Decomposing Berge Graphs Containing Proper Wheels **work in progress**

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Abstract

If a Berge graph contains certain wheels, then it contains a "good" skew partition.

1 Introduction

A graph G is *perfect* if, for all induced subgraphs of G, the size of a largest clique is equal to the chromatic number [1]. Lovász [8] showed that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. A graph is *minimally*

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imperfect if it is not perfect but all its proper induced subgraphs are. The only known minimally imperfect graphs are the odd holes and their complements. Berge [1] conjectured that there are no other (Strong Perfect Graph Conjecture). A graph is called *Berge* if it contains no odd hole or its complement. Every perfect graph is Berge. The Strong Perfect Graph Conjecture states that every Berge graph is perfect.

A graph G has a *skew partition* if the nodes $V(G)$ can be partitioned into nonempty sets A, B, C, D such that every node of A is adjacent to every node of B and there is no edge between C and D. Chvátal [4] conjectured that a minimally imperfect graph cannot have a skew partition. Chvátal [4] proved this when A or B has cardinality one (the star cutset lemma).

Hoàng [7] proved the conjecture for special types of skew partitions. A *T-cutset* is a skew partition with $u \in C$ and $v \in D$ such that every node of A is adjacent to both u and v.

Theorem 1 (Hoàng [7]) *No minimally imperfect graph has a T-cutset.*

This work was generalized by Robertson, Seymour, Thomas [10]. A skew partition (A, B, C, D) is *good* if $C \cup D$ contains a node u that is adjacent to every node of A or B.

Theorem 2 (Robertson, Seymour, Thomas [10]) *No minimally imperfect graph has a good skew partition.*

Chvátal's skew partition conjecture was solved recently in its generality:

Theorem 3 (Chudnovsky, Robertson, Seymour, Thomas [3]) *No minimally imperfect graph has a skew partition.*

In these notes, we show that, if a Berge graph contains certain types of induced subgraphs called wheels, then it has a good skew partition. This shows that no minimally imperfect graph can contain these types of wheels.

2 The Wonderful Lemma

Given a set $X \subset V(G)$ and a node $x \notin X$, we say that x is *universal for* X if x is adjacent to every node of X. We say that an edge $e = yz$ such that $y, z \notin X$, *sees* X if both y and z are universal for X.

Given a chordless path (or a hole) P in $G \setminus S$, we denote by $E_S(P)$ the set of edges in P that see S. $|P|$ denotes the length (number of edges) of P. $\mathrm{int}(P)$ denotes the set of internal nodes of P.

The following lemma, due to Roussel and Rubio [11], plays a fundamental role in this paper. This lemma was proved independently by Robertson, Seymour and Thomas [10], who named it *The Wonderful Lemma*.

Lemma 4 (Roussel and Rubio [11]) *Let G be a Berge graph where $V(G)$ can be partitioned into a co-connected set S and an odd chordless path $P = u, u', \ldots, v', v$ of length at least 3 such that u, v are both universal for S. Then one of the following holds:*

(i) An odd number of edges of P see S.

(ii) $|P| = 3$ and $S \cup \{u', v'\}$ contains an odd chordless anti-path between u' and v'.

(iii) $|P| \geq 5$ and there exist two nonadjacent nodes x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a chordless path.

Proof: The proof is by induction on $|S| + |P|$.

Note that, for every $x \in S$, there is an odd number of edges in $E(P)$ that see x, otherwise $V(P) \cup \{x\}$ contains an odd hole. We can therefore assume that $|S| \geq 2$.

Claim 1: Lemma 4 holds if $|P| = 3$.

If $|P| = 3$ and (i) does not hold, then S can be partitioned into 3 sets S_1, S_2 and S_3 such that every node in S_1 (resp. S_2) is adjacent to u' (resp. v') but not to v' (resp. u'), every node in S_3 is adjacent to u' and v', and both S_1 and S_2 are nonempty. Given two nodes $x_1 \in S_1$ and $x_2 \in S_2$ with minimum distance in $\bar{G}[S]$, let P' be a shortest x_1, x_2-anti-path in S, then $(x_1, P', x_2, u', v, u, v', x_1)$ is an anti-hole that is even if and only if P' has odd length. But then v', x_1, P', x_2, u' is a chordless odd anti-path in $S \cup \{u', v'\}$ and (ii) holds.

We may assume, then, that $|P| \geq 5$ and $|S| \geq 2$.

Claim 2: Lemma 4 holds if S contains two nonadjacent nodes x, x' such that $V(P) \setminus \{u, v\} \cup \{x, x'\}$ contains an odd chordless path P' between x and x'.

Assume, by contradiction, that such nodes a path P' between two nodes x and x' in S exists. If (iii) holds then we are done. Therefore x or x' must have a neighbor in the interior of P distinct from u' and v', so u or v has no neighbors in the interior of P', say, w.l.o.g., u. But then (u, x, P', x', u) is an odd hole, a contradiction.

Claim 3: The interior of P does not contain two adjacent nodes y, y' such that $S \cup \{y, y'\}$ contains a chordless odd anti-path P' between y and y'.

Assume not. Then, since $|P| \geq 5$, either u or v is adjacent to neither y nor y', say, w.l.o.g., u. But then (u, y, P', y', u) is an odd anti-hole, a contradiction.

Claim 4: For every co-connected nonempty subset S' of S, and for every odd subpath $P' = z, ..., z'$ of P such that z, z' are universal for S' and $G[S' \cup V(P')]$ is a proper subgraph of G, we may assume that $E_{S'}(P_{zz'})$ has odd cardinality.

Assume not. Then, by induction, either S' contains two nonadjacent nodes x, x' such that $V(P_{zz'}) \setminus \{z, z'\} \cup \{x, x'\}$ contains an odd path between x and x', and we are done by Claim 2, or the interior of $P_{zz'}$ contains two adjacent nodes y,y' such that $S' \cup \{y, y'\}$ contains a chordless odd anti-path between y and y', contradicting Claim 3.

Claim 5: No node in int(P) is universal for S.

Assume not. Then P can be partitioned into proper subpaths P_1,...,P_k such that, for every $1 \leq i \leq k$, $P_i = u_i, ..., u_{i+1}$, u_i is universal for S for every $1 \leq i \leq k+1$, $u_1 = u$, $u_{k+1} = v$ and no intermediate node of P_i is universal for S. Since P is an odd path, there is an odd number of paths P_i, $1 \leq i \leq k$ of odd length and, since (i) does not hold, $E_S(P)$ has even cardinality. Therefore there exists j, $1 \leq j \leq k$, such that P_j is an odd path of length at least 3, but $E_S(P_j) = 0$, contradicting Claim 4.

Let s_1, s_2 be two nodes with maximum distance in $\bar{G}[S]$, and let P' be a shortest anti-path between s_1 and s_2 contained in S. Let $S_1 = S \setminus s_1$, $S_2 = S \setminus s_2$ and $S' = S_1 \cap S_2$. By our choice of s_1 and s_2, S_1, S_2 and S' are all co-connected.

Claim 6: P' has odd length.

By Claim 4, $E_{S_i}(P)$ has odd cardinality, $i = 1, 2$, and, by Claim 5, no node universal for S_1 is also universal for S_2. Therefore, since $|P| \geq 5$, there exist two nonadjacent nodes z_1 and z_2 in the interior of P such that z_1 (resp. z_2)

is universal for S_1 (resp. S_2) but not for S_2 (resp. S_1). Since both z_1 and z_2 are universal for S', then, if P' has even length, $(z_1, s_1, P', s_2, z_2, z_1)$ is an odd anti-hole, a contradiction.

Since $E_{S_i}(P) \neq \emptyset$, $i = 1, 2$, then P can be partitioned into proper subpaths P_1,...,P_k where, for every $1 \leq i \leq k$, $P_i = u_i, ..., u_{i+1}$, u_i is universal for S_1 or S_2 for every $1 \leq i \leq k+1$, $u_1 = u$, $u_{k+1} = v$ and no node in int(P_i) is universal for S_1 or S_2.

Claim 7: There exists j, $1 \leq j \leq k$, such that P_j is an odd path of length at least 3, u_j is universal for S_1 and u_{j+1} is universal for S_2.

We first show that for any i, $1 \leq i \leq k$, if P_i has length 1 then $u_i u_{i+1} \in E_{S_1}(P) \cup E_{S_2}(P)$. Suppose otherwise. W.l.o.g. s_1 is adjacent to u_i but not u_{i+1} and s_2 is adjacent to u_{i+1} but not u_i. Since $|P| \geq 5$, then either u or v is adjacent to neither u_i nor u_{i+1}, say, w.l.o.g., u. But then, by Claim 6, $(u, u_{i+1}, s_1, P', s_2, u_i, u)$ is an odd anti-hole, a contradiction. Since P is an odd path, then there is an odd number of paths P_i, $1 \leq i \leq k$ of odd length. By Claim 4, $E_{S_i}(P)$ has odd cardinality for $i = 1, 2$. By Claim 5, $E_{S_1}(P) \cap E_{S_2}(P) = \emptyset$, so $E_{S_1}(P) \cup E_{S_2}(P)$ has even cardinality. Therefore there exists j, $1 \leq j \leq k$, such that P_j is an odd path of length at least 3. If both u_j and u_{j+1} are universal for S_1 (resp. S_2), then by Claim 4, $E_{S_1}(P_j)$ (resp. $E_{S_1}(P_j)$) has odd cardinality so, since $|P_j| \geq 3$, there is a node in the interior of P_j that is universal for S_1 (resp. S_2), a contradiction. Hence P_j satisfies Claim 8.

Claim 8: Lemma 4 holds if $|S| = 2$.

If $|S| = 2$ then, in the odd path P_j of Claim 7, u_j is adjacent to s_2, and u_{j+1} is adjacent to s_1, and no node in int(P_j) is adjacent to s_1 or s_2. Since G has no odd hole, s_1 is not adjacent to u_j and s_2 is not adjacent to u_{j+1}. But then $s_2, u_j, P_j, u_{j+1}, s_1$ is an odd path and we are done by Claim 2.

Claim 9: S is a stable set.

Consider the odd path P_j of Claim 7. Since $S' \neq \emptyset$, then by Claim 4, there is an odd number of edges in P_j that see S'. Hence, since $|P_j| \geq 3$, there exists a node z in the interior of P_j that is universal for S'. If S is not a stable set, P' is an odd anti-path of length at least 3, therefore (z, s_1, P', s_2, z) is an odd anti-hole, a contradiction.

Let $s_1, s_2, s_3 \in S$ and let $S_i = S \setminus s_i$, $i = 1, 2, 3$.

By Claim 4, $E_{S_i}(P)$ is odd, for $i = 1, 2, 3$, and, by Claim 5, given $e \in E_{S_i}(P)$, $e' \in E_{S_j}(P)$, for $1 \leq i < j \leq 3$, e and e' have no endnode in common, hence there must be some $k \in \{1,2,3\}$ and an edge in $yy' \in E_{S_k}(P)$such that $\{y,y'\} \cap \{u',v'\} = \emptyset$.

Assume y is closer to u in P than y'. Let z be the neighbor of s_k in P_{uy} closest to y and z' be the neighbor of s_k in $P_{y'v}$ closest to y'. By Claim 5, $y \neq z$ and $y' \neq z'$. $P_{zz'}$ is even, otherwise $(s_k, z, P_{zz'}, z', s_k)$ would be an odd hole, therefore either P_{zy} and $P_{yz'}$ are both odd paths, or $P_{zy'}$ and $P_{y'z'}$ are both odd paths. Let $w \in \{y, y'\}$ be such that P_{zw} and $P_{wz'}$ are both odd paths. Since P is an odd path, then either P_{uw} or P_{wv} has even length. Assume, w.l.o.g., that P_{uw} is an even path. Let G' be the graph induced by S, together with v and the nodes of P_{uw}, plus a new edge wv.

Claim 10: G' is a Berge graph.

Assume not. Then G' contains either an odd hole or an odd anti-hole. If G' contains an odd hole H, then H must contain wv (otherwise H would be an odd hole in G). Since v is universal for S, H must contain exactly one node in S, and such node must be s_k, since any other node in S is adjacent to both w and v. The only hole in G' containing s_k, w and v is $(z, P_{zw}, w, v, s_k, z)$, which, by construction, is even. If G' contains an odd anti-hole H, then H contains, at most, two nodes in S, since S is a stable set, and at most four nodes in P, since every set of nodes of P with at least five elements contains a stable set of size 3. But then H is a 5-anti-hole, therefore H is also a 5-hole.

By construction, since P_{uw} and P_{wv} have both length at least 2, G' has a number of nodes strictly smaller than G, while $P' = u, P_{uw}, w, v$ is an odd chordless path of length at least 3. Then, by induction, Lemma 4 holds for G'. Since, by Claim 5, there is no node in $\text{int}(P')$ universal for S, then either there exist two nodes x and x' in S such that $x, u', P_{u'w}, w, x'$ is a path, and we are done by Claim 2, or there exist two adjacent nodes t and t' in $\text{int}(P')$ such that $S \cup \{t, t'\}$ contains an odd anti-path, contradicting Claim 3. $\square$

The following is an easy consequence of Lemma 4.

Lemma 5 *Assume G is a Berge graph containing a co-connected set S and an odd chordless path $P = u, u', \ldots, v', v$ disjoint from S of length at least 3 such that u, v are both universal for the set S. Furthermore, assume that $G \setminus (S \cup V(P))$ contains a node w universal for S such that no intermediate node of P is adjacent to w. Then an odd number of edges of P see S.*

Proof: Assume not. Then, by Lemma 4, either $|P| = 3$ and $S \cup \{u', v'\}$ contains an odd anti-path Q between u' and v', or $|P| \geq 5$ and there exist two nonadjacent nodes x, x' in S such that $x, u', P_{u'v'}, v', x', w$ is a chordless path. In the first case, w, u', Q, v', w is an odd anti-hole, and in the other case $w, x, u', P_{u'v'}, v', x', w$ is an odd hole, a contradiction. $\Box$

3 Definitions

A *wheel*, denoted by (H, v), is a graph induced by a hole H and a node $v \notin V(H)$ having at least three neighbors in H. A wheel is *odd* if it contains an odd number of triangles. A wheel (H, v) is a *twin wheel* if v has exactly three neighbors in H and (H, v) contains exactly two triangles; the neighbor of v in H that is adjacent to all the other neighbors of v in H is said the *twin of v in H*. A wheel (H, v) is a *line wheel* if v has exactly four neighbors in H and (H, v) contains exactly two triangles and these two triangles have only the center v in common. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the nodes of H. A *triangle-free wheel* is a wheel containing no triangle. These four types of wheels are depicted in Figure 1, where solid lines represent edges and dotted lines represent paths. A *proper wheel* is a wheel that is not any of the above four types.

A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P^1 = x_1, \ldots, y$, $P^2 = x_2, \ldots, y$ and $P^3 = x_3, \ldots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P^i \setminus y$ and $P^j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by $\{x_1, x_2, x_3\}$. Also, at most one of the paths P^1, P^2, P^3 is an edge. We say that a graph G contains a $3PC(\Delta, .)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$.

Remark 6 *Since both odd wheels and $3PC(\Delta, \cdot)$'s contain an odd hole, they are never contained in a Berge graph as an induced subgraph.*

The following graphs will play an important role in this paper.

Definition 7 *A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths $P^1 = x_1, \ldots, y_1$, $P^2 = x_2, \ldots, y_2$ and $P^3 = x_3, \ldots, y_3$, having no common nodes and such that, for $i, j \in \{1, 2, 3\}$ distinct, x_i is not adjacent to y_j and the only adjacencies between nodes of $V(P^i) \setminus \{y_i\}$ and $V(P^j) \setminus \{y_j\}$ are the edges of the clique of size three induced by $\{x_1, x_2, x_3\}$ and the only adjacencies between nodes of $V(P^i) \setminus \{x_i\}$ and $V(P^j) \setminus \{x_j\}$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by $\{y_1, y_2, y_3\}$. We*

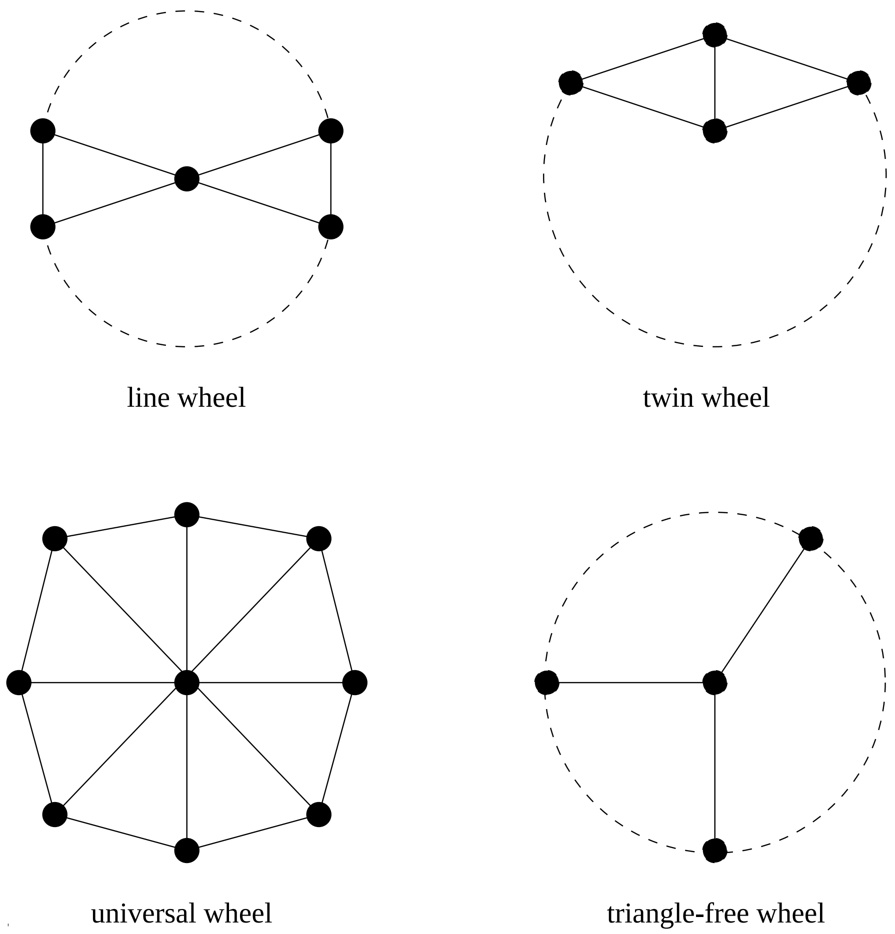

Figure 1: Wheels

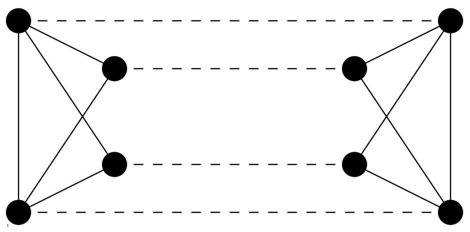

Figure 2: Connected diamonds

say that a graph G contains a $3PC(\Delta,\Delta)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1,x_2,x_3,y_1,y_2,y_3 \in V(G)$. We say that a $3PC(x_1x_2x_3, y_1y_2y_3)$ is long *if P_1, P_2 and P_3 are not all of length 1.*

Definition 8 Connected diamonds *consist of two node disjoint sets $\{a_1,\dots,a_4\}$ and $\{b_1,\dots,b_4\}$ each of which induces a diamond (the graph on four nodes with five edges) such that a_1a_4 and b_1b_4 are not edges, together with four chordless paths $P^1,\dots,P^4$ such that for $i=1,\dots,4$, P^i is a path between a_i and b_i. Paths $P^1,\dots,P^4$ are node disjoint and the only adjacencies between them are the edges of the two diamonds.*

Let H be a hole and let x_1,x_2,x_3,y_1,y_2,y_3 be distinct nodes of H such that x_2 is adjacent to x_1 and x_3, and y_2 is adjacent to y_1 and y_3. We say that (H,x,y) is a *double beetle* if x and y are not adjacent, x is adjacent to x_1,x_2,x_3 and y_2, and y is adjacent to y_1,y_2,y_3 and x_2. Note that a double beetle is a special case of connected diamonds.

Definition 9 *Given a graph G and $e=uv \in E(G)$, the graph G' obtained by* subdividing *e is the graph obtained from G by deleting the edge e and adding one node w adjacent only to u and v. Given two graphs G and G', G' is a* subdivision *of G if G' can be obtained from G by iteratively subdividing edges of G. We say that G' is a* bipartite subdivision *of G if G' is a bipartite graph that is a subdivision of G.*

A class of graphs that will play an important role in this paper is the class of line graphs of bipartite subdivisions of K_4 (the clique on four nodes). An example is depicted in Figure 3.

4 Hubs

Let H be a hole and $N \subseteq V(H)$. We say that two nodes of N are *consecutive* if at least one of the two subpaths of H joining them contains no node of N in its interior.

Theorem 10 *Let G be a Berge graph, H a hole of length at least 6, and S a co-connected set of nodes in $G \setminus V(H)$. One of the following holds:*

(1) an even number of edges of H see S, or

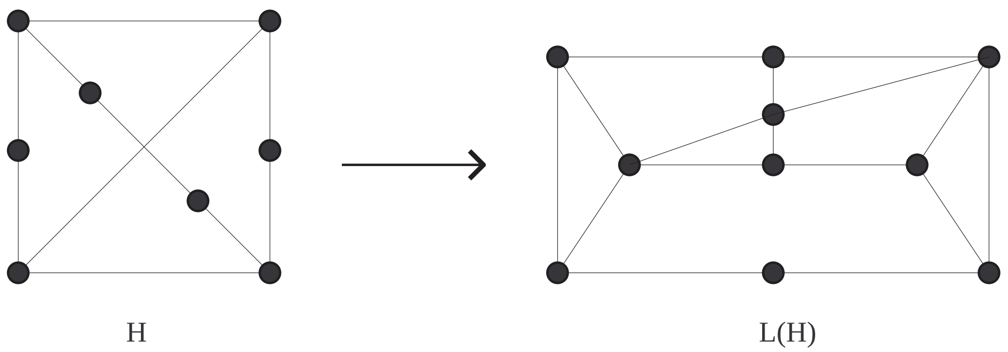

Figure 3: Bipartite subdivision of a K_4 and its line graph.

(2) S contains nonadjacent nodes x, y such that (H, x) and (H, y) are twin wheels and exactly one edge of H sees both x and y or

(3) S contains a node x with exactly 2 neighbors u and v in H, where u and v are adjacent.

Proof: The proof is by induction on $|S| + |H|$. When $|S| = 1$, the theorem is immediate, since we already observed that G cannot contain an odd wheel. We can therefore assume that S has at least 2 nodes. Also, by inductive hypothesis, for every co-connected set $S' \subset S$, $E_{S'}(H)$ is even, else (2) or (3) holds.

If $|E_S(H)|$ is even, then we are done. Hence, assume that $|E_S(H)|$ is odd and let $uv \in E_S(H)$.

Claim 1 *$E_S(H) = \{uv\}$ and no other node in H is universal for S.*
Assume not, then there exists an odd chordless subpath $P = x_1, \ldots, x_n$ of H such that $|P| \geq 3$, x_1 and x_n are both universal for S and no intermediate node of P is universal for S. Since P does not contain both u and v, let $w \in \{u, v\} \setminus V(P)$. Then the choice of S, P and w contradicts Lemma 5.

Let s_1 and s_2 be two nodes at maximum distance in $\bar{G}[S]$, and let P' be a shortest anti-path between s_1 and s_2 in S. Let $S_1 = S \setminus s_1$, $S_2 = S \setminus s_2$ and $S' = S_1 \cap S_2$. By our choice of s_1 and s_2, S_1, S_2 and S' are all co-connected.

Claim 2 *P' has odd length.*
Since $E_{S_i}(H) \setminus \{uv\} \neq \emptyset$, for $i = 1, 2$, and no node universal for S_1 in $V(H) \setminus \{u, v\}$ is also universal for S_2, then, since $|H| \geq 6$, there exist two

nonadjacent nodes z_1 and z_2 in $V(H) \setminus \{u, v\}$ such that z_1 (resp. z_2) is universal for S_1 (resp. S_2) but not for S_2 (resp. S_1). Therefore, if P' has even length, then $(z_1, s_1, P', s_2, z_2, z_1)$ is an odd anti-hole, a contradiction.

Let $u_1, \ldots, u_{k+1}$ be all the nodes of H that are universal for S_1 or S_2 in the order they appear going from u to v in $H \setminus uv$. By definition, $u_1 = u$, $u_{k+1} = v$. For every i, $1 \leq i \leq k$, let P_i be the path from u_i to u_{i+1} in $H \setminus uv$. Obviously, for every i, $1 \leq i \leq k$, no node in the interior of P_i is universal for S_1 or S_2. Since $E_{S_i}(H) \setminus \{uv\} \neq \emptyset$, $i = 1, 2$, then $k \geq 2$.

Claim 3 *There exists j, $1 \leq j \leq k$, such that P_j is an odd path of length at least 3, u_j is universal for S_1 but not for S_2 and u_{j+1} is universal for S_2 but not for S_1.*

For any i, $1 \leq i \leq k$, if P_i has length 1 then $u_i u_{i+1} \in (E_{S_1}(H) \cup E_{S_2}(H)) \setminus \{uv\}$, otherwise we may assume, w.l.o.g., that s_1 is adjacent to u_i but not u_{i+1} and s_2 is adjacent to u_{i+1} but not u_i. Since $|H| \geq 6$, then either u or v is not adjacent to u_i and u_{i+1}, say, w.l.o.g., u. But then, by Claim 2, $(u, u_{i+1}, s_1, P', s_2, u_i, u)$ is an odd anti-hole, a contradiction.

Since $H \setminus uv$ is an odd chordless path, then there is an odd number of paths P_i, $1 \leq i \leq k$ of odd length. By Claim 1, $E_{S_1}(H) \cap E_{S_2}(H) = \{uv\}$, so $E_{S_1}(H) \cup E_{S_2}(H) \setminus \{uv\}$ has even cardinality, therefore there exists j, $1 \leq j \leq k$, such that P_j is an odd path of length at least 3. If both u_j and u_{j+1} are universal for S_1 (resp. S_2), then by Lemma 5 applied to S_1 (resp. S_2), P_j and either node u or node v (since one of the two has no neighbor in the interior of P_j), P_j has an odd number of edges that see S_1 (resp. S_2), so there is a node in the interior of P_j that is universal for S_1 (resp. S_2), a contradiction. Hence P_j satisfies Claim 3.

Let u', v' be, respectively, the neighbors of u and v in $H \setminus uv$.

Claim 4 *Theorem 10 holds if $|S| = 2$.*

Assume $|S| = 2$. Let P_j be the path defined in Claim 3. If $u_j = u'$ and $u_{j+1} = v'$, then Theorem 10 (2) holds. Hence we may assume, w.l.o.g., $u' \neq u_j$, but then $(u, s_2, u_j, P_j, u_{j+1}, s_1, u)$ is an odd hole, a contradiction.

By Claim 4, we may assume $|S| \geq 3$

Claim 5 *S is a stable set.*

Since $S' \neq \emptyset$, then by Lemma 5 applied to S', P_j and u, there is an odd number of edges in P_j that see S'. Hence there exists a node z in the interior of P_j that is universal for S'. If S is not a stable set, P' is an odd anti-path of length at least 3, therefore (z, s_1, P', s_2, z) is an odd anti-hole, a contradiction.

Let $s_1, s_2, s_3 \in S$ and let $S_i = S \setminus s_i$, $i = 1, 2, 3$.

Since $E_{S_i}(H) \setminus \{uv\}$ has odd cardinality, $i = 1, 2, 3$, then, given $e \in E_{S_i}(H) \setminus \{uv\}$, $e' \in E_{S_j}(H) \setminus \{uv\}$, for $1 \leq i < j \leq 3$, by Claim 1 e and e' have no endnode in common, hence there exists $k \in \{1, 2, 3\}$ and an edge $yy' \in E_{S_k}(P)$ such that $\{y, y'\} \cap \{u', v'\} = \emptyset$. For every pair s, t of nodes of H, let us denote by H_{st} the path between s and t in $H \setminus uv$. Assume y is closer to u in $H \setminus uv$ than y'. Let z be the neighbor of s_k closest to y in H_{uy} and z' be the neighbor of s_k closest to y' in $H_{y'v}$. By Claim 1, $y \neq z$ and $y' \neq z'$. $H_{zz'}$ is even, otherwise $(s_k, z, H_{zz'}, z', s_k)$ would be an odd hole, therefore either H_{zy} and $H_{yz'}$ are both odd paths, or $H_{zy'}$ and $H_{y'z'}$ are both odd paths. Let $w \in \{y, y'\}$ be such that H_{zw} and $H_{wz'}$ are both odd paths. Since H is an even hole, then either H_{uw} or H_{wv} has even length. Assume, w.l.o.g., that H_{uw} is an even path. Let G' be the graph induced by S together with v and H_{uw}, plus a new edge wv. Let $H' = (u, H_{uw}, w, v, u)$; H' is an even hole in G'. In particular, H' must have length at least 6, otherwise z is adjacent to u, w is adjacent to z and, given any node s in S_k that is not adjacent to z, (s, w, z, s_k, v, s) is a 5-hole in G.

Claim 6 *G' is a Berge graph.*

Assume not. Then G' contains either an odd hole or an odd anti-hole. If G' contains an odd hole Q, then Q must contain wv, otherwise Q would be an odd hole in G. Also, Q must contain a node in S, otherwise $Q = H'$ that is an even hole. Since every node in S_k is adjacent to both w and v, Q must contain exactly one node in S, namely s_k. The only hole in G' containing s_k, w and v is $(z, P_{zy}, w, v, s_k, z)$, which, by construction, is even. If G' contains an odd anti-hole Q, then Q contains, at most, two nodes in S, since S is a stable set, and at most four nodes in H', since every subset of nodes of H' with at least five elements contains a stable set of size 3. But then Q is a 5-anti-hole, therefore Q is also a 5-hole, a contradiction.

Since, by construction, H_{uw} and H_{wv} have both length at least 2, H' has length strictly smaller than H. Therefore, by induction, Theorem 10 holds in G' for H' and S. Since $E_S(H') = \{uv\}$ and every node of S has at least three neighbors in H', then the only possibility is that z is adjacent to u and there exists a node s in S_k whose only neighbors in H' are u, v and w. But then, in G, $(z, H_{zw}, w, s, v, s_k, z)$ is an odd hole, a contradiction $\square$

Note that an edge set C of H of even cardinality induces a bicoloring of the nodes of H: two nodes of H are colored with distinct colors if and only if the subpaths of H connecting them contain an odd number of edges in C.

Definition 11 *Given a Berge graph G, a* hub *of G is a pair (H,S) where H is a hole of G of length at least* 6 *and S is a co-connected set in $G \setminus V(H)$ that sees a positive even number of edges of H. A* sector *of a hub (H,S) is a maximal subpath of H containing no edge of $E_S(H)$.*

Remark 12 *Let G be a Berge graph and (H,S) a hub of G. Then the endnodes of a sector are endnodes of edges of $E_S(H)$ and every sector of (H,S) has even length.*

Proof: By maximality in the definition of sector, every endnode of a sector must be an endnode of an edge in $E_S(H)$. Assume there exists a sector $P = x_1, \ldots, x_n$ of (H,S) of odd length. Let w be the endnode of some edge in $E_S(H)$ distinct from x_1 and x_n. Since both x_1 and x_n are universal for S and P has length at least 3, then by Lemma 5 applied to S, P and w, there is an odd number of edges of P that sees S, a contradiction. $\square$

Corollary 13 *Let G be a Berge graph and (H,S) be a hub of G. Let $y \in V(G) \setminus (V(H) \cup S)$ be a node that sees an odd number of edges in a sector of (H,S). Assume $S \cup y$ is co-connected. Then*

(i) y has exactly two neighbors in H and they are adjacent or

(ii) There exists $x \in S$ not adjacent to y such that (H,x) and (H,y) are twin wheels and exactly one edge of H sees both x and y or

(iii) S contains a node x not adjacent to y such that (H,y) and (H,x) are both line wheels and no edge of H sees both x and y or

(iv) $|H| = 6$, (H,y) is a line wheel and $S \cup y$ contains an odd chordless anti-path Q of length at least 3 *between y and a node x such that (H,x) is a line wheel, no edge of H sees both x and y and every intermediate node of Q is adjacent to every node in H.*

Proof: If y has exactly two neighbors in H then conclusion (i) holds. Assume then that y has at least 3 neighbors in H. If $E_{S \cup y}(H)$ has odd cardinality, then, by Theorem 10, conclusion (ii) holds. So $E_{S \cup y}(H)$ has even cardinality. Since there is an even number of edges of H that sees y and y sees an odd number of edges in some sector of (H,S), then there are at least 2 sectors $P = x_1, \ldots, x_h$ and $P' = x'_1, \ldots, x'_k$ of (H,S) such that an odd number of edges of P and P', respectively, sees y. Let y_1, y_2, (resp. y'_1, y'_2) be the neighbors of y in P (resp. P') closest to x_1 and x_h (resp. x'_1 and x'_k) respectively.

Since an odd number of edges of P sees y, then $P_{x_1y_1}$ and $P_{y_2x_h}$ have length of distinct parity. We can therefore assume that $P_{x_1y_1}$ has odd length and $P_{y_2x_h}$ has even length. Analogously, assume that $P'_{x'_1y'_1}$ has odd length and $P_{y'_2x'_k}$ has even length.
If y_1 and y_2 are nonadjacent, then $F = x_1, P_{x_1y_1}, y_1, y, y_2, P_{y_2x_h}, x_h$ is an odd path so, by Lemma 5 applied to S, F and x'_1, F has an odd number of edges that see S, contradicting either the definition of sector or the assumption that $S \cup y$ is co-connected. Hence y_1y_2 is an edge and, analogously, $y'_1y'_2$ is an edge. Let now $F = x_1, P_{x_1y_1}, y_1, y, y'_2, P_{y'_2x'_k}$. If F is a chordless path then F is odd and by Lemma 5 applied to S, F and x'_1, F has an odd number of edges that see S, a contradiction. Therefore F is not a chordless path, but then x_1 must be adjacent to x'_k. Analogously, by repeating the previous argument for $F' = x'_1, P_{x'_1y'_1}, y'_1, y, y_2, P_{y_2x_h}, x_h$ must be adjacent to x'_1. Therefore (H, y) is an L-wheel.
Case 1: $|H| > 6$

Then, w.l.o.g., $H' = (x'_1, P_{x'_1y'_1}, y'_1, y, y_2, P_{y_2x_h}, x_h, x'_1)$ is a hole of length at least 6. Since $E_S(H') = \{x'_1x_h\}$, Theorem 10 applies.

Case 1.1: Conclusion (3) of Theorem 10 holds.

Then there exists a node x in S such that the only neighbors of x in H' are x_h and x'_1. Since x sees an odd number of edges in a sector of (H, y), then, by the previous argument, (H, x) is an L-wheel and (iii) holds.

Case 1.2: Conclusion (2) of Theorem 10 holds.

Then there exists two nodes x and x' in S such that (H', x) and (H', x') are both twin wheels. Let w, w' be, respectively, the neighbors of x and x' in $V(H') \setminus \{x_hx'_1\}$ and let F be the path between w and w' induced by $V(H') \setminus \{x_h, x'_1\}$. Since F has odd length, $(x_1, x, w, F, w', x', x_1)$ is an odd hole, a contradiction.
Case 2: $|H| = 6$

Then $y_2 = x_h$ and $y'_2 = x'_k$. Since y_1 and y'_1 are not universal for S and $S \cup y$ is co-connected, let Q be a shortest anti-path in $S \cup y$ from y to a node x that is not adjacent to both y_1 and y'_1. Assume, w.l.o.g., that x is not adjacent to y_1, then (y, Q, x, y_1, x'_1, y) is an anti-hole, therefore Q must be an odd anti-path. If x is adjacent to y'_1, then $(y, Q, x, y_1, y'_1, x_1, y)$ is an odd anti-hole, a contradiction. Therefore (H, x) is a line wheel. If Q has length 1 then (iii) holds, else (iv) holds. □

5 Connections from blue to red sectors of a hub

Let P be a connected subgraph of $G \setminus (H \cup S)$. The *attachments* of P to H are the nodes of H adjacent to at least one node of P.

Theorem 14 *Let (H, S) be a hub of a Berge graph G. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node that is universal for S, such that x_1 has a blue neighbor in H and x_n has a red neighbor w.r.t. the bicoloring induced by $E_S(H)$ ($n = 1$ is allowed). If there exist consecutive attachments of P with distinct colors that are not adjacent, then one of the following holds.*

(a) There exists $y \in S$ such that $V(H) \cup V(P) \cup \{y\}$ induces the line graph of a bipartite subdivision of K_4.

(b) $n = 1$, $|H| = 6$, (H, x_1) is a line wheel and $S \cup x_1$ contains a chordless odd anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that (H, y) is a line wheel, no edge of H sees both x_1 and y and every intermediate node of Q is adjacent to every node in H.

(c) There exists $y \in S$ such that $V(H) \cup V(P) \cup \{y\}$ induces connected diamonds.

(d) $n = 1$ and there exists $y \in S$ nonadjacent to x_1 such that (H, x_1) and (H, y) are twin wheels and exactly one edge of H sees both x_1 and y.

(e) There exists $y \in S$ such that (H, y) is a twin wheel, no node of P is a neighbor of y, x_1 is adjacent to the twin of y in H and no other node in H while x_n is not adjacent to both the other neighbors of y in H.

(f) $n = 1$, H contains a subpath u, z, w, z', u' such that $E_S(H) = \{wz, wz'\}$, x_1 is adjacent to u, w and u' but not z and z', $S \cup x_1$ contains a chordless odd anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'.

(g) $n = 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent to u, w and w' but not z and z', $S \cup x_1$ contains an even anti-path Q between x_1 and a node $y \in S$ such that

y is nonadjacent to u and every intermediate node of Q is adjacent to u. Furthermore, every node in $V(H) \setminus \{z, z'\}$ that is universal for S is adjacent to x_1.

(h) *$n > 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent w and w' but not u, z and z', while x_n is adjacent to u but not w, z, w' and z'. Furthermore S contains two nonadjacent nodes y and y' such that the only neighbors of y in $V(P) \cup \{w, z, u, z', w'\}$ are u, z, z', w, w' while the only neighbors of y' in $V(P) \cup \{w, z, u, z', w'\}$ are x_1, z, z', w, w'.*

(k) *$n > 1$, $H = (v, w, z, u, z', w', v)$, $E_S(H) = \{wz, w'z'\}$, x_1 is adjacent only to v in H and x_n is adjacent only to u in H. Furthermore, S contains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'.*

Proof: Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H.

Case 1: x_1 or x_n sees an odd number of edges in some sector of (H, S).

Assume, w.l.o.g., that x_1 sees an odd number of edges in some sector of (H, S): then conclusion (i), (ii), (iii) or (iv) of Corollary 13 holds. If conclusion (ii) of Corollary 13 holds, then (d) holds. If conclusion (iii) of Corollary 13 holds, $n = 1$ and there exists y in S non adjacent to x_1 such that (H, x_1) and (H, y) are line wheels and no edge in H sees both x_1 and y, but then one can verify that $V(H) \cup \{x_1, y\}$ is the line graph of a bipartite subdivision of K_4, so (a) holds. If conclusion (iv) of Corollary 13 holds, then (b) holds. Therefore we can assume that conclusion (i) of Corollary 13 holds and x_1 has exactly two neighbors u, u' in H, u and u' are adjacent and they are both blue. If x_n has exactly one neighbor t in H, then there is a $3PC(x_1uu', t)$. If x_n has two neighbors in H that are not adjacent, then there is a $3PC(x_1uu', x_n)$. Hence x_n has exactly two neighbors v and v' in H and they are adjacent and both red. Assume that u and v are consecutive attachments of P and u', v' are consecutive attachments of P. W.l.o.g., u and v are non adjacent. Let H_{uv} and $H_{u'v'}$ be, respectively, the paths between u and v and between u' and v' in H such that no intermediate node of H_{uv} or $H_{u'v'}$ is an attachment of P. Since u and v are nonadjacent, then $H' = (u, H_{uv}, v, x_n, P, x_1)$ is a hole of length at least 6 and, since u and v

have distinct colors and no node in P is universal for S, an odd number of edges of H' see S. Also $H'' = (u', H_{u'v'}, v', x_n, P, x_1)$ is a hole (possibly of length 4) and an odd number of edges of H'' sees S. By Theorem 10, exactly one edge wz of H' and one edge of $w'z'$ of H'' sees S and one of the following cases holds.

Case 1.1: There exists a node $y \in S$ such that y has only two neighbors in H'.

But then y sees an odd number of edges in $H_{u'v'}$, so y must see exactly one edge in $H_{u'v'}$, otherwise $V(H_{u'v'}) \cup V(P) \cup \{y\}$ would induce an odd wheel. But then (H, y) is a line wheel and one can verify that $V(H) \cup V(P) \cup \{y\}$ induces the line graph of a bipartite subdivision of K_4, hence (a) holds.

Case 1.2: There exist non adjacent nodes $y, y' \in S$ such that (H', y) and (H', y') are twin wheels.

Let t and t' be the neighbors of y and y', respectively, in $V(H') \setminus \{w, z\}$. If u' and v' are nonadjacent, then at least one node among w' and z' has no neighbor in P, say w', but then $(V(H') \cup \{w', y, y'\}) \setminus \{w, z\}$ induces an odd hole, a contradiction. In particular, w.l.o.g. $t = u$ and $t' = v$, else (H, y) or (H, y') is an odd wheel. Since H' is even, P must be odd, therefore $(y, u, x_1, P, x_n, v', y)$ is an odd hole, a contradiction.

Case 2: Both x_1 and x_n see an even number of edges in every sector of (H, S).

Let u and v be two consecutive, nonadjacent attachments of P with distinct colors in the bicoloring of H induced by $E_S(H)$. Assume, w.l.o.g., v is adjacent to x_1 and u to x_n. Let H_{uv} be a subpath of H between u and v containing no attachments of P except u and v. Since u and v have distinct colors, H_{uv} contains an odd number of edges of $E_S(H)$, therefore the hole $H' = (x_1, P, x_n, u, H_{uv}, v, x_1)$ has an odd number of edges that see S, otherwise P would contain some node universal for S. By Theorem 10, H' must contain a unique edge of $E_S(H)$, say edge zw, and no node universal for S except z and w. Assume, w.l.o.g., that z is one endnode of the sector Z containing u, and let z' be the other endnode of Z. Let w' be the neighbor of z' in $V(H) \setminus V(Z)$; hence $z'w' \in E_S(H)$. Since H' is an even hole, H_{uv} has length of the same parity as P. Since u and v are nonadjacent, we may assume, w.l.o.g, that u and z are distinct. Let H_{uz} be the path between u and z in H_{uv} and H_{wv} be the path between w and v in H_{uv}.

Case 2.1: $w = w'$.

Then $w = w' = v$ and $E_S(H) = \{wz, wz'\}$.

Case 2.1.1: There exists a node $y \in S$ whose only neighbors in H' are w and z.

If (H, y) is a twin wheel, then case (e) applies. If (H, y) is not a twin wheel, y has at least a neighbor in $V(H) \setminus \{w, z, z'\}$. If u is the only neighbor of x_n in Z, then G contains a $3PC(zwy, u)$, hence x_n has a neighbor in Z distinct from u. Furthermore, since x_n sees an even number of edges in Z, x_n has a neighbor in Z that is not adjacent to u. If y has a neighbor in Z that is not adjacent to u, then there is a $3PC(zwy, x_n)$, hence y has a unique neighbor t in Z and t is adjacent to u. Furthermore, t is adjacent to x_n, else there is a $3PC(zwy, u)$. Let u' be the neighbor of x_n in Z closest to z', then $u' \neq t$. If u' is not adjacent to t, then there is a $3PC(x_n tu, y)$. So u' is adjacent to t and hence $V(H) \cup V(P) \cup \{y\}$ induces connected diamonds, so conclusion (c) holds.

Case 2.1.2: Every node in S has at least 3 neighbors in H'.

If $|H'| \geq 6$ then, by Theorem 10, S contains two nonadjacent nodes y and y' such that (H', y) and (H', y') are twin wheels and wz is the only edge of H' that sees both y and y'. But then $(V(H') \cup \{y, y'\}) \setminus \{w, z\}$ induces an odd path R between y and y' and (z', y, R, y', z') is an odd hole unless z' is adjacent to x_n. But then, since x_n sees an even number of edges in Z, H_{zu} must have even length. W.l.o.g. assume that y is not adjacent to x_1, then $(V(H_{uz}) \cup \{y, z', x_n\}) \setminus \{z\}$ induces an odd hole, a contradiction.
Hence $|H'| = 4$, so u is adjacent to z and $n = 1$. Let u' be the neighbor of x_1 in Z closest to z'. Then, since x_1 sees an even number of edges in Z and u is adjacent to z, u' and z' have odd distance in H. By repeating the previous argument on the hole H'' containing w, u' and x_1 in $V(Z) \cup \{x_1, w\}$ instead of H', we argue that u' and z' must be adjacent. Since u and u' are not universal for S, let Q be a shortest possible anti-path in $S \cup x_1$ between x_1 and a node y not adjacent to both u and u'. Assume, w.l.o.g, that y is not adjacent to u. Q must have odd length, or else (x_1, Q, y, u, z', x_1) is an odd anti-hole. Moreover, since every node in S has at least 3 neighbors in H', Q has length at least 3. Finally, if u' is adjacent to y, then $(x_1, Q, y, u, u', z, x_1)$ is an odd anti-hole, a contradiction. Hence conclusion (f) holds.

Case 2.2: $w \neq w'$.

Note that, since w' is universal for S and distinct from w and z, then w' is not in H_{uv}. Let s be the neighbor of x_n in Z closest to z' and let $H_{sz'}$ be the path between s and z' in Z. Since x_n sees an even number of edges in Z and

H_{zu} has length of the same parity as $H_{sz'}$. Let $F = w, H_{wv}, v, x_1, P, s, H_{sz'}, z'$. Since H' is an even hole and H_{zu} has the same length as $H_{sz'}$, F is an odd path between w and z'. If z is not adjacent to s then, by Lemma 5 applied to S, F and z, an odd number of edges of F see S, a contradiction. Hence u is the unique neighbor of x_n in Z and it is adjacent to z. Also, given any node t in $V(H) \setminus \{z, z', w\}$ universal for S, if t is not an attachment of P then, by Lemma 5 applied to S, F and t, an odd number of edges of F see S, a contradiction. In particular, w' must be adjacent to x_1 or to v.

If w' is adjacent to v then $F' = w', v, x_1, P, x_n, u, z$ is an odd path, therefore, by a similar argument, z' is adjacent to u and w is also adjacent to v (since x_1 sees an even number of edges in every sector, hence w cannot be adjacent to x_1). Therefore $|H| = 6$ and, since F' must have length at least 5, by Lemma 4 there exists two nonadjacent nodes nodes y and and y' in S such that y is adjacent to every node in H except v, y' is adjacent to every node in H except u and neither y nor y' has a neighbor in P, hence (k) holds.

If w' is adjacent to x_1 then $F' = w', x_1, P, x_n, u, z$ is an odd path, therefore, by the usual argument, z' is adjacent to u and w is adjacent to x_1. If $|F'| = 3$, then $n = 1$ and, by Lemma 4, there exists an odd anti-path x_1, Q, y, u between x_1 and u in $S \cup \{u, x_1\}$, hence case (g) holds. If $|F'| \geq 5$, then by Lemma 4 S contains two nonadjacent nodes y and y' such that y is adjacent to x_1, z, z', w, w' an no other node in $V(P) \cup \{w, z, u, z', w'\}$ while y' is adjacent to u, z, z', w, w' an no other node in $V(P) \cup \{w, z, u, z', w'\}$, hence case (h) holds. $\square$

Given a hub (H, S) and an edge $ab \in E_S(H)$, an *ear on ab* (with respect to (H, S)) is a chordless path $P = x_1, \dots, x_n$ in $G \setminus (V(H) \cup S)$ such that x_1 is adjacent to a, x_n is adjacent to b, no node in $V(H) \setminus \{a, b\}$ has a neighbor in P, no node of P is universal for S, and P is minimal with these properties.

Theorem 15 *Let (H, S) be a hub of a Berge graph G where S is maximal with the property that (H, S) is a hub. Let $P = x_1, \dots, x_n$ be a minimal chordless path in $G \setminus (H \cup S)$ containing no node universal for S such that x_1 has a blue neighbor in H and x_n has a red neighbor ($n = 1$ is allowed). If every pair of consecutive attachments of P with distinct colors are adjacent, then one of the following holds.*

(a) P is an ear on some edge of $E_S(H)$.

(b) $n > 1$, there exist two adjacent edges ab, bc of $E_S(H)$ such that b is the only neighbor of x_1 in H and x_n is adjacent to a, c and not to b.

Moreover, if $E_S(H) \supsetneq \{ab, bc\}$, then no node of P has a neighbor in $V(H) \setminus \{a, b, c\}$.

(c) $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the blue endnodes of the edges of H that see S (and possibly to other blue nodes of H), x_n is adjacent to all the red endnodes of the edges of H that see S (and possibly to other red nodes of H). If $n > 2$, then there exist nonadjacent $y, z \in S$ such that y is adjacent to x_1 and to no other node of P, and z is adjacent to x_n and to no other node of P. If $n = 2$, then $S \cup \{x_1, x_2\}$ contains an odd anti-path between x_1 and x_2.

Proof: Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H. Let a and b be two consecutive attachments of P with distinct colors. Then, by assumption, a and b are adjacent and $ab \in E_S(H)$. Assume, w.l.o.g., that a is adjacent to x_n and b is adjacent to x_1. Let c be the neighbor of b in $V(H) \setminus \{a\}$. If P has no neighbor in $V(H) \setminus \{a, b\}$, then P is an ear of ab and (a) occurs. Therefore we may assume, w.l.o.g., that x_n has a neighbor in $V(H) \setminus \{a, b\}$. Note that $n > 1$, otherwise either $S \cup x_1$ sees a positive even number of edges of H, contradicting the maximality of S, or ab is the only edge of H that sees $S \cup x_1$, and by Theorem 10 there exists $y \in S$ nonadjacent to x_1 such that (H, x_1) and (H, y) are twin wheels and exactly one edge of H sees both x_1 and y, thus contradicting the assumption that every two consecutive attachments of P with distinct colors are adjacent. Therefore x_1 has only blue neighbors and x_n has only red neighbors. If x_n sees an odd number of edges in some sector of (H, S) then, by Corollary 13, the only neighbors of x_n in H are a and the neighbor d of a in $V(H) \setminus \{b\}$. If x_1 has no neighbor in $V(H) \setminus \{b\}$, then G contains a $3PC(x_n ad, b)$. If x_1 has two nonadjacent neighbors in H, then G contains a $3PC(x_n ad, x_1)$. Therefore x_1 is adjacent to b, c and no other node in H. But then c and d are consecutive, non adjacent attachments of P with distinct colors in the bicoloring of H induced by $E_S(H)$, a contradiction. Therefore x_n sees an even number of edges in every sector of (H, S) and, by the same argument, also x_1 sees an even number of edges in every sector of (H, S).

We may assume that x_n has at least as many neighbors in H as x_1 does. If $E_S(H) = \{ab, bc\}$ then (b) holds. Next we show that if x_n has no neighbor in $H \setminus \{a, c\}$, then (b) holds. Suppose that x_n has no neighbor in $H \setminus \{a, c\}$. Then x_n is adjacent to c. If x_1 has no neighbors in $H \setminus b$ then (b) holds.

Otherwise , x_1 has exactly two neighbors in H, b and say d. Since all pairs of consecutive attachments of P having distinct colors are adjacent, then a, d and c, d are adjacent, hence $|H| = 4$, contradicting the assumption that (H, S) is a hub. Now we may assume that (b) does not hold, hence there exists a red sector $Z = z_1, \ldots, z_k$ of (H, S) such that $\{a, c\} \neq \{z_1, z_k\}$ and such that x_n has a neighbor in $V(Z) \setminus \{a, c\}$. Assume, w.l.o.g, that $z_1 \notin \{a, c\}$ and x_n has a neighbor in $V(Z) \setminus \{z_k\}$. Let z_i be the neighbor of x_n of lowest index in Z, and let $H_{z_1 z_i}$ be the subpath between z_1 and z_i in Z. Note that $i < k$. Since x_n sees an even number of edges in every sector of (H, S) and x_n has only red neighbors in H, then $H_{z_1 z_i}$ has even length (since x_n is adjacent to a) and also z_k and z_i have even distance in Z, hence they are not adjacent. Moreover, $H' = (a, b, x_1, P, x_n, a)$ is an even hole, therefore P is an odd path. But then $F = b, x_1, P, x_n, z_i, H_{z_1 z_i}, z_1$ is an odd chordless path. If there exists a node w universal for S in $V(H) \setminus \{a, b, z_1\}$ that has no neighbor in the interior of F, then Lemma 5 applied to S, F and w implies that there exists an odd number of edges in F that see S, a contradiction. Therefore every node universal for S in $V(H) \setminus \{a, b, z_1\}$ is adjacent either to x_1 or to x_n. Let t be the unique blue neighbor of z_1 in H. Note that t is adjacent to x_1. Since t and z_i are consecutive attachments of P, they must be adjacent. So x_n is adjacent to z_1. Hence every node of H that is universal for S must be adjacent to x_1 or x_n. In particular, x_1 is adjacent to all the blue endnodes of the edges of H that see S, x_n is adjacent to all the red endnodes of the edges of H that see S. If $n > 2$, then F has length at least 5 and by Lemma 4 there exist nonadjacent $y, z \in S$ such that y is adjacent to x_1 and to no other node of P, and z is adjacent to x_n and to no other node of P. If $n = 1$, then $|F| = 3$ and, by Lemma 4, $S \cup \{x_1, x_2\}$ contains an odd anti-path between x_1 and x_2. So conclusion (c) holds. $\square$

In the bicoloring of H induced by $E_S(H)$, we say that a node u of H is an *inner* blue (resp. red) node if both neighbors of u in H are blue (resp. red).

Theorem 16 *Let (H, S) be the hub of a Berge graph G. Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node universal for S such that x_1 has a red neighbor, no other node of P has a red neighbor and*

x_n has a blue neighbor b in H so that neither of the neighbors of b in H is a red neighbor of x_1. Then one of the following holds:

(a) P has two consecutive attachments of different colors that are nonadjacent, and P is of one of the types in Theorem 14 (a)-(c) or (f)-(k).

(b) There exist two adjacent edges ab_1, ab_2 of $E_S(H)$ such that a is the only red neighbor of x_1 in H and at least one node of P is adjacent to both b_1 and b_2. If $E_S(H) \underset{\neq}{\supseteq} \{ab_1, ab_2\}$ or if S contains a node s with no neighbors in P, then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.

(c) $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the red endnodes of the edges of H that see S and the node x_j of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If $j > 2$, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and to no other node of $P_{x_1x_j}$, and z is adjacent to x_j and to no other node of $P_{x_1x_j}$. If $j = 2$, then $S \cup \{x_1, x_2\}$ contains an odd chordless anti-path between x_1 and x_2.

Note that every path $P = x_1, \ldots, x_n$ such that x_1 has a red neighbor and x_n has an inner blue neighbor contains a subpath as in the hypothesis of Theorem 16.

Proof: Let x_j be the node of P of lowest index having a blue neighbor. If the path $P_{x_1x_j}$ has consecutive attachments of distinct colors that are not adjacent, then $P_{x_1x_j}$ satisfies the hypothesis of Theorem 14, hence one the cases (a)-(c) or (f)-(k) of Theorem 14 apply (cases (d) and (e) cannot occur since S does not contain any center of a twin wheel w.r.t. H). Since in any of these cases x_j has a blue neighbor that is not adjacent to any red neighbor of x_1 in H, then $j = n$ and case (a) holds .

Hence we may assume that every pair of consecutive attachments with distinct colors of $P_{x_1x_j}$ are adjacent, so case (a)-(c) of Theorem 15 occur. If case (c) occurs, then case (c) of Theorem 16 holds and we are done. Hence we may assume that case (a) or (b) of Theorem 15 holds. In particular, x_1 has a unique red neighbor, say a and, given b_1 and b_2 the two neighbors of a in H, ab_1 sees S and x_j is adjacent to b_1. Since x_n has a blue neighbor in H neither of whose neighbors in H is a red neighbor of x_1, $n > 1$.

Claim 1 ab_2 sees S and b_2 has a neighbor in P.

Let t be the attachment of P in $V(H) \setminus \{a, b_1\}$ that is closest to a in the path induced by $V(H) \setminus \{b_1\}$. Since a is the unique red attachment of P, then t is blue. If $t = b_2$ then ab_2 sees S and we are done. Assume then that $t \neq b_2$, hence no neighbor of t in H is a red neighbor of x_1 so t is adjacent to x_n and no other node in P. Let H_{b_2t} be the path between b_2 and t in the graph induced by $V(H) \setminus \{b_1\}$, and let $H' = (a, x_1, P, x_n, t, H_{b_2t}, b_2, a)$. Then H' is an hole of length at least 6 and, since a and t have distinct colors in the bicoloring of H induced by $E_S(H)$ and no node in P is universal for S, an odd number of edges of H' sees S, therefore, by Theorem 10, exactly one edge of H' sees S and no node of H' is universal for S except the endnodes of such edge. Since a is universal for S, then the unique edge in H' that sees S must be ab_2. Also, by Theorem 10, we have two possibilities.

Case 1: There exists a node $y \in S$ such that the only neighbors of y in H' are a and b_2.

Then t is not adjacent to b_1, otherwise (H, y) would be a twin wheel. Let $Z = z_1, \ldots, z_k$ be the path induced by $V(H) \setminus (V(H_{b_2t}) \cup \{a, b_1\})$, where z_1 is adjacent to t and z_k is adjacent to b_1. Since (H, y) is not a twin wheel, then y has a neighbor in Z. If x_n does not have a neighbor in Z, then there is a $3PC(yab_2, t)$. If both y and x_n have a neighbor in Z distinct from z_1, then there is a $3PC(yab_2, x_n)$. Note that b_1 has a neighbor in $V(P) \setminus \{x_1\}$, otherwise $(y, b_1, x_1, P, x_n, t, H_{b_2t}, b_2, y)$ is an odd hole.

If x_n has no neighbor in Z except z_1, then t and z_1 are the only neighbors of x_n in H, otherwise (H, x_n) is an odd wheel. Since b_1 has a neighbor in $V(P) \setminus \{x_1\}$, then there is a $3PC(x_ntz_1, b_1)$.

Hence x_n has a neighbor in $V(Z) \setminus \{z_1\}$, therefore the only neighbor of y in Z is z_1. Also x_n is adjacent to z_1 otherwise there is a $3PC(yab_2, t)$. Consider now the hole $H'' = (z_1, y, a, x_1, P, x_n, z_1)$. Since b_1 sees at least one edge in H'' and b_1 has at least one neighbor in $V(P) \setminus \{x_1\}$, then either (H', b_1) or (H'', b_1) is an odd wheel since b_1 sees in H'' exactly one edge more than in H'.

Case 2: S contains two nonadjacent nodes y and z such that the only neighbors of y in H' are a, b_2 and x_1 and the only neighbors of z in H' are a, b_2 and the node $c \neq a$ adjacent to b_2 in H' .

Then t is not adjacent to b_1, otherwise (H, y) would be a twin wheel. Let $Z = z_1, \ldots, z_k$ be the path induced by $V(H) \setminus (V(H_{b_2t}) \cup \{a, b_1\})$, where z_1 is adjacent to t and z_k is adjacent to b_1. Since (H, y) is not a twin wheel, then

y has a neighbor in Z. Also, since (H, z) is not an odd wheel, also z has a neighbor in Z. Let p and q be two neighbors in Z of y and z respectively with minimum distance in Z. Let Z_{pq} be the path between p and q in Z. Z_{pq} is an even path, otherwise $(a, y, p, Z_{pq}, q, z, a)$ would be an odd hole. If b_1 has a neighbor in $P \setminus x_1$, then $(P \setminus x_1) \cup H_{b_2 t} \cup \{y, z, b_1\}$ contains a $3PC(b_2 zc, b_1)$. So x_1 is the unique neighbor of b_1 in P. If x_n has no neighbors in Z, then $H \cup P$ induces a $3PC(x_1 ab_1, t)$. If z_1 is not the unique neighbor of x_n in Z, then $H \cup P$ contains a $3PC(x_1 ab_1, x_n)$. So z_1 is the unique neighbor of x_n in Z. If Z_{pq} contains z_1, then $V(Z_{pq}) \cup V(P) \cup \{y, z, a\}$ induces a $3PC(x_1 ay, z_1)$. Otherwise, $V(P) \cup (V(H_{b_2 t}) \setminus b_2) \cup V(Z_{pq}) \cup \{y, z\}$ induces an odd hole. This concludes the proof of Claim 1.

Claim 2 There exists a node in P that is adjacent to both b_1 and b_2.

Assume not. Let x_k be the node of P of lowest index that is adjacent to b_2. Since we assumed that the node x_j of lowest index in P adjacent to some blue node is adjacent to b_1, then $k > j$.

Case 1: x_1 is the unique neighbor of b_1 in $P_{x_1 x_k}$.

Then x_k must be adjacent to the neighbor c of b_2 in $V(H) \setminus \{a\}$ and to no other node in $V(H) \setminus \{b_2, c\}$, or else there is either a $3PC(ab_1 x_1, b_2)$ or a $3PC(ab_1 x_1, x_k)$. Let $F = b_1, x_1, P_{x_1 x_k}, x_k, b_2$. F is an odd path and b_1 and b_2 are universal for S. Since P does not contain any node universal for S, then conclusion (ii) or (iii) of Lemma 4 holds.

If conclusion (ii) holds, then F has length 3 and $S \cup \{x_1, x_2\}$ contains an odd anti-path Q between x_1 and x_2. Since no node of $V(H) \setminus \{a, b_1, b_2, c\}$ is adjacent to x_1 or x_2 and a is universal for all intermediate nodes of Q, then we can apply Lemma 5 in $\overline{G}$ to the set $V(H) \setminus \{a, b_1, b_2, c\}$, the path Q and the node a. Therefore there must exists an intermediate node y of Q with no neighbors in $V(H) \setminus \{a, b_1, b_2, c\}$. But then the only neighbors of y in H are a, b_1 and b_2 and (H, y) is a twin wheel, a contradiction.

If conclusion (iii) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and no other node of $P_{x_1 x_k}$ while z is adjacent to x_k and no other node of $P_{x_1 x_k}$. Since S does not contain any center of twin wheels w.r.t. H, then y and z must have neighbors in $V(H) \setminus \{a, b_1, b_2, c\}$. Let p and q be two neighbors of y and z, respectively, that are closest possible in $V(H) \setminus \{a, b_1, b_2, c\}$ and let Z be the path between p and q in the graph induced by $V(H) \setminus \{a, b_1, b_2, c\}$. Z must have even length otherwise (a, y, p, Z, q, z, a) is an odd hole, but then $(y, x_1, P_{x_1 x_k}, x_k, z, q, Z, p, y)$ is an odd hole, a contradiction.

Case 2: b_1 has a neighbor in $P_{x_2x_k}$.

Then $k > 2$ and $H' = (a, x_1, P_{x_1x_k}, x_k, b_2, a)$ is a hole of length at least 6. The only edge of H' that sees S is ab_2 hence conclusion (2) or (3) of Theorem 10 holds.

If conclusion (2) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and no other node of $P_{x_1x_k}$ while z is adjacent to x_k and no other node of $P_{x_1x_k}$, but then there exists a $3PC(zb_2x_k, b_1)$.

If conclusion (3) holds, then S contains a node y whose only neighbors in H' are a and b_2. Let P' be the shortest path between x_1 and y in the graph induced by $(V(P) \cup V(H) \cup \{y\}) \setminus \{a, b_1, b_2\}$. Then $H'' = (a, x_1, P', y, a)$ is a hole. Both b_1 and b_2 see the edge ay of H'', both b_1 and b_2 have a neighbor in $P_{x_1x_j}$ and y is not adjacent to x_k, therefore by Theorem 10 b_1 and b_2 see an even number of edges in H'', but then there exists a node of P that is adjacent to both b_1 and b_2.

This concludes the proof of Claim 2.

Claim 3 If $E_S(H) \supsetneqq \{ab_1, ab_2\}$ then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.

Assume that $E_S(H) \supsetneqq \{ab_1, ab_2\}$. Suppose it is not the case that an odd number of edges of Q see both b_1 and b_2. Let x_l be the node of highest index that is adjacent to both b_1 and b_2. Then $l > 1$. Suppose l is odd. Then $F = a, x_1, P_{x_1x_l}, x_l$ is an odd path and hence by Lemma 4 applied to F and set $\{b_1, b_2\}$, b_1 is adjacent to x_1, x_l and no other node in $P_{x_1x_l}$ while b_2 is adjacent to x_{l-1}, x_l and no other node in $P_{x_1x_l}$. But then $(V(H) \cup V(P_{x_1x_{l-1}})) \setminus \{a\}$ induces an odd hole, a contradiction. Therefore l is even. Let x_h and x_k be the nodes of highest index adjacent to, respectively, b_1 and b_2. W.l.o.g., $h \leq k$. We want to show that $P_{x_lx_h}$ has even length. Assume not, then $l < h$, therefore, by definition of l, h and k, $h < k$. Since $P_{x_lx_h}$ has odd length, then b_1 must see an odd number of edges of $P_{x_lx_h}$. Let $l = k_1 \leq \ldots \leq k_m = k$ be all the indexes between l and k such that b_2 is adjacent to x_{k_i}. Then there exists i, $1 \leq i \leq m-1$ such that b_1 sees an odd number of edges in $P_{x_{k_i}x_{k_{i+1}}}$. But then $P_{x_{k_i}x_{k_{i+1}}}$ has length at least 2 and $C = (b_2, x_{k_i}, P_{x_{k_i}x_{k_{i+1}}}, x_{k_{i+1}}, b_2)$ is an hole, therefore b_1 sees exactly one edge uv in C, and $V(C) \cup \{a, b_1\}$ induces a $3PC(b_1uv, b_2)$, a contradiction. Hence we have proven that $a, x_1, P_{x_1x_h}, x_h$ has even length.

Case 1: x_n sees an odd number of edges in some sector of (H, S).

Since x_n has only blue neighbors in H, by Corollary 13, x_n has exactly two neighbors u and v in H and they are adjacent. Suppose x_n is not adjacent to b_2. If $h < k$ then there is a $3PC(x_n uv, b_2)$. If $h = k$ then there is a $3PC(x_n uv, x_k)$. So x_n is adjacent to b_2. $P_{x_h x_n}$ has odd length, else $(V(H) \cup V(P_{x_h x_n})) \setminus \{a, b_2\}$ induces an odd hole. Let c be the neighbor of b_2 in $H \setminus a$. Then c is adjacent to x_n. Let z be the endnode distinct from b_2 of the sector Z of (H, S) containing c, and let F be the path between c and z in Z. Since $E_S(H) \supsetneqq \{ab_1, ab_2\}$, then $z \neq b_1$. Moreover F has odd length, therefore $R = a, x_1, P, x_n, c, F, z$ has odd length. Let w be the neighbor of z in $V(H) \setminus V(Z)$, then $zw \in E_S(H)$ and, by Lemma 5 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

Case 2: x_n sees an even number of edges in every sector of (H, S).

Let u be the neighbor of x_n closest to b_1 in the graph induced by $V(H) \setminus \{a, b_2\}$ and H_{ub_1} be the path between u and b_1 in the graph induced by $V(H) \setminus \{a, b_2\}$. We want to show that $P_{x_h x_n}$ has length of the same parity as the length of H_{ub_1}. If not then $u \neq b_1$ and $x_h \neq x_n$, but then $(b_1, x_h, P_{x_h x_n}, x_n, u, H_{ub_1}, b_1)$ is an odd hole. Let z be the endnode distinct from b_1 and b_2 of the sector Z of (H, S) containing u (the existence of such a node is guaranteed by the hypothesis $E_S(H) \supsetneqq \{ab_1, ab_2\}$). Let u' be the neighbor of x_n closest to z in Z and let F be the path between u' and z in Z. Since x_n sees an even number of edges in Z, then H_{ub_1} and F have lengths of the same parity, therefore $R = a, x_1, P, x_n, u', F, z$ has odd length. Let w be the neighbor of z in $V(H) \setminus V(Z)$, then $zw \in E_S(H)$ and, by Lemma 5 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

This concludes the proof of Claim 3.

Claim 4 If S contains a node s with no neighbors in P, then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.

Let F be the shortest path between x_1 and s in the graph induced by $(V(H) \cup V(P) \cup \{s\}) \setminus \{a, b_1, b_2\}$. Then $H' = (s, a, x_1, F, s)$ is a hole. Since as sees both b_1 and b_2 and there exists a further node in P that is adjacent to both b_1 and b_2 then, by Theorem 10, H' contains an even number of edges that see both b_1 and b_2, but then $Q = a, x_1, P, x_n$ has an odd number of edges that see both b_1 and b_2. This concludes the proof of Claim 4. □

6 Ears on isolated edges of a hub

Given an hub (H, S) in a Berge graph G, an edge uv in $E_S(H)$ is *isolated* if no other edge in $E_S(H)$ is adjacent to uv.

Lemma 17 *Let (H, S) be a hub of a Berge graph G such that H contains an edge uv in $E_S(H)$ that is isolated. Assume that S is maximal with such a property. Let $P = x_1, ..., x_n$ be an ear on uv. Let $Q = y_1, ..., y_m$ be a minimal path in $G \setminus (V(H) \cup V(P) \cup S)$ such that y_1 has a neighbor in P and y_m has a neighbor in the interior of some sector of (H, S). Then Q contains a node that is universal for S.*

Proof: By contradiction, let $Q = y_1, ..., y_m$ be a minimal path in $G \setminus (V(H) \cup V(P) \cup S)$ such that y_1 has a neighbor in P, y_m has a neighbor in the interior of some sector of (H, S) and no node in Q is universal for S. Note that we only need to prove the statement in the case in which Q does not contain any node whose only neighbors in H are u and v. In fact, if Q contains such a node and y_i is the node of highest index whose only neighbors are u and v, then $P' = y_i$ is an ear on uv and $Q' = y_{i+1}, Q_{y_{i+1}y_m}, y_m$ is a path such that y_{i+1} has a neighbor in P' and y_m has a neighbor in the interior of some sector of (H, S) but no node of Q' is adjacent to u, v and no other node of H. Let us assume, then, that Q does not contain any node whose only neighbors in H are u and v.

Claim 1: No node in Q is adjacent to both u and v.

Assume there exists i, $1 \leq i \leq m$, such that y_i is adjacent to u and v. Since y_i is not universal for S, then $S \cup y_i$ is co-connected. By the maximality of S, $(H, S \cup y_i)$ is not a hub, hence uv is the only edge of H that sees $S \cup y_i$. Since uv is isolated, S does not contain any center of a twin wheel w.r.t. H, hence, by Theorem 10, y_i is adjacent only to u and v in H, a contradiction.

Claim 2: Let y_i be a node with a neighbor in H distinct from v (resp. u). Let s be the neighbor of y_i closest to u (resp. v) in $V(H) \setminus \{v\}$ (resp. $V(H) \setminus \{u\}$) and assume that no node in $Q_{y_1y_{i-1}}$ has a neighbor closer to u (resp. v) in $V(H) \setminus \{v\}$ (resp. $V(H) \setminus \{u\}$) than s. Then s and u (resp. v) have the same color.

Assume, w.l.o.g., that y_i has a neighbor in H distinct from v. By contradiction, assume s and u have distinct colors, then $s \neq u$. Let w and w' be the endnodes of the sector Z of (H, S) containing s and assume w is closer to u in $V(H) \setminus \{v\}$ than w'. Since uv is isolated, then w is not adjacent to u. Let F be the shortest path between w and u in $V(Z) \cup V(Q_{y_1y_i}) \cup V(P) \cup \{u\}$ and F' be the path between u and w in $V(H) \setminus \{v\}$. Since $H' = (u, F', w, F, u)$ is a hole, then F and F' have length of the same parity. Since w and u

have distinct colors in the bicoloring of H induced by $E_S(H)$, then F' has odd length, therefore F is an odd chordless path. Since F' is odd and uv is isolated, F' contains a node t, distinct from u and v, that is universal for S. Lemma 5 applied to S, F and t implies that F contain an odd number of edges that see S, a contradiction.

Let y_j be the node of Q of lowest index such that y_j has a neighbor in H distinct from u and v. Let s be the neighbor of y_j closest to v in $V(H) \setminus \{u\}$ and t be the neighbor of y_j closest to u in $V(H) \setminus \{v\}$.

Claim 3: *st is an edge of H that sees S and $st \neq uv$. Furthermore, $P = x_1$ and no node in $Q_{y_1 y_{j-1}}$ has a neighbor in H.*

By Claim 2 applied to y_j, s has the same color of v and t has the same color of u in the bicoloring induced on H by $E_S(H)$. By Claim 1, either $s \neq v$ or $t \neq u$. Assume, w.l.o.g., that $u \neq t$. Assume s and t are nonadjacent. Then s and t are consecutive neighbors of y_j with distinct colors in H that are nonadjacent, therefore we can apply Theorem 14 to the path consisting of y_j. Since $E_S(H)$ contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 14 holds.

Case 1: Case (a) or (b) of Theorem 14 holds.

Then $E_S(H)$ consists of two nonadjacent edges uv and $u'v'$ while (H, y_j) is a line wheel. Assume v and v' have the same color. By symmetry, we may assume that $u \neq t$ and v' is not adjacent to y_j. Let F be the shortest path between u and y_j in $V(P) \cup V(Q_{y_1 y_j}) \cup \{u\}$ and let F' be the path between u and t in $V(H) \setminus \{v\}$. Since $u \neq t$, $H' = (u, F', t, y_j, F, u)$ is a hole, hence F' has distinct parity from F. But then, since y_j sees an odd number of edges in the sector of (H, S) with endnodes u and u', the shortest path F'' from u to u' in $(V(H) \cup V(P) \cup V(F)) \setminus \{v, v', t\}$ has odd length. By Lemma 5 applied to S, F'' and v', an odd number of edges of F'' see S, a contradiction.

Case 2: Case (g) of Theorem 14 holds.

Then $s = v$, u and t are adjacent and H contains a path v, u, t, u', v' where $u'v'$ sees S and y_j is adjacent to v, t, v' but not to u or u'. Let F be the shortest path between u and y_j in $V(P) \cup V(Q_{y_1 y_j}) \cup \{u\}$. Since $H' = (u, t, y_j, F, u)$ is a hole, F has even parity, but then u, F, y_j, v' is an odd chordless path and Lemma 5 applied to S, u, F, y_j, v' and u', implies that an odd number of edges of F see S, a contradiction.

Therefore s and t must be adjacent and, since they have distinct colors, st sees S. To conclude the proof of Claim 3, let $F = v_1, \ldots, v_k$ be a shortest path in $V(Q_{y_1y_j}) \cup V(P)$ such that $v_k = y_j$ and v_1 is adjacent to u or v . If v_1 is not adjacent to both u and v, say v_1 is not adjacent to v, then $V(H) \cup V(F)$ induces a a $3PC(sty_j, u)$, a contradiction. Therefore $P = x_1$, $v_1 = x_1$ and no node in $Q_{y_1y_{j-1}}$ has a neighbor in H. This concludes the proof of Claim 3.

Let H_{ut} be the path in $V(H) \setminus \{v\}$ between u and t and H_{vs} be the path in $V(H) \setminus \{u\}$ between v and s. Note that H_{ut} and H_{vs} have both even length. Let y_k be the node of lowest index in Q such that $k > j$ and y_k has a neighbor in $V(H) \setminus \{s, t\}$.

Claim 4: y_k has a neighbor both in $V(H_{ut}) \setminus \{t\}$ and in $V(H_{vs}) \setminus \{s\}$.

Assume, w.l.o.g, that y_k has a neighbor in H_{ut} distinct from t and let p be the neighbor of y_k closest to u in H_{ut} (possibly $u = p$). By Claim 2, p and u must have the same color. Let F be the shortest path between p and s in $V(Q_{y_jy_k}) \cup \{p, s\}$ and let F' be the path between u and p in H_{ut}. If y_k has no neighbors in $V(H_{vs}) \setminus \{s\}$, then $H' = (u, F', p, F, s, H_{vs}, v, u)$ is a hole, then $R = u, F', p, F, s$ is an odd path so, by Lemma 5 applied to S, R and v, R contains an odd number of edges that see S. Since u and p have the same color, then S sees an even number of edges of F', therefore S must see an odd number of edges of F, a contradiction.

Let p be the neighbor of y_k closest to u in H_{ut} and let q be the neighbor of y_k closest to v in H_{vs}. By Claim 1 and Claim 4, p and q are nonadjacent and, by Claim 2, p has the same color of u and q has the same color of v. We can also assume, w.l.o.g., that $u \neq p$.

Then p and q are consecutive neighbors of y_k with distinct colors in H that are nonadjacent, therefore we can apply Theorem 14 to the path consisting of y_k. Since $E_S(H)$ contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 14 holds.

Case 1: Case (a) or (b) of Theorem 14 holds.

Then $E_S(H)$ consists only of uv and st. Note that st is an isolated edge of $E_S(H)$, $P' = y_j$ is an ear of st and S is maximal with this property. Moreover $Q' = Q_{y_{j+1}y_k}$ is a path in $G \setminus (V(H) \cup V(P') \cup S)$ such that y_{i+1} has a neighbor in P' and y_k has a neighbor in the interior of a sector of (H, S). But now P' and Q' contradict Claim 3.

Case 2: Case (g) of Theorem 14 holds.

Then $q = v$, u and p are adjacent and H contains a path v, u, p, u', v' where $u'v'$ sees S and y_j is adjacent to v, p, v' but not to u or u'.

We have two cases:

Case 2.1: $u'v' \neq st$.

Then $u'v'$ is not adjacent to st, since v' is in H_{ut} and v' and t have distinct colors. Let F be the shortest path between u and y_k in $V(P) \cup V(Q_{y_1 y_k}) \cup \{u\}$. Since $H' = (u, p, y_k, F, u)$ is a hole, then F is even, but then u, F, y_k, v' is an odd chordless path and Lemma 5 applied to S, u, F, y_k, v' and u', implies that an odd number of edges of F see S, a contradiction.

Case 2.2: $u'v' = st$.

Then $u' = t$ and $v' = s$. Let F be the shortest path between t and y_k in $V(Q_{y_j y_k}) \cup \{t\}$. Since $H' = (t, p, y_k, F, t)$ is a hole, then F is even, but then t, F, y_k, v is an odd chordless path and Lemma 5 applied to S, t, F, y_k, v and u, implies that an odd number of edges of F see S, a contradiction.

□

Theorem 18 *Let (H, S) be a hub of a Berge graph. If G contains an ear P on an isolated edge uv of $E_S(H)$, then G has a skew partition.*

Proof: Let A be a maximal set containing S such that (H, A) is a hub and uv sees A. Assume that u is colored red in the bicoloring of (H, A) induced by $E_A(H)$. Let B be the set containing all the endnodes of the edges of $E_A(H)$ and all the nodes in $G \setminus (V(H) \cup A)$ that are universal for A. If $G \setminus (A \cup B)$ is not connected, then G contains a skew-partition. Assume that $G \setminus (A \cup B)$ is connected, then there exists a minimal path $Q = y_1, ..., y_m$ in $G \setminus (V(H) \cup V(P) \cup A \cup B)$ such that y_1 has a neighbor in P and y_m has a neighbor in the interior of some sector of (H, A), but such a path would contradict Lemma 17. □

7 Hubs in graphs containing no "large" line graphs

ASSUMPTION: Throughout this section, we will assume that G is a Berge graph such that G and $\overline{G}$ contain no long $3PC(\Delta, \Delta)$ and no line graph of a bipartite subdivision of K_4.

Lemma 19 *Let (H, S) be a hub of a Berge graph G such that G and $\overline{G}$ contain no long $3PC(\Delta, \Delta)$ and no line graph of a bipartite subdivision of K_4. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node that is universal for S, such that x_1 has a blue neighbor* *in H and x_n has a red neighbor ($n = 1$ is allowed). If there exist consecutive attachments of P with distinct colors that are not adjacent, then one of the following holds.*

(a) $|H| = 6$, $n = 1$ and there exists $y \in S$ such that $V(H) \cup \{x_1, y\}$ induces a double beetle.

(b) $n = 1$ and there exists $y \in S$ nonadjacent to x_1 such that (H, x_1) and (H, y) are twin wheels and exactly one edge of H sees both x_1 and y.

(c) There exists $y \in S$ such that (H, y) is a twin wheel, no node of P is a neighbor of y, x_1 is adjacent to the twin of y in H and no other node in H while x_n is not adjacent to both the other neighbors of y in H.

Proof: Assume not, then P is of one of the types (a)-(c) or (f)-(k) of Theorem 14. If P is of type (c), then $V(H) \cup V(P) \cup \{y\}$ contains a long $3PC(\Delta, \Delta)$ unless $n = 1$ and $|H| = 6$, so case (a) of Lemma 19 holds. P cannot be of type (a) by assumption. If P is of type (b), then $n = 1$, $|H| = 6$, (H, x_1) is a line wheel and $S \cup x_1$ contains an odd chordless anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that (H, y) is a line wheel, no edge of H sees both x_1 and y and every intermediate node of Q is adjacent to every node in H. One can verify that $\overline{G}[V(H) \cup V(Q)]$ is the line graph of a bipartite subdivision of K_4. If P is of type (f), then $n = 1$, H contains a subpath u, z, w, z', u' such that $E_S(H) = \{wz, wz'\}$, x_1 is adjacent to u, w and u' but not z and z', $S \cup x_1$ contains an odd chordless anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'. One can verify that $\overline{G}[V(Q) \cup \{u, z, z', u'\}]$ is a $3PC(uu'y, z'zx_1)$, and such $3PC(\Delta, \Delta)$ is long since Q has length at least 3. If P is of type (g), then $n = 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent to u, w and w' but not z and z', $S \cup x_1$ contains an even chordless anti-path Q between x_1 and a node $y \in S$ such that y is nonadjacent to u and every intermediate node of Q is adjacent to u. One can verify that $\overline{G}[V(Q) \cup \{w, z, u, z', w'\}]$ is a $3PC(ww'u, z'zx_1)$, which is long since Q has positive even length. If P is of type (h), then $n > 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent w and w' but not u, z and z', while x_n is adjacent to u but not w, z, w' and z'. Furthermore S contains two nodes y and y' such that the only neighbors of y in $V(P) \cup \{w, z, u, z', w'\}$ are u, z, z', w, w' while the only neighbors of y' in $V(P) \cup \{w, z, u, z', w'\}$ are x_1, z, z', w, w'. One can verify that $G[V(P) \cup \{y, y', u, z, w'\}]$ is a long $3PC(uyz, x_1w'y')$. If P is of type (k), then $H = (v, w, z, u, z', w', v)$, $E_S(H) = \{wz, w'z'\}$, x_1 is adjacent only to v in H and x_n is adjacent only to u in H. Furthermore, S contains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'. One can verify that $G[V(P) \cup \{y, y', u, v, z, w'\}]$ is a long $3PC(uyz, vw'y')$. $\square$

Lemma 20 *Let (H, S) be the hub of a Berge graph G such that G and $\overline{G}$ contain no long $3PC(\Delta, \Delta)$ and no line graph of a bipartite subdivision of K_4. Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node universal for S such that x_1 has a red neighbor, no other node of P has a red neighbor and x_n has a blue neighbor whose neighbors in H are not red neighbors of x_1. Then one of the following holds:*

(1) There exist two adjacent edges ab_1, ab_2 of $E_S(H)$ such that a is the only red neighbor of x_1 in H and at least one node of P is adjacent to both b_1 and b_2. If $E_S(H) \supsetneqq \{ab_1, ab_2\}$ or if S contains a node s with no neighbors in P, then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.

(2) $|H| = 6$, $n = 1$ and there exists $y \in S$ such that $V(H) \cup \{x_1, y\}$ induces a double beetle.

Proof: Obviously, one of the conclusions of Theorem 16 must occur. If conclusion (a) of Theorem 16 holds, then by Lemma 19 conclusion (2) holds (since S does not contain any center of a twin wheel) and we are done. If conclusion (b) holds, then conclusion (1) holds and we are done.

So we may assume that conclusion (c) of Theorem 16 holds. Then $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the red endnodes of the edges of H that see S and the node x_j of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If $j > 2$, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and to no other node of $P_{x_1x_j}$, and z is adjacent to x_j and to no other node of $P_{x_1x_j}$. If $j = 2$, then $S \cup \{x_1, x_2\}$ contains an odd chordless anti-path between x_1 and x_2.

Let uv and $u'v'$ be two nonadjacent edges of $E_S(H)$ and assume, w.l.o.g., that x_1 is adjacent to u and u' and x_j is adjacent to v and v'. If $j > 2$ then $G[V(P_{x_1x_j}) \cup \{y, z, u, v'\}]$ is a long $3PC(x_1yu, x_jv'z)$. If $j = 2$ then $\overline{G}[V(Q) \cup \{u, u', v, v'\}]$ is a long $3PC(x_1vv', x_2u'u)$. $\square$

7.1 Good hubs

We say that a hub (H, S) is *good* if H has an inner blue node and an inner red node w.r.t. the bicoloring induced on H by $E_S(H)$. Equivalently, given the maximal paths $P^1, \ldots, P^k$ induced by the endnodes of the edges of $E_S(H)$, (H, S) is a good hub if and only if there exists i, $1 \leq i \leq k$, such that P^i has odd length.

Lemma 21 *Let (H, S) be a good hub of a Berge graph G such that G and $\overline{G}$ contain no long $3PC(\Delta, \Delta)$ and no line graph of a bipartite subdivision of K_4. Let $y \in G \setminus (V(H) \cup S)$ be a node such that $(H, S \cup y)$ is a hub. Then either $(H, S \cup y)$ is a good hub or $V(H) \cup y$ contains a hole H' such that (H', S) is a good hub with $E_S(H') \subsetneqq E_S(H)$.*

Proof: Since (H, S) is a good hub, by Lemma 19 every pair of consecutive neighbors of y in H with distinct colors are adjacent. Assume $(H, S \cup y)$ is not a good hub. Let $P^1, \ldots, P^k$ be the maximal paths induced by the endnodes of the edges of $E_S(H)$ and assume, w.l.o.g, that $P^1 = y_1, ..., y_m$ has odd length. If y has no neighbor in P^1, then P^1 is contained in a sector $Q = s, ..., t$ of (H, y), therefore, given $H' = (y, s, Q, t, y)$, (H', S) is a good hub and $E_S(H') \subsetneqq E_S(H)$. Therefore we may assume that y has a neighbor in P^1. Let r be the neighbor of y closest to y_1 in P^1 and s be the neighbor of y closest to y_m in P^1 (possibly $r = s$). Since $(H, S \cup y)$ is not a good hub, then y sees an even number of edges of P^1, therefore P^1_{rs} has even length. Since P^1 has odd length, we can assume, w.l.o.g., that $P^1_{sy_m}$ has odd length. Let $Q = s, ..., t$ be the sector of (H, y) containing $P^1_{sy_m}$, then, given $H' = (y, s, Q, t, y)$, (H', S) is a good hub and $E_S(H') \subsetneqq E_S(H)$ (since $(H, S \cup y)$ is a hub). $\square$

Theorem 22 *Let G be a Berge graph such that G and $\overline{G}$ contain no long $3PC(\Delta, \Delta)$ and no line graph of a bipartite subdivision of K_4. If G contains a good hub (H, S), then G has a good skew partition.*

Proof: Assume that, among all the good hubs contained in G, (H, S) is chosen so that $E_S(H)$ is minimal (i.e. there is no good hub (H', S') such that $E_{S'}(H') \subsetneq E_S(H)$). Let A be a maximal set containing S such that (H, A) is a hub. Then, by Lemma 21 and by the minimality assumption on $E_S(H)$, (H, A) is a good hub and $E_A(H) = E_S(H)$. Let B be the set containing all the nodes that are universal for A in $G \setminus (V(H) \cup A)$ and all the blue endnodes of the edges in $E_S(H)$. If in $G \setminus (A \cup B)$ the red nodes of H are in distinct connected components than the blue nodes of H, then G has a skew partition. Otherwise there exists a chordless path $P = x_1, ..., x_n$ in $G \setminus (V(H) \cup A)$ containing no node universal for S such that x_1 is adjacent to a red node of H, no other node of P has a red node of H and x_n is adjacent to an inner blue node of H. Let j be the node of P with lowest index that is adjacent to a blue node b in H so that neither of the neighbors of b in H is a red neighbor of x_1. Then either conclusion (1) or (2) of Lemma 20 holds for $P_{x_1x_j}$. Conclusion (2) cannot hold since (H, A) is a good hub. Hence conclusion (1) holds, so there exist two adjacent edges ab_1, ab_2 of $E_A(H)$ such that a is the only red neighbor of x_1 in H and at least one node of $P_{x_1x_j}$ is adjacent to both b_1 and b_2. Since (H, A) is a good hub, $E_A(H) \supsetneq \{ab_1, ab_2\}$ so by Lemma 20 the path $Q = a, x_1, \ldots, x_j$ contains an odd number of edges that see both b_1 and b_2. If $j = 1$, then $(H, A \cup x_1)$ is a hub, contradicting the maximality of A. Therefore $j > 1$ and there exists a node x_i, $i < j$, adjacent to b_1 and b_2 and to no other node in $V(H) \setminus \{a, b_1, b_2\}$. Thus $(V(H) \cup \{x_i\}) \setminus \{a\}$ induces a hole H' and (H', A) is a good hub with $E_A(H') \subsetneq E_S(H)$, contradicting the minimality of $E_S(H)$.

Hence G contains a skew partition (A, B, C, D) where C contains all the red nodes of H and D contains all the inner blue nodes of H (w.r.t. the bicoloring induced on H by $E_A(H)$). Let u be any red endpoint of some edge in $E_A(H)$, then $u \in C$ and u is universal for A, hence (A, B, C, D) is a good skew partition. $\square$

Recently, Chudnovsky, Robertson, Seymour and Thomas [3] showed that a minimally imperfect graph cannot contain a long $3PC(\Delta, \Delta)$ or the line graph of a bipartite subdivision of K_4. This result, together with Theorems 2 and 22, implies the following.

Theorem 23 *No minimally imperfect graph contains a good hub.*

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