r+3-2rd}{2rb+3}a. \end{aligned}$$

To prove (a), it suffices to show the following claim:

Claim 1. There is no integer point in Q_1^d .

Proof. By contradiction, suppose $\bar{v} \in Q_1^d \cap \mathbf{Z}^{n-1}$. Then there must exist a vector $v' \equiv \beta_0 a$, where $0 < \beta_0 < 1$, such that $\|\bar{v} - v'\|_\infty \equiv \max_{1 \leq i \leq n-1} |\bar{v}_i - v'_i| \leq \frac{b+d-\frac{1}{2}}{2b^2-b+1} \leq \frac{b-\frac{1}{2}}{2b^2-b+1} < \frac{1}{2b}$. From the construction of P , it is easy to see that $0 < \bar{v}_i < a_i$ for $i = 1, 2, \dots, n-1$. Because the greatest common divisor of a_1, a_2, \dots, a_{n-1} is 1, there exists no integer point on the line segment connecting 0 and a except for the two end points. Therefore, there exists $1 \leq k \leq n-2$ such that $(\bar{v}_k, \bar{v}_{n-1})$ is not on the line segment connecting $(0, 0)$ and (a_k, a_{n-1}) in \mathbf{R}^2 . This is because otherwise if $(\bar{v}_i, \bar{v}_{n-1})$ is on the line segment connecting $(0, 0)$ and (a_i, a_{n-1}) for every $1 \leq i \leq n-2$, then $\frac{\bar{v}_i}{a_i} = \frac{\bar{v}_{n-1}}{a_{n-1}}$ for every i , implying that \bar{v} is on the line segment connecting 0 and a , which contradicts the previous statement. Now, to derive contradiction to the inequality $\|\bar{v} - v'\|_\infty < \frac{1}{2b}$, we show below that $\|(\bar{v}_k, \bar{v}_{n-1}) - (v'_k, v'_{n-1})\|_\infty = \max(|\bar{v}_k - v'_k|, |\bar{v}_{n-1} - v'_{n-1}|) \geq \frac{1}{2(b-1)}$.

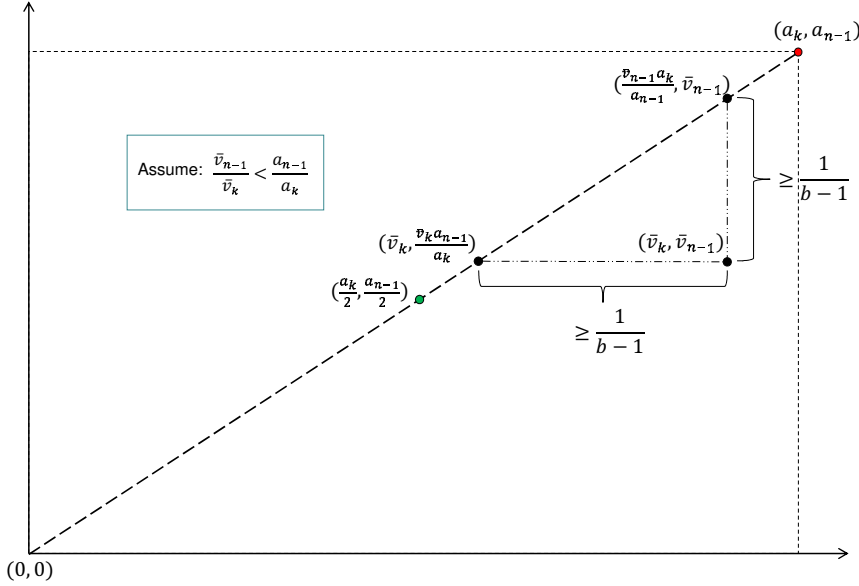


Fig. 1 Proving Claim 1 in the proof of Lemma 3

Let L denote the line segment connecting $(0, 0)$ and (a_k, a_{n-1}) in \mathbf{R}^2 ; see Fig.1 for illustration. We know that (v'_k, v'_{n-1}) is on L . Because the integer points between 0 and (a_k, a_{n-1}) that are not on L are symmetric across $(\frac{a_k}{2}, \frac{a_{n-1}}{2})$, we may assume without loss of generality that $\frac{\bar{v}_{n-1}}{\bar{v}_k} < \frac{a_{n-1}}{a_k}$. It is not hard to see that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\bar{v}_k, \bar{v}_{n-1})$ is attained at a point on the segment L connecting $(\frac{v_{n-1}a_k}{a_{n-1}}, \bar{v}_{n-1})$ and $(\bar{v}_k, \frac{\bar{v}_k a_{n-1}}{a_k})$. Since $\bar{v}_k > \frac{v_{n-1}a_k}{a_{n-1}}$, $\|(\bar{v}_k, \bar{v}_{n-1}) - (\frac{v_{n-1}a_k}{a_{n-1}}, \bar{v}_{n-1})\|_\infty \geq \frac{1}{a_{n-1}} \geq \frac{1}{b-1}$. Because $a_{n-1} \geq a_k$, $\|(\bar{v}_k, \bar{v}_{n-1}) - (\bar{v}_k, \frac{\bar{v}_k a_{n-1}}{a_k})\|_\infty \geq \frac{1}{b-1}$. So it follows that the shortest distance under $\|\cdot\|_\infty$ between a point on L and $(\bar{v}_k, \bar{v}_{n-1})$

is no less than $\frac{1}{2(b-1)}$. Therefore, $\|(\bar{v}_k, \bar{v}_{n-1}) - (v'_k, v'_{n-1})\|_\infty \geq \frac{1}{2(b-1)}$. Claim 1 is proved.

Next we prove (b). Because $v_1, v_2, \dots, v_n, v_{n+3} \in \{x \in \mathbf{R}^n : x_n < 1\}$ and $v_{n+2} \in \{x \in \mathbf{R}^n : x_n > 1\}$, we know by calculation that P_2^d , where d is an integer in $[1, nb + 1]$, is the convex hull of the $n + 1$ vectors:

$$\begin{aligned} & \left(\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_1 + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_2, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_3, \dots, \right. \\ & \left. \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_{n-1}, d \right), \\ & \left(\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_1, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_2 + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_3, \dots, \right. \\ & \left. \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_{n-1}, d \right), \\ & \dots\dots\dots \\ & \left(\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_1, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_2, \dots, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_{n-2}, \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_{n-1} + \right. \\ & \left. \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}, d \right), \\ & \left(\frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_1, \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_2, \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_3, \dots, \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_{n-1}, d \right), \\ & \left(\frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} a_1, \frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} a_2, \frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} a_3, \dots, \frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} a_{n-1}, d \right). \end{aligned}$$

So we just need to prove that the convex hull of the following $n + 1$ vectors in \mathbf{R}^{n-1} , denoted by \bar{P}_2^d , contains no integer points:

$$\begin{aligned} & \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} e_1, \\ & \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} e_2, \\ & \dots\dots\dots \\ & \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} e_{n-1}, \\ & \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a, \\ & \frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} a. \end{aligned}$$

The following properties can be verified by calculation, using $d \leq nb + 1$ and $b > r$:

1. $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} < \frac{(r-1)(\frac{1}{2r} - d)}{(r-1)b + 1 - \frac{1}{2r}} < \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} < 0$, and each of the three terms strictly decreases as d increases.
2. For $d = kb + h$, where integers k and h satisfy $0 \leq k \leq \lfloor r - 1 \rfloor = n$ and $0 < h < b$, $\frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_i < -ka_i$ and $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_i + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < -ka_i$ for $i = 1, 2, \dots, n - 1$.
3. For $d = kb$, where integer k satisfies $1 \leq k \leq \lfloor r - 1 \rfloor = n$, $\frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}} a_i < -(k - 1)a_i$, $-ka_i < \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_i$, and $\frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1} a_i + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < -(k - 1)a_i$ for $i = 1, 2, \dots, n - 1$.

By the above property 1, $\tilde{P}_2^d \subseteq Q_2^d$, where Q_2^d is the convex hull of the $n+1$ vectors:

$$\begin{aligned} y_0 &\equiv \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a, \\ y_1 &\equiv \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_1, \\ y_2 &\equiv \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_2, \\ &\dots\dots\dots \\ y_{n-1} &\equiv \frac{(r-1)(1-2bd)}{2(r-1)b^2+2b-1}a + \frac{(r-1)b+1-d}{2(r-1)b^2+2b-1}e_{n-1}, \\ y_n &\equiv \frac{(r-1)(\frac{1}{2} + \frac{3}{2r} - d)}{(r-1)b + \frac{1}{2} - \frac{3}{2r}}a. \end{aligned}$$

To prove (b), it suffices to show that Q_2^d contains no integer points. Given the properties 2 and 3 and the fact that $\frac{(r-1)b+1-d}{2(r-1)b^2+2b-1} < \frac{1}{2b}$, the proof is very similar to that of Claim 1. The lemma is proved. \square

In conclusion, we obtain the following theorem as result of Lemmas 1-3.

Theorem 2 *Given a rational polyhedron containing no integer point, it is \mathcal{NP} -complete to decide whether its Gomory-Chvátal closure is empty.*

4 Deciding Emptiness of the $\{-1, 0, 1\}$ -Closure (or $\{0, 1\}$ -Closure) of a Rational Polyhedron with No Integer Point

In this section, we first prove \mathcal{NP} -completeness of deciding emptiness of the $\{-1, 0, 1\}$ -closure of a rational polyhedron containing no integer point, and then prove the same for the $\{0, 1\}$ -closure as a corollary.

Theorem 3 *Given a rational polyhedron containing no integer point, it is \mathcal{NP} -complete to decide whether its $\{-1, 0, 1\}$ -closure is empty.*

Proof We will prove the theorem by polynomially reducing the following partition problem, which is known to be \mathcal{NP} -complete [12], to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for a polyhedron P with no integer points.

Partition Problem: Given a finite set of positive integers $S = \{a_i\}_{i=1}^s$, is there a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$?

Let $b \equiv \frac{1}{2} \sum_{1 \leq i \leq s} a_i$. We assume without loss of generality that $a_i < b$ for $i = 1, 2, \dots, s$ and that the greatest common divisor of a_1, a_2, \dots, a_s is 1. We also assume without loss of generality that $s \geq 8$ (because the Partition Problem with fixed s can be formulated as an integer program with only s binary variables, which can be solved in polynomial time [21]).

First, we prove that b can be assumed to be greater than or equal to $s+3$. Note that a Partition Problem with $b < s+3$ can be polynomially converted to another Partition Problem with $S' = S \cup \{\sum_{1 \leq i \leq s} a_i + 1, \sum_{1 \leq i \leq s} a_i + 1\}$.

Let $s' = |S'|$. So $s' = s + 2$. It is easy to see that the Partition Problem with S has a feasible partition if and only if the Partition Problem with S' has a feasible partition. Let $b' \equiv b + (\sum_{1 \leq i \leq s} a_i + 1)$. Then $b' = \frac{3}{2} \sum_{1 \leq i \leq s} a_i + 1 \geq \frac{3s}{2} + 1 = s + 5 + (\frac{s}{2} - 4) \geq (s + 2) + 3 = s' + 3$.

Now we consider the Partition Problem with $s = n - 1$, $n \geq 9$ and $b \geq n + 2$. Let $r = n + 1 + \frac{1}{2b}$. So r is a rational number satisfying $r < b$ and $rb \notin \mathbf{Z}_+$. We only need to show that the Partition Problem can be polynomially reduced to the problem of deciding whether $P'_{\{-1,0,1\}} = \emptyset$ for the same polyhedron P as constructed in the proof of Lemma 2, *i.e.*, the convex hull of the $n + 3$ vectors: $v_1 = (\frac{1}{2b}, 0, \dots, 0, \frac{1}{2b})$, $v_2 = (0, \frac{1}{2b}, 0, \dots, 0, \frac{1}{2b})$, \dots , $v_{n-1} = (0, \dots, 0, \frac{1}{2b}, \frac{1}{2b})$, $v_n = (0, \dots, 0, \frac{1}{2} + \frac{3}{2r})$, $v_{n+1} = (a_1, a_2, \dots, a_{n-1}, -b + \frac{1}{2})$, $v_{n+2} = ((1-r)a_1, (1-r)a_2, \dots, (1-r)a_{n-1}, (r-1)b + 1)$, and $v_{n+3} = (0, \dots, 0, \frac{1}{2r})$. Recall from Lemma 3 that the polyhedron P contains no integer point. It suffices to prove the following two claims.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$ only if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Consider an inequality $cx \leq q$, where $c = (c_1, c_2, \dots, c_n)$, $c_i = -1$ for $i \in K$, $c_i = 0$ for $i \in S \setminus K$, $c_n = -1$, and $q = -\frac{1}{2b}$. One can easily verify: $-1 < cv_i < -\frac{1}{2b}$ for $1 \leq i \leq n-1$, $cv_n = -\frac{1}{2} - \frac{3}{2r}$, and because $\sum_{i \in K} a_i = b$, $cv_{n+1} = -\frac{1}{2}$ and $cv_{n+2} = -1$. Hence, $cx \leq q$ is valid for P . Since $c_i = -1$ or 0 , the inequality $cx \leq \lfloor q \rfloor (= -1)$ is a $\{-1, 0, 1\}$ -cut of P . From the value of cv_i for $1 \leq i \leq n+2$, we see that v_{n+2} is the only vector in P that satisfies the inequality $cx \leq \lfloor q \rfloor$. Since $rb \notin \mathbf{Z}_+$ and $b = \sum_{i \in K} a_i$, there exists some $j \in K$ satisfying $ra_j \notin \mathbf{Z}_+$. Consider the inequality $fx \leq g$, where $fx = -x_j$ and $g = (r-1)a_j$. Apparently, $g > 0$ and $g \notin \mathbf{Z}_+$. In addition, $fv_i < 0$ for $1 \leq i \leq n+1$ and $fv_{n+2} = g > 0$. It is obvious that the inequality $fx \leq \lfloor g \rfloor (= \lfloor (r-1)a_j \rfloor)$ is a $\{-1, 0, 1\}$ -cut of P and that v_{n+2} violates this inequality. Therefore, $P'_{\{-1,0,1\}} = \emptyset$ and Claim 1 is proved.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$ if $P'_{\{-1,0,1\}} = \emptyset$.

Proof. Let $v_0 \equiv (0, 0, \dots, 0, \frac{1}{2r} + \frac{1}{2}) \in P$. Since $P'_{\{-1,0,1\}} = \emptyset$, v_0 violates an inequality $cx \leq \lfloor q \rfloor$, where $c = (c_1, c_2, \dots, c_n) \in \{-1, 0, 1\}^n$ and $cx \leq q$ is valid for P . It is easy to see that $c_n \neq 0$. Otherwise, $0 = cv_0 \leq q$, which contradicts that v_0 violates $cx \leq \lfloor q \rfloor$.

Now we show that $\sum_{1 \leq i \leq n-1} c_i a_i = c_n b$. If $c_n = 1$, then $cv_0 = \frac{1}{2r} + \frac{1}{2}$. In this case, if $\sum_{1 \leq i \leq n-1} c_i a_i \geq c_n b + 1$, then $\frac{3}{2} \leq cv_{n+1} \leq q$; if $\sum_{1 \leq i \leq n-1} c_i a_i \leq c_n b - 1$, then $n+1 < r \leq cv_{n+2} \leq q$. Hence, $cv_0 < 1 < q$, contradicting $cv_0 > \lfloor q \rfloor$. If $c_n = -1$, then $cv_0 = -\frac{1}{2r} - \frac{1}{2}$. In this case, if $\sum_{1 \leq i \leq n-1} c_i a_i \geq c_n b + 1$, then $\frac{1}{2} \leq cv_{n+1} \leq q$; if $\sum_{1 \leq i \leq n-1} c_i a_i \leq c_n b - 1$, then $n-1 < r-2 \leq cv_{n+2} \leq q$. So, $cv_0 < 0 < q$, a contradiction to $cv_0 > \lfloor q \rfloor$.

It is true that $c_n = -1$. Otherwise, $cv_0 = \frac{1}{2r} + \frac{1}{2}$ and $cv_{n+2} = 1 \leq q$, contradicting that v_0 violates $cx \leq \lfloor q \rfloor$.

We now claim that $c_i \leq 0$ for $1 \leq i \leq n - 1$. Otherwise, suppose $c_j = 1$ for some $1 \leq j \leq n - 1$. Then $cv_j = 0$. Since $cv_0 = -\frac{1}{2r} - \frac{1}{2}$ and $0 = cv_j \leq q$, a contradiction similar to the early ones can be derived. Therefore, Claim 2 is proved.

Using the same approach as shown in the end of the proof of Lemma 2, it is straight to polynomially obtain a description of P in the form of $\tilde{A}x \leq \tilde{b}$, where $\tilde{A} \in \mathbf{Z}^{m \times n}$ and $\tilde{b} \in \mathbf{Z}^m$, from the vectors v_1, v_2, \dots, v_{n+3} . The theorem is proved. \square

Corollary 1 *Given a rational polyhedron containing no integer point, it is \mathcal{NP} -complete to decide whether its $\{0, 1\}$ -closure is empty.*

Proof The proof is similar to that of Theorem 3, hence we omit the details and only point out the differences.

The Partition Problem is polynomially reduced to the problem of deciding whether $P'_{\{0,1\}} = \emptyset$ for the polyhedron $P \subseteq \mathbf{R}^n$ that is the convex hull of the $n+3$ vectors: $v_1 = (-\frac{1}{2b}, 0, \dots, 0, -\frac{1}{2b})$, $v_2 = (0, -\frac{1}{2b}, 0, \dots, 0, -\frac{1}{2b})$, \dots , $v_{n-1} = (0, \dots, 0, -\frac{1}{2b}, -\frac{1}{2b})$, $v_n = (0, \dots, 0, -\frac{1}{2} - \frac{3}{2r})$, $v_{n+1} = (-a_1, -a_2, \dots, -a_{n-1}, b - \frac{1}{2})$, $v_{n+2} = ((r-1)a_1, (r-1)a_2, \dots, (r-1)a_{n-1}, (1-r)b - 1)$, and $v_{n+3} = (0, \dots, 0, -\frac{1}{2r})$.

Claim 1. There is a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$ only if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 1 is similar to that of Claim 1 in the proof of Theorem 3 except that $c_i = 1$ for $i \in K$, $c_i = 0$ for $i \in S \setminus K$, $c_n = 1$, and $fx = x_j$.

Claim 2. There is a subset $K \subseteq S$ such that $\sum_{i \in K} a_i = \sum_{i \in S \setminus K} a_i$ if $P'_{\{0,1\}} = \emptyset$.

The proof of Claim 2 is similar to but simpler than that of Claim 2 in the proof of Theorem 3. Here are two differences: First, we let $v_0 \equiv (0, 0, \dots, 0, -\frac{1}{2r} - \frac{1}{2})$. Second, to show by contradiction that $\sum_{1 \leq i \leq n-1} c_i a_i = c_n b$, we only consider the case that $c_n = 1$, and a contradiction can be derived due to $cv_0 < 0 < q$. \square

5 Conclusion

In this paper, we raised two questions, (Q1) and (Q2), about integer linear programs with the property that the Gomory-Chv atal closure of the linear constraints is identical to their integer hull. To answer (Q2), we proved that the problem of deciding whether the Gomory-Chv atal closure of a rational polyhedron P is empty is \mathcal{NP} -complete, even when P is known to contain no integer point. Similar results are also proved for the $\{-1, 0, 1\}$ -closure and $\{0, 1\}$ -closure of polyhedron. There are several questions related to (Q2) to which we have not found answer yet. In particular, (Q2) is open when we restrict our attention to polyhedra in the unit cube (denoted by $[0, 1]^n$). We state three natural questions of this type:

- (i) Is it \mathcal{NP} -complete to decide whether $P' = \emptyset$ for a polytope $P \subseteq [0, 1]^n$ that contains no integer point?
- (ii) Is it \mathcal{NP} -complete to decide whether $P'_{\{-1,0,1\}} = \emptyset$ for a polytope $P \subseteq [0, 1]^n$ that contains no integer point?
- (iii) Is it \mathcal{NP} -complete to decide whether $P'_{\{0,1\}} = \emptyset$ for a polytope $P \subseteq [0, 1]^n$ that contains no integer point?

Given a rational polyhedron P with the promise that $P_I = \emptyset$, Theorem 2 shows that it is \mathcal{NP} -complete to decide whether $P' = \emptyset$. However the following question, which is closely related to (Q1), is still open: Given a rational polyhedron P with the promise that $P' = P_I$, can one decide in polynomial time whether $P' = P_I = \emptyset$, or is this an “intermediate problem” in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ that is not polynomially solvable?

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