Every nontrivial facet-defining inequality for the corner polyhedron is an intersection cut

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Abstract

Intersection cuts were introduced by Balas and the corner polyhedron by Gomory. It is a classical result that intersection cuts are valid for the corner polyhedron. In this paper we show that, conversely every nontrivial facet-defining inequality for the corner polyhedron is an intersection cut.

Keywords: integer programming, cutting plane, corner polyhedron, intersection cut

We consider a mixed integer linear set

$$\begin{array}{ll} Ax = b & \\ x_j \in \mathbb{Z} & \text{for } j = 1, \dots, p \\ x_j \geq 0 & \text{for } j = 1, \dots, n \end{array} \tag{1}$$

where $p \leq n$, the matrix $A \in \mathbb{Q}^{m \times n}$ and the column vector $b \in \mathbb{Q}^m$. We assume that A has full row rank m. Given a feasible basis B, let $N = \{1, \dots, n\} \setminus B$ index the nonbasic variables. We rewrite the system $Ax = b$ as

$$x_i \quad = \quad \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B. \tag{2}$$

where $\bar{b}_i \geq 0,\ i \in B$.

The corner polyhedron introduced by Gomory [3] is obtained from (1) by dropping the nonnegativity restriction on all the basic variables x_i, $i \in B$, in (2). Note that in this

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relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij}x_j$ for all $i \in B \cap \{p+1,\ldots,n\}$ since these variables x_i are continuous and only appear in one equation and no other constraint. Therefore from now on we assume that all basic variables in (2) are integer variables, i.e. $B \subseteq \{1,\ldots,p\}$.

Therefore the relaxation of (1) introduced by Gomory is the mixed integer set

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j\in N} \bar{a}_{ij}x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1,\ldots,p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{3}$$

The convex hull of the feasible solutions to (3) is called the *corner polyhedron* relative to the basis B and it is denoted by corner(B).

Let $P(B)$ be the linear relaxation of (3). $P(B)$ is a polyhedron whose vertices and extreme rays are simple to describe. Indeed, $\bar{x}_i = \bar{b}_i$, for $i \in B$, $\bar{x}_j = 0$, for $j \in N$ is the unique vertex of $P(B)$. The recession cone of $P(B)$ is defined by the following system of inequalities

$$\begin{array}{rcll} x_i & = & -\sum_{j\in N} \bar{a}_{ij}x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Since the projection of this cone onto $\mathbb{R}^N$ is defined by the inequalities $x_j \geq 0$, $j \in N$ and variables x_i, $i \in B$ are defined by the above equations, its extreme rays are the vectors satisfying at equality all but one nonnegativity constraints. Thus there are $|N|$ extreme rays, $\bar{r}^j$ for $j \in N$, defined by

$$\bar{r}^j_h = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } j = h, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases} \tag{4}$$

Remark 1. *The vectors $\bar{r}^j$, $j \in N$ are linearly independent. Hence $P(B)$ is an $|N|$-dimensional polyhedron whose affine hull is defined by the equations $x_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij}x_j$ for $i \in B$. Therefore the projection of $P(B)$ onto $\mathbb{R}^N$ is an $|N|$-dimensional polyhedron that completely determines $P(B)$.*

The rationality assumption of the matrix A will be used in the proof of the next lemma. We denote the affine hull of a set by aff$(\cdot)$.

Lemma 2. *If* aff$(P(B))$ *contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then* corner$(B)$ *is an $|N|$-dimensional polyhedron. Otherwise* corner(B) *is empty.*

Proof. Since corner(B) is contained in aff$(P(B))$, corner(B) is empty when aff$(P(B))$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Next we assume that aff$(P(B))$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and we show that corner(B) is an $|N|$-dimensional polyhedron. We first show that corner(B) is nonempty.

Let $x' \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ belong to aff$(P(B))$. Then $x'_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij}x'_j$ for $i \in B$.

Let N^- be the subset of indices in $j \in N$ such that $x'_j < 0$. If N^- is empty, x' belongs to corner(B). Let $D \in \mathbb{Z}^+$ be such that $D\bar{a}_{ij} \in \mathbb{Z}$ for all $i \in B$ and $j \in N^-$. Define the point x'' as follows

$$x''_j = x'_j,\ j \in N \setminus N^-; \quad x''_j = x'_j - D\lfloor \frac{x'_j}{D} \rfloor,\ j \in N \setminus N^-; \quad x''_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij}x''_j.$$

Since x'' belongs to corner(B), this shows that corner(B) is nonempty.

Since $P(B)$ is a rational polyhedron, by a theorem of Meyer [4], the recession cones of $P(B)$ and corner(B) coincide. Since the dimension of both $P(B)$ and its recession cone is $|N|$ and corner$(B) \subseteq P(B)$, the dimension of corner(B) is $|N|$. □

$P(B)$ coincides with corner(B) when $\bar{x}_i \in \mathbb{Z}$ for $i = 1, \dots, p$. If this is not the case, $\bar{x}$ does not belong to corner(B) and we address the problem of finding valid inequalities for the set (1) that are violated by the point $\bar{x}$. Balas [2] proposed the following construction to generate valid inequalities for the corner polyhedron that cut off the basic solution $\bar{x}$.

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that the interior of C contains the point $\bar{x}$. Assume that the interior of C contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. In particular C does not contain any feasible point of (3) *in its interior. For each of the $|N|$ extreme rays of* corner(B)*, define*

$$\alpha_j = \max\{\alpha \geq 0 : \ \bar{x} + \alpha \bar{r}^j \in C\}. \tag{5}$$

Since $\bar{x}$ is in the interior of C, $\alpha_j > 0$. When the half-line $\{\bar{x} + \alpha \bar{r}^j : \alpha \geq 0\}$ intersects the boundary of C, then α_j is finite, the point $\bar{x} + \alpha_j \bar{r}^j$ belongs to the boundary of C and the semi-open segment $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < 1\}$ is contained in the interior of C. When r_j belongs the recession cone of C, we have $\alpha_j = +\infty$. Define $\frac{1}{+\infty} = 0$. The inequality

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1 \tag{6}$$

is the intersection cut *defined by C.*

Theorem 3. *(Balas [2]) Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut* (6) *defined by C is a valid inequality for* corner(B).

Proof. The set of points of the linear relaxation $P(B)$ of corner(B) that are cut off by (6) is $S := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \text{ for } i = 1, \dots, q, \ x_j \geq 0, \ j \in N, \ \sum_{j \in N} \frac{x_j}{\alpha_j} < 1\}$. We will show that S is contained in the interior of C. Since the interior of C does not contain a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, the result will follow.

Consider polyhedron $\bar{S} := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \text{ for } i = 1, \dots, q, \ x_j \geq 0, \ j \in N, \ \sum_{j \in N} \frac{x_j}{\alpha_j} \leq 1\}$. By Remark 1, $\bar{S}$ is a $|N|$-dimensional polyhedron with vertices $\bar{x}$ and $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite and extreme rays $\bar{r}_j$ for $\alpha_j = +\infty$. Since the vertices of $\bar{S}$ that lie on the hyperplane $\{x \in \mathbb{R}^n : \sum_{j \in N} \frac{x_j}{\alpha_j} = 1\}$ are the points $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite, every point in S can be expressed as a convex combination of points in the segments $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < 1\}$ for α_j finite, plus a conic combination of extreme rays $\bar{r}_j$, for $\alpha_j = +\infty$. Since the interior of C contains the segments $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < 1\}$ for α_j finite and the rays $\bar{r}_j$ for $\alpha_j = +\infty$ belong to the recession cone of C, the set S is contained in the interior of C. □

Theorem 3 shows that intersection cuts are valid for corner(B). The following theorem provides a converse statement, namely that corner(B) is defined by the intersection cuts.

Since equations $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i \in B$ define the affine hull of $P(B)$ and corner$(B) \subseteq P(B)$, every valid inequality for corner(B) can be written as $\sum_{j \in N} \gamma_j x_j \geq \delta$ in terms of the variables x_j for $j \in N$ only.

Assume that corner(B) is nonempty and that $\sum_{j\in N}\gamma_j x_j \geq \delta$ is a valid inequality for corner(B). Then $\gamma_j \geq 0$ for all $j \in N$. Indeed, if $\gamma_{j^*} < 0$ for some $j^* \in N$, then consider $\bar{r}^{j^*}$ defined in (4). We have $\sum_{j\in N}\gamma_j \bar{r}^{j^*}_j = \gamma_{j^*} < 0$, hence $\min\{\sum_{j\in N}\gamma_j x_j \,|\, x \in \text{corner}(B)\}$ is unbounded, because corner(B) is nonempty and $\bar{r}^{j^*}$ is a rational ray in the recession cone of corner(B).

If $\delta \leq 0$, the inequality $\sum_{j\in N}\gamma_j x_j \geq \delta$ is *trivial* since it is implied by the nonnegativity constraints $x_j \geq 0,\ j \in N$. If $\delta > 0$, multiplying by δ^{-1} both sides of $\sum_{j\in N}\gamma_j x_j \geq \delta$, we may assume without loss of generality that $\delta = 1$. Hence every nontrivial valid inequality for corner(B) can be written in the form $\sum_{j\in N}\gamma_j x_j \geq 1$. We say that such an inequality is *minimal* if there is no other valid inequality $\sum_{j\in N}\gamma'_j x_j \geq 1$ for corner(B) such that $\gamma'_j \leq \gamma_j$ for all $j \in N$, and the inequality is strict for at least one index $j \in N$.

Theorem 4. *If* corner(B) *is empty, the inequality* $\sum_{j\in N} 0x_j \geq 1$ *is an intersection cut for* corner(B)*. If* corner(B) *is nonempty, let* $\sum_{j\in N}\gamma_j x_j \geq 1$ *be a nontrivial minimal valid inequality for* corner(B) *with rational coefficients. Then* $\sum_{j\in N}\gamma_j x_j \geq 1$ *is an intersection cut.*

Proof. Assume first that corner(B) is empty. By Lemma 2, aff$(P(B))$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. By the "Mixed Integer Farkas' Lemma" [1], there exist an equation $\sum_{i=1}^p a_i x_i = b$ where $a_i \in \mathbb{Z}$ $i = 1,\dots p$ and $b \in \mathbb{R}\setminus\mathbb{Z}$ such that aff$(P(B)) \subseteq \{x \in \mathbb{R}^n :\ \sum_{i=1}^p a_i x_i = b\}$. (This is a certificate that aff$(P(B))$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$).

Therefore $\{x \in \mathbb{R}^n :\ \lfloor b \rfloor \leq \sum_{i=1}^p a_i x_i \leq \lceil b \rceil\}$ is a full-dimensional convex set whose interior contains aff$(P(B))$ (and hence $\bar{x}$) but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Since $P(B) \subseteq \text{aff}(P(B)) \subseteq C$, for every ray r^j of the recession cone of $P(B)$, $\max\{\alpha \geq 0 :\ \bar{x} + \alpha\bar{r}^j \in C\} = +\infty$ for every r^j in the recession cone of $P(B)$. Hence the intersection cut defined by C is $\sum_{j\in N} 0x_j \geq 1$, showing that corner$(B)$ is empty.

Assume now that corner(B) is nonempty and consider the polyhedron

$$S = \{x \in \mathbb{R}^n \,|\, \sum_{j\in N}\gamma_j x_j \leq 1,\ x_j \geq 0 \text{ for } j \in N,\ x_i = \bar{b}_i - \sum_{j\in N}\bar{a}_{ij}x_j \text{ for } i \in B\}.$$

(1) No face F of S containing $\bar{x}$ has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior (including the improper face $F = S$).

Indeed, let $\tilde{x}$ be a point of S in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. Since $S \subseteq P(B)$, $\tilde{x}$ belongs to corner(B) and since $\sum_{j\in N}\gamma_j x_j \geq 1$ is a valid inequality for corner(B), then $\sum_{j\in N}\gamma_j \tilde{x}_j = 1$.

Let F be a face of S containing $\bar{x}$, and suppose that $\tilde{x}$ is in the relative interior of F. Then there exists a scalar $\lambda > 1$ such that $z = \bar{x} + \lambda(\tilde{x} - \bar{x})$ is in F. Since $\bar{x}_j = 0,\ j \in N$ and $\sum_{j\in N}\gamma_j \tilde{x}_j = 1$, then $\sum_{j\in N}\gamma_j z_j > 1$. This contradicts the fact that $z \in S$ and (1) is proven.

Let $\tilde{S} = S + L$ where $L = \{0\}^p \times \mathbb{R}^{n-p}$. Since S is a rational polyhedron and the lineality space of $\tilde{S}$ contains L, the polyhedron $\tilde{S}$ can be expressed as $\tilde{S} = \{x \in \mathbb{R}^n \,|\, \sum_{j=1}^p c^i_j x_j \leq d_i,\ i = 1,\dots,t\}$ for some integral vectors $c^1,\dots,c^t \in \mathbb{Z}^p$ and $d_1,\dots,d_t \in \mathbb{Z}$. (Indeed $\{x \in \mathbb{R}^p \,|\, \sum_{j=1}^p c^i_j x_j \leq d_i,\ i = 1,\dots,t\}$ is the projection of S onto $\mathbb{R}^p$.)

(2) No face of $\tilde{S}$ containing $\bar{x}$ has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior.

Let $\tilde{F}$ be a face of $\tilde{S}$ and let $F = S \cap \tilde{F}$. Then F is a face of S and $\tilde{F} = F + L$. Therefore since L is in the lineality space of $\tilde{F}$, we have $\text{relint}(\tilde{F}) = \text{relint}(F) + L$, where $\text{relint}(\cdot)$ denotes the relative interior of a set.

Assume $\tilde{F}$ contains $\bar{x}$. Since $\bar{x}$ belongs to S and $F = S \cap \tilde{F}$, we have $\bar{x} \in F$.

Assume $\text{relint}(\tilde{F})$ contains $\tilde{x}$ in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. Then $\tilde{x} + L$ is contained in $\text{relint}(\tilde{F}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Since $\text{relint}(\tilde{F}) = \text{relint}(F) + L$, we have $\tilde{x} + L$ contains a point in $\text{relint}(F) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, a contradiction to (1). This proves (2).

(3) There exists a convex set $K \subset \mathbb{R}^p$ with no point of $\mathbb{Z}^p$ in its interior such that the set $C := K \times \mathbb{R}^{n-p}$ contains $\bar{x}$ in its interior and $\tilde{S} \subseteq C$.

Assume, without loss of generality, that $\bar{x}$ satisfies at equality the first h constraints defining $\tilde{S}$ (possibly $h = 0$), and none of the other constraints. That is

$$\begin{array}{rcll} \sum_{j=1}^p c_j^i \bar{x}_j & = & d_i & i = 1, \ldots, h; \\ \sum_{j=1}^p c_j^i \bar{x}_j & < & d_i & i = h+1, \ldots, t. \end{array}$$

Define $d'_i = d_i + 1$ for $i = 1, \ldots, h$ and $d'_i = d_i$ for $i = h+1, \ldots, t$, and let $K = \{x \in \mathbb{R}^p \mid \sum_{j=1}^p c_j^i x_j \leq d'_i, \ldots, i = 1, \ldots, t\}$ and $C = K \times \mathbb{R}^{n-p}$. Note that $C = \{x \in \mathbb{R}^n \mid \sum_{j=1}^p c_j^i x_j \leq d'_i, \ldots, i = 1, \ldots, t\}$. By construction, $\bar{x}$ is in the interior of C and $\tilde{S} \subseteq C$.

We only need to show that K contains no point of $\mathbb{Z}^p$ in its interior. Suppose not. Then there exists $\tilde{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ such that $\tilde{x}$ is in the interior of C. Hence $\sum_{j=1}^p c_j^i \tilde{x}_j < d'_i$ for $i = 1, \ldots, t$. By definition of d' and since $\tilde{x}_j \in \mathbb{Z}$ for $j = 1, \ldots, p$, we have $\sum_{j=1}^p c_j^i \tilde{x}_j \leq d_i$ for $i = 1, \ldots, h$, and $\sum_{j=1}^p c_j^i \tilde{x}_j < d_i$ for $i = h+1, \ldots, t$. Let J be the set of indices i such that $\sum_{j=1}^p c_j^i \tilde{x}_j = d_i$, and $\tilde{F} = \{x \in \tilde{S} \mid \sum_{j=1}^p c_j^i \tilde{x}_j = d_i, \ i \in J\}$. By construction, $\tilde{F}$ is a face of $\tilde{S}$ containing $\bar{x}$, and $\tilde{x}$ is a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in the relative interior of $\tilde{F}$, a contradiction to (2) and this proves (3).

For $h \in N$, let β_h be the largest β such that $\bar{x} + \beta \bar{r}^h$ is in S. Since $1 = \sum_{j \in N} \gamma_j (\bar{x}_j + \beta_h \bar{r}_j^h) = \gamma_h \beta_h$, then $\gamma_h = \beta_h^{-1}$.

Let α_j be the largest scalar such that $\bar{x} + \alpha_j \bar{r}^j$ is in C, $j \in N$. Since $S \subseteq \tilde{S} \subseteq C$, $\alpha_j \geq \beta_j$ for every $j \in N$, hence $\gamma_j \geq \alpha_j^{-1}$. Therefore the intersection cut defined by C, namely $\sum_{j \in N} \alpha_j^{-1} x_j \geq 1$, dominates the inequality $\sum_{j \in N} \gamma_j x_j \geq 1$. Since the latter is minimal, $\gamma_j = \alpha_j^{-1}$, $j \in N$. □

Corollary 5. *Assume that* corner(B) *is nonempty. Then every nontrivial facet defining inequality for* corner(B) *is an intersection cut.*

Proof. Let $\sum_{j \in N} \gamma_j x_j \geq 1$ be a facet defining inequality for corner(B) and let F be the corresponding facet of corner(B).

We first show that $\sum_{j \in N} \gamma_j x_j \geq 1$ is a minimal inequality. Let $\sum_{j \in N} \gamma'_j x_j \geq 1$ be a valid inequality for corner(B) such that $\gamma'_j \leq \gamma_j$ for all $j \in N$. The face F' of corner(B) that it defines contains F. Since F' is distinct from corner(B) and F is a facet, we have $F' = F$. By the unique representation of facets it follows that $\gamma'_j = \gamma_j$ for all $j \in N$. Therefore $\sum_{j \in N} \gamma_j x_j \geq 1$ is a minimal inequality.

Furthermore, by Meyer's theorem, every facet defining inequality of corner(B) has rational coefficients. Now the corollary follows from Theorem 4. □

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