CUT-GENERATING FUNCTIONS

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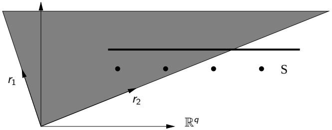
The problem

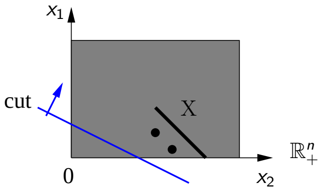
This talk deals with sets of the form

$$X := \left\{x \in \mathbb{R}^n_+ \ : \ Rx \in S\right\}$$

where

$$R = [r_1, \ldots, r_n] \text{ is a real } q \times n \text{ matrix,}$$
$$S \subset \mathbb{R}^q \text{ is a nonempty closed set with } 0 \notin S.$$





Since $0 \notin S$, the closed convex hull of X does not contain 0. We are interested in *separating* 0 from X, which we write as

$$c^\top x \geqslant 1\,, \quad \text{for all } x \in X.$$

Motivation arising in integer programming

Start from a polyhedron

$$P = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m \,:\, Ax + y = b\,\right\}$$

and assume that $b \notin \mathbb{Z}^m$.

Example 1 The set of interest is $P \cap \{\mathbb{Z}^n_+ \times \mathbb{Z}^m\}$.
I.e. we want $(x, y = b - Ax)$ such that $x \in \mathbb{Z}^n_+$ and $b - Ax \in \mathbb{Z}^m$.
The convex hull of this set is Gomory's corner polyhedron 1969.

This problem fits our framework if we set

$$q = n + m\,, \quad R = \begin{bmatrix} I \\ -A \end{bmatrix}, \quad S = \left\{ \begin{matrix} \mathbb{Z}^n \\ \mathbb{Z}^m \end{matrix} \right\} - \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

Since $b \notin \mathbb{Z}^m$, this S is a closed set not containing the origin.

Motivation arising in mixed integer programming

Start again from a polyhedron

$$P = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m \;:\; Ax + y = b\,\right\}$$

and again assume that $b \notin \mathbb{Z}^m$.

Example 2 Andersen, Louveaux, Weismantel and Wolsey 2007
The set of interest is $P \cap \{\mathbb{R}^n_+ \times \mathbb{Z}^m\}$,
I.e. we want $(x, y = b - Ax)$ such that $x \in \mathbb{R}^n_+$ and $b - Ax \in \mathbb{Z}^m$.

This fits our model by taking

$$q = m\,, \quad R = -A\,, \quad S = \mathbb{Z}^m - b$$

Motivation arising from complementary slackness

Still using $$P = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m \,:\, Ax + y = b\,\right\}$$

let $$E \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$$

and $$C := \{y \in \mathbb{R}^m_+ \,:\, y_i y_j = 0\,, (i, j) \in E\}.$$

The set of interest is then $P \cap (\mathbb{R}^n_+ \times C)$.

It can be modeled in our framework where

$$q = m\,, \quad R = -A\,, \quad S = C - b;$$

Cuts have been used for complementarity problems of this type, for example in Judice, Sherali, Ribeiro, Faustino 2006

The problem

We will retain from the above examples the asymmetry between
S – a very particular and highly structured set –
and R – an arbitrary matrix.

Keeping this in mind, we will consider that (q, S) *is given and fixed*, while (n, R) *is instance-dependent data*.

A number of papers have appeared in recent years, dealing with the above problem with various special forms for S:

Andersen, Louveaux, Weismantel and Wolsey IPCO 2007

Dey and Wolsey SIOPT 2010

Basu, Conforti, Cornuéjols and Zambelli SIDMA 2010.

Cut-generating functions

Let S be fixed. Consider a function

$$\rho : \mathbb{R}^q \mapsto \mathbb{R}$$

that produces coefficients $c_j := \rho(r_j)$ of a cut $c^\top x \geqslant 1$ valid for $X(R, S)$ for any choice of n and $R = [r_1 \ldots r_n]$.

In summary, we require our ρ to satisfy

$$\forall R = [r_1 \ldots r_n], \quad x \in X \quad \Longrightarrow \quad \sum_{j=1}^{n} \rho(r_j) x_j \geqslant 1.$$

Such a ρ can then justifiably be called a *cut-generating function*.

Sufficiency of cut-generating functions

Cut-generating functions are defined assuming that S is fixed but R can vary arbitrarily.

What happens if both S and R are fixed?
A natural question is whether, for every cut $c^\top x \geqslant 1$ that is valid for $X(R,S)$, there exists some cut-generating function ρ such that $\rho(r_j) \leq c_j$.

THEOREM Cornuejols, Wolsey, Yildiz 2013
Suppose $S \subset \text{cone}(R)$. Then any valid inequality $c^\top x \geq 1$ separating 0 from X is dominated by one obtained from a cut-generating function.

Next we show that the (vast!) class of cut-generating functions from $\mathbb{R}^q$ to $\mathbb{R}$ can be drastically reduced.

Cut-generating functions

Let $\bar{\rho}(r) := \inf_{K,\alpha}\left\{\sum_{k=1}^{K}\alpha_k\rho(r_k) \ : \ \sum_{k=1}^{K}\alpha_k r_k = r,\ \alpha_k \geqslant 0\right\}$.

THEOREM

If ρ is a cut-generating function, then $\bar{\rho}$ is nowhere $-\infty$ and is again a cut-generating function.

The function $\bar{\rho}$ is *sublinear* (convex and positively homogeneous). Sublinear functions are continuous.
Because $\bar{\rho} \leq \rho$, the theorem shows that sublinear functions suffice to generate all relevant cuts; a fairly narrow class indeed, which is fundamental in convex analysis.

Sublinear functions are in correspondence with closed convex sets and in our context, such a correspondence is based on the mapping $\rho \mapsto V$ defined by

$$V := \left\{r \in \mathbb{R}^q \ : \ \rho(r) \leqslant 1\right\}.$$

S-free sets

The set V turns out to be a cornerstone: the theorem below establishes a correspondence between cut-generating functions and the so-called *S-free sets*.

DEFINITION

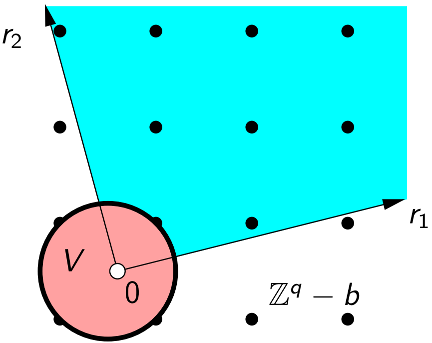
Given a closed set $S \subset \mathbb{R}^q$ not containing the origin, a closed convex neighborhood V of $0 \in \mathbb{R}^q$ is called S-free if its interior contains no point in S.

THEOREM

Let ρ be a sublinear function and $V := \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. Then ρ is a cut-generating function if and only if V is S-free.

Example of an S-free set

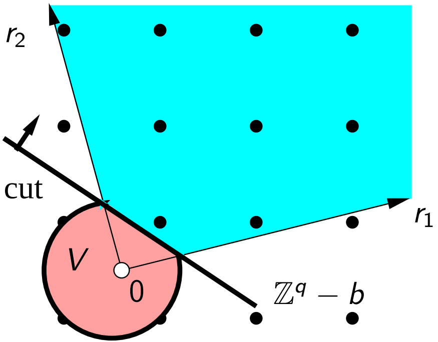
Assume $b \notin \mathbb{Z}^q$ and $S := \mathbb{Z}^q - b$. Want to cut off the point $x = 0$.



Convex set V is S-free: $0 \in int(V)$ and no point of S is in $int(V)$.

Example of an S-free set

Assume $b \notin \mathbb{Z}^q$ and $S := \mathbb{Z}^q - b$. Want to cut off the point $x = 0$.



Compute intersection of the rays with the boundary of V.
Cut defined by these points is valid: $\rho(r_1)x_1 + \rho(r_2)x_2 \geq 1$.
Here $\rho(\frac{r_1}{4}) = \rho(\frac{r_2}{4}) = 1$. The cut is $4x_1 + 4x_2 \geq 1$.

Representation

As a result, cut-generating functions can alternatively be studied from a geometric point of view, involving sets V instead of functions ρ. This situation, common in convex analysis, is often very fruitful. However, there is a difficulty here: the mapping $\rho \mapsto V$ is many-to-one and therefore has no inverse.

DEFINITION

Let $V \subset \mathbb{R}^q$ be a closed convex neighborhood of the origin. A *representation* of V is a sublinear function ρ satisfying $V = \left\{r \in \mathbb{R}^q \,:\, \rho(r) \leqslant 1\right\}$.

A cut-generating function is a representation of an S-free set. Among the several representations of an S-free set V, we are interested in the small ones.

Main results

We extend the results in

Dey and Wolsey SIOPT 2010

Basu, Conforti, Cornuéjols and Zambelli SIDMA 2010

Basu, Cornuéjols and Zambelli JOCA 2011

- We show that the representations of V have a unique maximal element γ_V (the gauge of V introduced by Minkowski) and a unique minimal element μ_V, which is *the* relevant inverse of $\rho \mapsto V$ for our purpose.
- Then we study the correspondence $V \leftrightarrow \mu_V$. We show that different concepts of minimality come into play for ρ. Geometrically they correspond to different concepts of maximality for V. We also show that they coincide in a number of cases.

Support function

The *support function* of a set $G \subset \mathbb{R}^q$ is

$$\sigma_G(r) := \sup_{d \in G} d^\top r.$$

It is easily seen to be sublinear, to grow when G grows, but to remain unchanged if G is replaced by its closed convex hull: $\sigma_G = \sigma_{\overline{\text{conv}}(G)}$. Conversely, any sublinear function ρ is the support function of a closed convex set, unambiguously defined by

$$G := \left\{d \in \mathbb{R}^n \,:\, d^\top r \leqslant \rho(r) \text{ for all } r \in \mathbb{R}^q\right\}.$$

Besides, the *polar* of G

$$G^\circ := \left\{r \in \mathbb{R}^q \,:\, d^\top r \leqslant 1 \text{ for any } d \in G\right\} = \{r \in \mathbb{R}^q \,:\, \sigma_G(r) \leqslant 1\}$$

is also a closed convex set. And it is a neighborhood of the origin when σ_G is finite-valued (i.e. when G is bounded).

Thus the support function of G represents its polar G°.

Minimal representation

The following geometric objects turn out to be relevant:

$\hat{V}^\circ := \{d \in V^\circ \ : \ \sigma_V(d) = 1\}$,
$V^\bullet := \overline{\mathrm{conv}}(\hat{V}^\circ)$.

Let $\mu_V := \sigma_{\hat{V}^\circ} = \sigma_{V^\bullet}$

PROPOSITION Basu, Cornuéjols and Zambelli JOCA 2011
Any sublinear function ρ representing V satisfies $\rho \geqslant \mu_V$.

THEOREM
A sublinear function ρ represents V if and only if it satisfies

$$\mu_V \leqslant \rho \leqslant \gamma_V.$$

Minimal cut-generating functions and maximal S-free sets

Definition

A cut-generating function ρ is *minimal* if any cut-generating function $\rho' \leqslant \rho$ is ρ itself.

A minimal cut-generating function is certainly a smallest representation of some set V. But this set is special:

Take for example $S = \{1\} \subset \mathbb{R}$, $V = [-1, +1]$;
$\rho(r) := |r|$ is the unique representation of V but ρ is not minimal: $\rho'(r) := \max\{0, r\}$ is also a cut-generating function; it represents the larger set $V' =]-\infty, +1]$.

A smaller ρ describes a larger V; so the above definition has its geometrical counterpart:

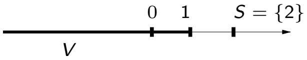
Definition

An S-free set V is called *maximal* if any S-free set $V' \supset V$ is V itself.

An example

Actually, this "duality" is deceiving, as the two definitions do not match: the set represented by a minimal cut-generating function need not be maximal. Here is a trivial example.

Example



With $n = 1$, the set $V =]-\infty, 1]$ (represented by $\rho(r) = r$) is $\{2\}$-free but is obviously not maximal.
However $\rho(r) = r$ is a minimal cut-generating function.

So a subtlety is necessary, indeed the smallest representation of a maximal V enjoys a stronger property than minimality.

Strongly minimal cut-generating functions

DEFINITION

A cut-generating function ρ is called strongly minimal if any cut-generating function $\rho' \leqslant \gamma_{V(\rho)}$ satisfies $\rho' \geqslant \rho$.

Strong minimality turns out to be *the* appropriate definition in general:

THEOREM

An S-free set V is maximal if and only if its smallest representation μ_V is a strongly minimal cut-generating function.

Asymptotically maximal S-free sets

Then comes a natural question: how maximal are the S-free sets represented by minimal cut-generating functions? For this, we introduce one more concept:

DEFINITION

An S-free set V is called *asymptotically maximal* if any S-free set $V' \supset V$ has the same recession cone: $V'_\infty = V_\infty$.

It allows a partial answer to the question.

THEOREM

The S-free set represented by a minimal cut-generating function is asymptotically maximal.

When does minimality imply strong minimality?

The following theorem provides two favorable cases when this implication holds.

THEOREM
Suppose $0 \in \hat{S} := \overline{\mathrm{conv}} S$; denote by L the lineality space of the S-free set V and assume that μ_V is minimal.
Then μ_V is strongly minimal in either of the following situations

(i) $V_\infty \cap \hat{S}_\infty = \{0\}$ (in particular S bounded),

(ii) $V_\infty \cap \hat{S}_\infty = L \cap \hat{S}_\infty$ and $\hat{S} = G + \hat{S}_\infty$ with G bounded.

This theorem generalizes several earlier results. The special case where S is a finite set of points in $\mathbb{Z}^q - b$ was first considered by Johnson 1981 and more recently by Dey and Wolsey 2010.
Part (ii) was proven by Dey and Wolsey 2010 and Basu, Conforti, Cornuéjols and Zambelli 2010 in the special case where $S = P \cap (\mathbb{Z}^q - b)$ for some rational polyhedron P.