Disjunctive Cuts for Cross-Sections of the Second-Order Cone

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June 10, 2014, revised May 25, 2015

Abstract

In this paper we study general two-term disjunctions on affine cross-sections of the second-order cone. Under some mild assumptions, we derive a closed-form expression for a convex inequality that is valid for such a disjunctive set, and we show that this inequality is sufficient to characterize the closed convex hull of all two-term disjunctions on ellipsoids and paraboloids and a wide class of two-term disjunctions—including split disjunctions—on hyperboloids. Our approach relies on the work of Kılınç-Karzan and Yıldız which considers general two-term disjunctions on the second-order cone.

Keywords: Mixed-integer conic programming, second-order cone programming, cutting planes, disjunctive cuts

1 Introduction

In this paper we consider the mixed-integer second-order conic set

$$S := \{x \in \mathbb{L}^n : \ Ax = b, \ x_j \in \mathbb{Z} \ \forall j \in J\}$$

where $\mathbb{L}^n$ is the n-dimensional second-order cone $\mathbb{L}^n := \{x \in \mathbb{R}^n : \ \|(x_1; \ldots; x_{n-1})\| \le x_n\}$, A is an $m \times n$ real matrix of full row rank, d and b are real vectors of appropriate dimensions, $J \subseteq \{1, \ldots, n\}$, and $\|.\|$ denotes the Euclidean norm. The set S appears as the feasible solution set or a relaxation thereof in mixed-integer second order cone programming problems. Because the structure of S can be very complicated, a first approach to solving

$$\sup\left\{d^\top x : \ x \in S\right\}. \tag{1}$$

entails solving the relaxed problem obtained after dropping the integrality requirements on the variables:

$$\sup\left\{d^\top x : \ x \in C\right\} \ \text{ where } \ C := \{x \in \mathbb{L}^n : \ Ax = b\}.$$

The set C is called the natural *continuous relaxation* of S. Unfortunately, the continuous relaxation C is often a poor approximation to the mixed-integer conic set S, and tighter formulations are needed for the development of practical strategies for solving (1). An effective way to improve the approximation quality of the continuous relaxation C is to strengthen it with additional convex inequalities that are valid for S but not for the whole of C. Such valid inequalities can be

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derived by exploiting the integrality of the variables x_j, $j \in J$, and enhancing C with linear *two-term disjunctions* $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ that are satisfied by all solutions in S. Valid inequalities that are obtained from disjunctions using this approach are known as *disjunctive cuts*. In this paper we study two-term disjunctions on the set C and give closed-form expressions for the strongest disjunctive cuts that can be obtained from such disjunctions.

Disjunctive cuts were introduced by Balas in the context of mixed-integer linear programming [3] and have since been the cornerstone of theoretical and practical achievements in integer programming. There has been a lot of recent interest in extending disjunctive cutting-plane theory from the domain of mixed-integer linear programming to that of mixed-integer conic programming [2, 7, 9, 11, 12, 18]. Kılınç-Karzan [13] studied minimal valid linear inequalities for general disjunctive conic sets and showed that these are sufficient to describe the associated closed convex hull under a mild technical assumption. Bienstock and Michalka [6] studied the characterization and separation of linear inequalities that are valid for the epigraph of a convex, differentiable function whose domain is restricted to the complement of a convex set. On the other hand, several papers in the last few years have focused on deriving closed-form expressions for nonlinear convex inequalities that fully describe the convex hull of a disjunctive second-order conic set in the space of the original variables. Dadush et al. [10] and Andersen and Jensen [1] derived split cuts for ellipsoids and the second-order cone, respectively. Modaresi et al. extended these results to split disjunctions on cross-sections of the second-order cone [16] and compared the effectiveness of split cuts against conic MIR inequalities and extended formulations [15]. For disjoint two-term disjunctions on cross-sections of the second-order cone and under the assumption that $\{x \in C : l_1^\top x = l_{1,0}\}$ and $\{x \in C : l_2^\top x = l_{2,0}\}$ are bounded, Belotti et al. [4, 5] proved that there exists a unique cone which describes the convex hull of the disjunction. They also identified a procedure for identifying this cone when C is an ellipsoid. Using the structure of minimal valid linear inequalities, Kılınç-Karzan and Yıldız [14] derived a family of convex inequalities which describes the convex hull of a general two-term disjunction on the whole second-order cone. In this paper, we pursue a similar goal: We study general two-term disjunctions on a cross-section C of the second-order cone, namely $C = \{x \in \mathbb{L}^n : Ax = b\}$. Given a disjunction $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ on C, we let

$$C_1 := \left\{x \in C : \ l_1^\top x \geq l_{1,0}\right\} \quad \text{and} \quad C_2 := \left\{x \in C : \ l_2^\top x \geq l_{2,0}\right\}.$$

In order to derive the tightest disjunctive cuts that can be obtained for S from the disjunction $C_1 \cup C_2$, we study the closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$. In particular, we are interested in convex inequalities that may be added to the description of C to obtain a characterization of $\overline{\text{conv}}(C_1 \cup C_2)$. Our starting point is the paper [14] about two-term disjunctions on the second-order cone $\mathbb{L}^n$. We extend the main result of [14] to cross-sections of the second-order cone. Such cross-sections include ellipsoids, paraboloids, and hyperboloids as special cases. Our results generalize the work of [10, 16] on split disjunctions on cross-sections of the second-order cone and [4] on disjoint two-term disjunctions on ellipsoids. We note here that general results on convexifying the intersection of a cross-section of the second-order cone with a non-convex cone defined by a single homogeneous quadratic were recently obtained independently in [8].

We first show in Section 2 that the continuous relaxation C can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. In Section 3, we give a complete description of the convex hull of a homogeneous two-term disjunction on the whole second-order cone. In Section 4, we prove our main result, Theorem 3, characterizing $\overline{\text{conv}}(C_1 \cup C_2)$ under certain conditions. We end the paper with two examples which illustrate the applicability of Theorem 3.

Throughout the paper, we use $\operatorname{conv} K$, $\overline{\operatorname{conv}}\, K$, $\operatorname{cone} K$, and $\operatorname{span} K$ to refer to the convex hull, closed convex hull, conical hull, and linear span of a set K, respectively. We also use $\operatorname{bd} K$, $\operatorname{int} K$, and $\dim K$ to refer the boundary, interior, and dimension of K. The *dual cone* of $K \subseteq \mathbb{R}^n$ is $K^* := \{\alpha \in \mathbb{R}^n : x^\top \alpha \geq 0 \; \forall x \in K\}$. The second-order cone $\mathbb{L}^n$ is self-dual, that is, $(\mathbb{L}^n)^* = \mathbb{L}^n$. Given a vector $u \in \mathbb{R}^n$, we let $\tilde{u} := (u_1; \ldots; u_{n-1})$ denote the subvector obtained by dropping its last entry.

2 Intersection of the Second-Order Cone with an Affine Subspace

In this section, we show that the continuous relaxation C can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. Let $E := \{x \in \mathbb{R}^n : Ax = b\}$ so that $C = \mathbb{L}^n \cap E$. We are going to use the following lemma to simplify our analysis.

Lemma 1. *Let V be a p-dimensional linear subspace of $\mathbb{R}^n$. The intersection $\mathbb{L}^n \cap V$ is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^p$.*

See Section 2.1 of [5] for a similar result. We do not give a formal proof of Lemma 1 but just note that it can be obtained by observing that the second-order cone is the conic hull of a (one dimension smaller) sphere, and that the intersection of a sphere with an affine space is either empty, a single point (when the affine space intersects the sphere but not its interior), or a lower dimensional sphere of the same dimension as the affine space (when the affine space intersects the interior of the sphere).

Lemma 1 implies that, when $b = 0$, C is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m}$. The closed convex hull $\overline{\operatorname{conv}}(C_1 \cup C_2)$ can be described easily when C is a single point or a half-line. Furthermore, the problem of characterizing $\overline{\operatorname{conv}}(C_1 \cup C_2)$ when C is a bijective linear transformation of $\mathbb{L}^{n-m}$ can be reduced to that of convexifying an associated two-term disjunction on $\mathbb{L}^{n-m}$. We refer the reader to [14] for a detailed study of the closed convex hulls of two-term disjunctions on the second-order cone.

In the remainder, we focus on the case $b \neq 0$. Note that, whenever this is the case, we can permute and normalize the rows of (A, b) so that its last row reads $(a_m^\top, 1)$, and subtracting a multiple of $(a_m^\top, 1)$ from the other rows if necessary, we can write the remaining rows of (A, b) as $(\tilde{A}, 0)$. Therefore, we can assume without any loss of generality that all components of b are zero except the last one. Isolating the last row of (A, b) from the others, we can then write

$$E = \left\{ x \in \mathbb{R}^n : \; \tilde{A}x = 0, \; a_m^\top x = 1 \right\}.$$

Let $V := \{x \in \mathbb{R}^n : \tilde{A}x = 0\}$. By Lemma 1, $\mathbb{L}^n \cap V$ is the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m+1}$. Again, the first two cases are easy and not of interest in our analysis. In the last case, we can find a matrix D whose columns form an orthonormal basis for V and define a nonsingular matrix H such that $\{y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n\} = H\mathbb{L}^{n-m+1}$. Then we can represent C equivalently as

$$\begin{aligned} C &= \left\{ x \in \mathbb{L}^n : \; x = Dy, \, a_m^\top x = 1 \right\} \\ &= D \left\{ y \in \mathbb{R}^{n-m+1} : \; Dy \in \mathbb{L}^n, \, a_m^\top Dy = 1 \right\} \\ &= D \left\{ y \in \mathbb{R}^{n-m+1} : \; y \in H\mathbb{L}^{n-m+1}, \, a_m^\top Dy = 1 \right\} \\ &= DH \left\{ z \in \mathbb{L}^{n-m+1} : \; a_m^\top DHz = 1 \right\}. \end{aligned}$$

The set $C = \mathbb{L}^n \cap E$ is a bijective linear transformation of $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$. Furthermore, the same linear transformation maps any two-term disjunction in $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$ to a two-term disjunction in C and vice versa. Thus, without any loss of generality, we can take $m = 1$ in (1) and study the problem of describing $\overline{\text{conv}}(C_1 \cup C_2)$ where

$$\begin{gathered} C = \left\{ x \in \mathbb{L}^n : \; a^\top x = 1 \right\}, \\ C_1 = \left\{ x \in C : \; l_1^\top x \geq l_{1,0} \right\}, \quad \text{and} \quad C_2 = \left\{ x \in C : \; l_2^\top x \geq l_{2,0} \right\}. \end{gathered} \tag{2}$$

In Section 4 we will give a full description of $\overline{\text{conv}}(C_1 \cup C_2)$ under certain conditions.

3 Homogeneous Two-Term Disjunctions on the Second-Order Cone

In this section, we study the convex hull of a homogeneous two-term disjunction $c_1^\top x \geq 0 \vee c_2^\top x \geq 0$ on the second-order cone. Let

$$Q_1 := \left\{ x \in \mathbb{L}^n : \; c_1^\top x \geq 0 \right\} \quad \text{and} \quad Q_2 := \left\{ x \in \mathbb{L}^n : \; c_2^\top x \geq 0 \right\}. \tag{3}$$

The main result of this section characterizes $\text{conv}(Q_1 \cup Q_2)$. Note that Q_1 and Q_2 are closed, convex, pointed cones; therefore, $\text{conv}(Q_1 \cup Q_2)$ is always closed (see, e.g., Rockafellar [17, Corollary 9.1.3]).

When $Q_1 \subseteq Q_2$, we have $\text{conv}(Q_1 \cup Q_2) = Q_2$. Similarly, when $Q_1 \supseteq Q_2$, we have $\text{conv}(Q_1 \cup Q_2) = Q_1$. In the remainder of this section, we focus on the case where $Q_1 \not\subseteq Q_2$ and $Q_1 \not\supseteq Q_2$.

Assumption 1. *$Q_1 \not\subseteq Q_2$ and $Q_1 \not\supseteq Q_2$.*

We also make the following technical assumption.

Assumption 2. *$Q_1 \cap \text{int}\,\mathbb{L}^n \neq \emptyset$ and $Q_2 \cap \text{int}\,\mathbb{L}^n \neq \emptyset$.*

This assumption will be useful later when we use Theorem 1 whose proof relies on conic duality.

By Assumption 1, we have $Q_1, Q_2 \subsetneq \mathbb{L}^n$, and by Assumption 2, we have that Q_1 and Q_2 are full-dimensional. This implies $c_1, c_2 \notin \pm\mathbb{L}^n$, or equivalently $\|\tilde{c}_i\|^2 > c_{i,n}^2$, for $i \in \{1, 2\}$. By scaling c_1 and c_2 with appropriate positive scalars if necessary, we may assume without any loss of generality that

$$\|\tilde{c}_1\|^2 - c_{1,n}^2 = \|\tilde{c}_2\|^2 - c_{2,n}^2 = 1. \tag{4}$$

These have the following consequences.

Remark 1. *Let c_1 and c_2 satisfy* (4)*. Then*

$$\begin{aligned}\mathcal{M} &:= \|\tilde{c}_1\|^2 - c_{1,n}^2 - \left(\|\tilde{c}_2\|^2 - c_{2,n}^2\right) = 0,\\ \mathcal{N} &:= \|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_{1,n} - c_{2,n})^2 = 2 - 2\left(\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n}\right).\end{aligned}$$

Remark 2. *Let Q_1 and Q_2, defined as in* (3)*, satisfy Assumption 1. Then we have $c_1 - c_2 \notin \pm\mathbb{L}^n$. Indeed, $c_1 - c_2 \in \mathbb{L}^n$ implies that $(c_1 - c_2)^\top x \geq 0$ for all $x \in \mathbb{L}^n$, and this implies $C_1 \subseteq C_2$; similarly, $c_2 - c_1 \in \mathbb{L}^n$ implies $C_2 \subseteq C_1$. Hence,*

$$\mathcal{N} = \|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_{1,n} - c_{2,n})^2 > 0.$$

The following result from [14] gives a valid convex inequality for $\operatorname{conv}(Q_1 \cup Q_2)$.

Theorem 1 ([14], Theorem 3 and Remark 2)**.** *Let Q_1 and Q_2 be defined as in* (3)*. Suppose Assumptions 1 and 2 hold. Then the inequality*

$$-(c_1 + c_2)^\top x \leq \sqrt{\left((c_1 - c_2)^\top x\right)^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \tag{5}$$

is valid for $\operatorname{conv}(Q_1 \cup Q_2)$*. Furthermore, this inequality is convex in $\mathbb{L}^n$.*

The next proposition shows that (5) can be written in conic quadratic form in $\mathbb{L}^n$ except in the region where both clauses of the disjunction are satisfied. Its proof is a simple extension of the proofs of Propositions 3 and 4 in [14] and therefore omitted. Let

$$r := \begin{pmatrix} \tilde{c}_1 - \tilde{c}_2 \\ -c_{1,n} + c_{2,n} \end{pmatrix}.$$

Proposition 1 ([14], Propositions 3 and 4)**.** *Let Q_1 and Q_2 be defined as in* (3)*. Suppose Assumptions 1 and 2 hold. Let $x' \in \mathbb{L}^n$ be such that $c_1^\top x' \leq 0$ or $c_2^\top x' \leq 0$. Then the following statements are equivalent:*

i) x' satisfies (5)*.*

ii) x' satisfies the conic quadratic inequality

$$\mathcal{N}x - 2(c_1^\top x)r \in \mathbb{L}^n. \tag{6}$$

iii) x' satisfies the conic quadratic inequality

$$\mathcal{N}x + 2(c_2^\top x)r \in \mathbb{L}^n. \tag{7}$$

Remark 3. *When c_1 and c_2 satisfy* (4)*, the inequalities* (6) *and* (7) *describe a cylindrical second-order cone whose lineality space contains* $\operatorname{span}\{r\}$*. This follows from Remark 1 by observing that*

$$\mathcal{N} = 2 - 2\left(\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n}\right) = 2c_1^\top r = -2c_2^\top r.$$

The next theorem is the main result of this section. It shows that the inequality (5) is in fact sufficient to describe $\operatorname{conv}(Q_1 \cup Q_2)$ when c_1 and c_2 are scaled so that they satisfy (4). Because this assumption is without any loss of generality, our result settles one of the cases left open by Kılınç-Karzan and Yıldız [14], where the right-hand-sides of both terms of the disjunction are zero in (3).

Theorem 2. *Let Q_1 and Q_2 be defined as in* (3)*. Suppose Assumptions 1 and 2 hold. Assume that c_1 and c_2 have been scaled so that they satisfy* (4)*. Then*

$$\operatorname{conv}(Q_1 \cup Q_2) = \{x \in \mathbb{L}^n : x \textit{ satisfies } (5)\}\,. \tag{8}$$

Proof. Let D denote the set on the right-hand side of (8). We already know that (5) is valid for $\operatorname{conv}(Q_1 \cup Q_2)$. Hence, $\operatorname{conv}(Q_1 \cup Q_2) \subseteq D$. Let $x' \in D$. If $x' \in Q_1 \cup Q_2$, then clearly $x' \in \operatorname{conv}(Q_1 \cup Q_2)$. Therefore, suppose $x' \in \mathbb{L}^n \setminus (Q_1 \cup Q_2)$ is a point that satisfies (5). By Proposition 1, x' satisfies

$$\mathcal{N}x' - 2(c_1^\top x')r \in \mathbb{L}^n \quad \text{and} \quad \mathcal{N}x' + 2(c_2^\top x')r \in \mathbb{L}^n.$$

We are going to show that x' belongs to $\operatorname{conv}(Q_1 \cup Q_2)$.

By Remarks 2 and 3, $0 < \mathcal{N} = 2c_1^\top r = -2c_2^\top r$. Let

$$\begin{aligned} \alpha_1 &:= \frac{-c_1^\top x'}{c_1^\top r}, \qquad \alpha_2 := \frac{-c_2^\top x'}{c_2^\top r}, \\ x_1 &:= x' + \alpha_1 r, \quad x_2 := x' + \alpha_2 r. \end{aligned} \tag{9}$$

It is not difficult to see that $c_1^\top x_1 = c_2^\top x_2 = 0$. Furthermore, $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. Therefore, the only thing we need to show is $x_1, x_2 \in \mathbb{L}^n$. By Remark 3

$$\mathcal{N}r - 2(c_1^\top r)r = \mathcal{N}r + 2(c_2^\top r)r = 0.$$

Hence,

$$\begin{aligned} \mathcal{N}x_1 - 2(c_1^\top x_1)r &= \mathcal{N}x' - 2(c_1^\top x')r \in \mathbb{L}^n \quad \text{and} \\ \mathcal{N}x_2 + 2(c_2^\top x_2)r &= \mathcal{N}x' + 2(c_2^\top x')r \in \mathbb{L}^n. \end{aligned}$$

Now observing that $c_1^\top x_1 = c_2^\top x_2 = 0$ and $\mathcal{N} > 0$ shows $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in Q_1$ and $x_2 \in Q_2$. $\square$

In the next section, we will show that the inequality (5) can also be used to characterize $\overline{\operatorname{conv}}(C_1 \cup C_2)$ where C_1 and C_2 are defined as in (2).

4 Two-Term Disjunctions on Cross-Sections of the Second-Order Cone

4.1 The Main Result

Consider C, C_1, and C_2 defined as in (2). The set C is an ellipsoid when $a \in \operatorname{int}\mathbb{L}^n$, a paraboloid when $a \in \operatorname{bd}\mathbb{L}^n$, a hyperboloid when $a \notin \pm\mathbb{L}^n$, and empty when $a \in -\mathbb{L}^n$. In this section, we prove our main result, Theorem 3, which characterizes $\overline{\operatorname{conv}}(C_1 \cup C_2)$ under some mild conditions.

When $C_1 \subseteq C_2$, we have $\overline{\operatorname{conv}}(C_1 \cup C_2) = C_2$. Similarly, when $C_1 \supseteq C_2$, we have $\overline{\operatorname{conv}}(C_1 \cup C_2) = C_1$. In the remainder we concentrate on the case where $C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.

Assumption 3. *$C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.*

We also make the following assumption.

Assumption 4. *$C_1 \cap \operatorname{int} \mathbb{L}^n \neq \emptyset$ and $C_2 \cap \operatorname{int} \mathbb{L}^n \neq \emptyset$.*

This assumption will be useful later when we again use Theorem 1 whose proof relies on conic duality. The following simple observation underlies our approach.

Observation 1. *Let C, C_1, and C_2 be defined as in (2). Then $C_1 = \{x \in C : (\beta_1 l_1 + \gamma_1 a)^\top x \geq \beta_1 l_{1,0} + \gamma_1\}$ for any $\beta_1 > 0$ and $\gamma_1 \in \mathbb{R}$. Similarly, $C_2 = \{x \in C : (\beta_2 l_2 + \gamma_2 a)^\top x \geq \beta_2 l_{2,0} + \gamma_2\}$ for any $\beta_2 > 0$ and $\gamma_2 \in \mathbb{R}$.*

Observation 1 allows us to conclude

$$C_1 = \left\{x \in C : \ (l_1 - l_{1,0} a)^\top x \geq 0\right\} \quad \text{and} \quad C_2 = \left\{x \in C : \ (l_2 - l_{2,0} a)^\top x \geq 0\right\}.$$

By Assumption 3, we have $C_1, C_2 \subsetneq C$, and by Assumption 4, we have $\dim C_1 = \dim C_2 = n-1$. This implies $l_i - l_{i,0} a \notin \pm \mathbb{L}^n$, or equivalently $\|\tilde{l}_i - l_{i,0}\tilde{a}\|^2 > (l_{i,n} - l_{i,0} a_n)^2$, for $i \in \{1, 2\}$. Let

$$c_i := \lambda_i (l_i - l_{i,0} a) \ \text{ where } \ \lambda_i := \frac{1}{\sqrt{\|\tilde{l}_i - l_{i,0}\tilde{a}\|^2 - (l_{i,n} - l_{i,0} a_n)^2}} \quad \text{for } \ i \in \{1, 2\}. \tag{10}$$

Because $\lambda_1, \lambda_2 > 0$, we can write

$$C_1 = \left\{x \in C : \ c_1^\top x \geq 0\right\} \quad \text{and} \quad C_2 = \left\{x \in C : \ c_2^\top x \geq 0\right\}.$$

This scaling ensures that c_1 and c_2 satisfy (4).

Let Q_1 and Q_2 be the relaxations of C_1 and C_2 to the whole cone $\mathbb{L}^n$:

$$Q_1 := \left\{x \in \mathbb{L}^n : \ c_1^\top x \geq 0\right\} \quad \text{and} \quad Q_2 := \left\{x \in \mathbb{L}^n : \ c_2^\top x \geq 0\right\}.$$

It is clear that Q_1 and Q_2 satisfy Assumptions 1-2 because C_1 and C_2 satisfy Assumptions 3-4. Define $\mathcal{N}$, $\mathcal{M}$, and r as in Section 3 using c_1 and c_2. Noting that Q_1 and Q_2 satisfy Assumptions 1-2 and c_1 and c_2 satisfy (4), all results of Section 3 hold for Q_1 and Q_2. In particular, Theorem 1 implies that the inequality (5) is valid for $\overline{\text{conv}}(C_1 \cup C_2)$. In Theorem 3, we are going to show that (5) is also sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$ when the sets C_1 and C_2 satisfy certain conditions. The proof of Theorem 3 requires the following technical lemma.

Lemma 2. *Let C_1 and C_2 be defined as in (2). Suppose Assumptions 3 and 4 hold. Let c_1 and c_2 be defined as in (10). Suppose $a^\top r \neq 0$, and let $x^* := \frac{r}{a^\top r}$. Let $x' \in C \setminus (C_1 \cup C_2)$ satisfy (5).*

a) If $a^\top r > 0$, then $c_1^\top (x' - x^*) < 0$. If in addition

$$\begin{array}{c} (a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \\ (-a + \text{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset, \end{array} \tag{11}$$

then $c_2^\top (x' - x^*) \geq 0$.

b) If $a^\top r < 0$, then $c_2^\top (x' - x^*) < 0$. If in addition

$$\begin{array}{c} (a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \\ (-a + \text{cone}\{c_1\}) \cap -\mathbb{L}^n \neq \emptyset, \end{array} \tag{12}$$

then $c_1^\top (x' - x^*) \geq 0$.

Proof. By Remarks 2 and 3, $\mathcal{N} = 2c_1^\top r = -2c_2^\top r > 0$. From this, we get

$$\mathcal{N}x^* - 2(c_1^\top x^*)r = \frac{1}{a^\top r}\left(\mathcal{N} - 2c_1^\top r\right) r = 0, \tag{13}$$

$$\mathcal{N}x^* + 2(c_2^\top x^*)r = \frac{1}{a^\top r}\left(\mathcal{N} + 2c_2^\top r\right) r = 0. \tag{14}$$

Furthermore, $a^\top x' = a^\top x^* = 1$.

a) Having $x' \notin C_1$ implies $c_1^\top x' < 0$. Furthermore, it follows from $c_1^\top r = \frac{\mathcal{N}}{2} > 0$ that

$$c_1^\top x^* = \frac{c_1^\top r}{a^\top r} > 0.$$

Thus, we get $c_1^\top(x' - x^*) < 0$.

Now suppose $(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$. Then there exist $\lambda \geq 0$ and $0 \leq \theta \leq 1$ such that $a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. The point x' does not belong to either C_1 or C_2 and satisfies (5). By Proposition 1, it satisfies (7) as well. Using (14), we can write

$$\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r \in \mathbb{L}^n. \tag{15}$$

Because $\mathbb{L}^n$ is self-dual, we get

$$\begin{aligned}
0 &\leq (a + \lambda(\theta c_1 + (1-\theta)c_2))^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda(\theta c_1 + (1-\theta)c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(c_1 - c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) + \lambda c_2^\top(x' - x^*)(\mathcal{N} + 2c_2^\top r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(c_1 - c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(\mathcal{N}(c_1 - c_2)^\top(x' - x^*) + 2c_2^\top(x' - x^*)(c_1 - c_2)^\top r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(\mathcal{N}(c_1 + c_2)^\top(x' - x^*)) \\
&= (2a^\top r + \lambda\theta\mathcal{N})c_2^\top(x' - x^*) + \lambda\theta\mathcal{N}c_1^\top(x' - x^*)
\end{aligned}$$

where we have used $a^\top(x' - x^*) = 0$ to obtain the first equality, $\mathcal{N} + 2c_2^\top r = 0$ to obtain the third equality, and $(c_1 - c_2)^\top r = \mathcal{N}$ to obtain the fifth equality. Now it follows from $2a^\top r + \lambda\theta\mathcal{N} > 0$, $c_1^\top(x' - x^*) < 0$, and $\lambda\theta\mathcal{N} \geq 0$ that $c_2^\top(x' - x^*) \geq 0$.

Now suppose $(-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$, and let $\lambda \geq 0$ and $0 \leq \theta \leq 1$ be such that $-a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. By Proposition 1, x' satisfies (6), and using (13), we can write

$$\mathcal{N}(x' - x^*) - 2c_1^\top(x' - x^*)r \in \mathbb{L}^n.$$

As before, because $\mathbb{L}^n$ is self-dual, we get

$$0 \leq (-a + \lambda(\theta c_1 + (1-\theta)c_2))^\top(\mathcal{N}(x' - x^*) - 2c_1^\top(x' - x^*)r).$$

The right-hand side of this inequality is identical to

$$(2a^\top r + \lambda(1-\theta)\mathcal{N})c_1^\top(x' - x^*) + \lambda(1-\theta)\mathcal{N}c_2^\top(x' - x^*).$$

It follows from $2a^\top r + \lambda(1-\theta)\mathcal{N} > 0$, $c_1^\top(x' - x^*) < 0$, and $\lambda(1-\theta)\mathcal{N} \geq 0$ that $c_2^\top(x' - x^*) \geq 0$.

Finally suppose $(-a + \text{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset$, and let $\theta \geq 0$ be such that $-a + \theta c_2 \in -\mathbb{L}^n$. Then using (15),

$$\begin{aligned}
0 &\geq (-a + \theta c_2)^\top (\mathcal{N}(x' - x^*) + 2c_2^\top (x' - x^*) r) \\
&= -2c_2^\top (x' - x^*) a^\top r + \theta c_2^\top (x' - x^*)(\mathcal{N} + 2c_2^\top r) \\
&= -2c_2^\top (x' - x^*) a^\top r.
\end{aligned}$$

It follows from $a^\top r > 0$ that $c_2^\top (x' - x^*) \geq 0$.

b) If $a^\top r < 0$, then $a^\top(-r) > 0$. Since $-r = \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, part (b) follows from part (a) by interchanging the roles of C_1 and C_2.

$\square$

In the next result we show that the inequality (5) is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$ when conditions (11) and (12) hold.

Theorem 3. *Let C_1 and C_2 be defined as in* (2)*. Suppose Assumptions 3 and 4 hold. Let c_1 and c_2 be defined as in* (10)*. Suppose also that one of the following conditions is satisfied:*

a) $a^\top r = 0$,

b) $a^\top r > 0$ *and* (11) *holds,*

c) $a^\top r < 0$ *and* (12) *holds.*

Then

$$\overline{\text{conv}}(C_1 \cup C_2) = \{x \in C : \ x \text{ satisfies } (5)\}. \tag{16}$$

Proof. Let D denote the set on the right-hand side of (16). The inequality (5) is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ by Theorem 1. Hence, $\overline{\text{conv}}(C_1 \cup C_2) \subseteq D$. Let $x' \in D$. If $x' \in C_1 \cup C_2$, then clearly $x' \in \overline{\text{conv}}(C_1 \cup C_2)$. Therefore, suppose $x' \in C \setminus (C_1 \cup C_2)$ is a point that satisfies (5). By Proposition 1, it satisfies (6) and (7) as well. We are going to show that in each case x' belongs to $\overline{\text{conv}}(C_1 \cup C_2)$.

a) Suppose $a^\top r = 0$. By Remarks 2 and 3, $\mathcal{N} = 2c_1^\top r = -2c_2^\top r > 0$. Define α_1, α_2, x_1, and x_2 as in (9). It is not difficult to see that $a^\top x_1 = a^\top x_2 = 1$ and $c_1^\top x_1 = c_2^\top x_2 = 0$. Furthermore, $x' \in \text{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. One can show that $x_1, x_2 \in \mathbb{L}^n$ using the same arguments as in the proof of Theorem 2. This proves $x_1 \in C_1$ and $x_2 \in C_2$.

b) Suppose $a^\top r > 0$ and (11) holds. Let $x^* := \frac{r}{a^\top r}$. Then by Lemma 2, $c_1^\top (x' - x^*) < 0$ and $c_2^\top (x' - x^*) \geq 0$.

   First, suppose $c_2^\top (x' - x^*) > 0$, and let

$$\begin{aligned}
\alpha_1 &:= \frac{-c_1^\top x'}{c_1^\top (x' - x^*)}, & \alpha_2 &:= \frac{-c_2^\top x'}{c_2^\top (x' - x^*)}, \\
x_1 &:= x' + \alpha_1 (x' - x^*), & x_2 &:= x' + \alpha_2 (x' - x^*).
\end{aligned} \tag{17}$$

As in part a), $a^\top x_1 = a^\top x_2 = 1$, $c_1^\top x_1 = c_2^\top x_2 = 0$, and $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_1 < 0 < \alpha_2$. To show $x_1, x_2 \in \mathbb{L}^n$, first note $\mathcal{N}x^* - 2(c_1^\top x^*)r = \mathcal{N}x^* + 2(c_2^\top x^*)r = 0$ as in (13) and (14). Using this and $c_1^\top x_1 = c_2^\top x_2 = 0$, we get

$$\mathcal{N}x_1 = \mathcal{N}x_1 - 2(c_1^\top x_1)r = (1+\alpha_1)(\mathcal{N}x' - 2(c_1^\top x')r),$$
$$\mathcal{N}x_2 = \mathcal{N}x_2 + 2(c_2^\top x_2)r = (1+\alpha_2)(\mathcal{N}x' + 2(c_2^\top x')r).$$

Clearly, $1 + \alpha_2 > 0$, so $\mathcal{N}x_2 \in \mathbb{L}^n$. Furthermore,

$$1 + \alpha_1 = \frac{-c_1^\top x^*}{c_1^\top (x' - x^*)} = \frac{-c_1^\top r}{(a^\top r) c_1^\top (x' - x^*)} = \frac{-\mathcal{N}}{2(a^\top r) c_1^\top (x' - x^*)} > 0$$

where we have used the relationships $\mathcal{N} > 0$, $a^\top r > 0$, and $c_1^\top (x' - x^*) < 0$ to reach the inequality. It follows that $\mathcal{N}x_1 \in \mathbb{L}^n$ as well. Because $\mathcal{N} > 0$, we get $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in C_1$ and $x_2 \in C_2$.

Now suppose $c_2^\top (x' - x^*) = 0$, and define α_1 and x_1 as in (17). All of the arguments that we have just used to show $\alpha_1 < 0$ and $x_1 \in C_1$ continue to hold. Using $\mathcal{N}x^* + 2c_2^\top x^* r = 0$, we can write

$$\mathcal{N}(x' - x^*) = \mathcal{N}(x' - x^*) + 2c_2^\top (x' - x^*) r \in \mathbb{L}^n.$$

Because $\mathcal{N} > 0$, we get $x' - x^* \in \mathbb{L}^n$. Together with $c_2^\top (x' - x^*) = 0$ and $a^\top (x' - x^*) = 0$, this implies $x' - x^* \in \operatorname{rec} C_2$. Then $x' = x_1 - \alpha_1 (x' - x^*) \in C_1 + \operatorname{rec} C_2$ because $\alpha_1 < 0$. The claim now follows from the fact that the last set is contained in $\overline{\operatorname{conv}}(C_1 \cup C_2)$ (see, e.g., [17, Theorem 9.8]).

c) Suppose $a^\top r < 0$ and (12) holds. Since $-r := \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, part (c) follows from part (b) by interchanging the roles of C_1 and C_2.

□

The following result shows that when C is an ellipsoid or a paraboloid, the closed convex hull of any two-term disjunction can be obtained by adding the cut (5) to the description of C.

Corollary 1. *Let C_1 and C_2 be defined as in* (2)*. Suppose Assumptions 3 and 4 hold. Let c_1 and c_2 be defined as in* (10)*. If $a \in \mathbb{L}^n$, then* (16) *holds.*

Proof. The result follows from Theorem 3 after observing that conditions (11) and (12) are trivially satisfied for any c_1 and c_2 when $a \in \mathbb{L}^n$. □

The case of a split disjunction is particularly relevant in the solution of mixed-integer second-order cone programs, and it has been studied by several groups recently, in particular Dadush et al. [10], Andersen and Jensen [1], Belotti et al. [4], and Modaresi et al. [16]. Theorem 3 has the following consequence for a split disjunction.

Corollary 2. *Consider C_1 and C_2 defined by a split disjunction on C as in* (2)*. Suppose Assumptions 3 and 4 hold. Let c_1 and c_2 be defined as in* (10)*. Then* (16) *holds.*

Proof. Let $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ define a split disjunction on C with $l_2 = -tl_1$ for some $t > 0$. Then we have $tl_{1,0} > -l_{2,0}$ so that $C_1 \cup C_2 \neq C$. Let λ_1, λ_2, c_1, and c_2 be defined as in (10). Let $\theta_2 := \frac{1}{\lambda_2(tl_{1,0}+l_{2,0})}$ and $\theta_1 := \frac{t\lambda_2\theta_2}{\lambda_1}$. Then

$$a + \theta_1 c_1 + \theta_2 c_2 = a + \lambda_2\theta_2(t(l_1 - l_{1,0}a) + (l_2 - l_{2,0}a)) = 0 \in \mathbb{L}^n.$$

The result now follows from Theorem 3 after observing that $\theta_1, \theta_2 \geq 0$ implies that conditions (11) and (12) are satisfied. □

When the sets C_1 and C_2 do not intersect, except possibly on their boundary, Proposition 1 says that (5) can be expressed in conic quadratic form and directly implies the following result.

Corollary 3. *Let C_1 and C_2 be defined as in* (2)*. Suppose Assumptions 3 and 4 hold. Let c_1 and c_2 be defined as in* (10)*. Suppose that one of the conditions a), b), or c) of Theorem 3 holds. Suppose, in addition, that*

$$\{x \in C : \ c_1^\top x > 0,\ c_2^\top x > 0\} = \emptyset.$$

Then

$$\begin{aligned}\overline{\operatorname{conv}}(C_1 \cup C_2) &= \{x \in C : \ x \text{ satisfies } (6)\} \\ &= \{x \in C : \ x \text{ satisfies } (7)\}.\end{aligned}$$

Remark 4. *Conditions* (11) *and* (12) *are directly related to the sufficient conditions that guarantee the closedness of the convex hull of a two-term disjunction on $\mathbb{L}^n$ explored in [14]. In particular, one can show that the convex hull of a disjunction $h_1^\top x \geq h_{1,0} \vee h_2^\top x \geq h_{2,0}$ on the whole second-order cone $\mathbb{L}^n$ is closed if*

i) $h_{1,0} = h_{2,0} \in \{\pm 1\}$ *and there exists* $0 < \mu < 1$ *such that* $\mu h_1 + (1-\mu)h_2 \in \mathbb{L}^n$*, or*

ii) $h_{1,0} = h_{2,0} = -1$ *and* $h_1, h_2 \in -\operatorname{int}\mathbb{L}^n$*.*

In our present context, exploiting (*i*) *and* (*ii*) *after letting $h_i := a + \theta_i c_i$ and $h_{i,0} := 1$ (or, $h_i := -a + \theta_i c_i$ and $h_{i,0} := -1$) for some $\theta_i > 0$ leads to* (11) *and* (12)*.*

4.2 Two Examples

In this section we illustrate Theorem 3 with two examples.

4.2.1 A Two-Term Disjunction on a Paraboloid

Consider the disjunction $-2x_1 - x_2 - 2x_4 \geq 0 \vee x_1 \geq 0$ on the paraboloid $C := \{x \in \mathbb{L}^4 : x_1 + x_4 = 1\}$. Let $C_1 := \{x \in C : \ -2x_1 - x_2 - 2x_4 \geq 0\}$ and $C_2 := \{x \in C : \ x_1 \geq 0\}$. Noting that C is a paraboloid and C_1 and C_2 are disjoint, we can use Corollary 3 to characterize $\overline{\operatorname{conv}}(C_1 \cup C_2)$ with a conic quadratic inequality:

$$\overline{\operatorname{conv}}(C_1 \cup C_2) = \left\{x \in C : \ 3x + x_1(-3;-1;0;2) \in \mathbb{L}^4\right\}.$$

Figure 1 depicts the paraboloid C in mesh and the disjunction $C_1 \cup C_2$ in blue. The conic quadratic disjunctive cut added to convexify this set is shown in red.

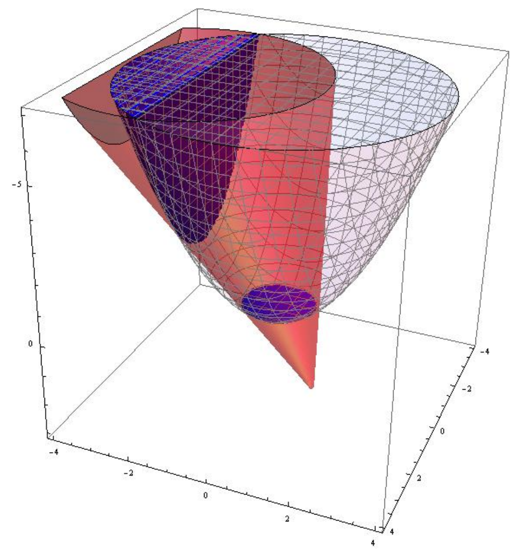

Figure 1: The disjunctive cut obtained from a two-term disjunction on a paraboloid.

4.2.2 A Two-Term Disjunction on a Hyperboloid

Consider the disjunction $-2x_1 - x_2 \geq 0 \vee \sqrt{2}x_1 - x_3 \geq 0$ on the hyperboloid $C := \{x \in \mathbb{L}^3 : x_1 = 2\}$. Let $C_1 := \{x \in C : -2x_1 - x_2 \geq 0\}$ and $C_2 := \{x \in C : \sqrt{2}x_1 - x_3 \geq 0\}$. Note that, in this setting,

$$a^\top r = \frac{1}{10}(1;0;0)^\top \left(-2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5\right) < 0,$$

but none of the conditions (12) are satisfied. The conic quadratic inequality

$$(5 + 2\sqrt{10})x + (\sqrt{2}x_1 - x_3)\left(-2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5\right) \in \mathbb{L}^3 \tag{18}$$

of Theorem 3 is valid for $C_1 \cup C_2$ but not sufficient to describe its closed convex hull. Indeed, the inequality $x_2 \leq 2$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ but is not implied by (18). Figure 2 depicts the hyperboloid C in mesh and the disjunction $C_1 \cup C_2$ in blue. The conic quadratic disjunctive cut (18) is shown in red.

Acknowledgments

We would like to thank Fatma Kılınç-Karzan for the fruitful discussions that led to the results presented in this paper. This work was supported in part by NSF grant CMMI1263239 and ONR grant N00014-12-10032.

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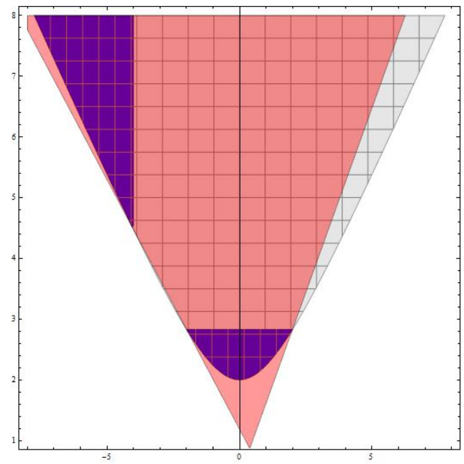

Figure 2: The disjunctive cut obtained from a two-term disjunction on a hyperboloid.

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